**Quadratic divergences and dimensional regularisation
in two and four dimensions.**

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Abstract

We calculate the quadratic divergences through four loops in renormalisable scalar theories in four dimensions and sigma models in two dimensions. A previous conjecture concerning the relationship between quadratic divergences and scale invariance is found to fail at the four loop level.

1. Introduction

The discovery that quantum field theories (QFT) led to divergent quantities when pushed beyond the tree approximation was a source of considerable disappointment to the pioneers. While subsequently the development of the machinery of renormalisation showed that unambiguous physical predictions were nevertheless accessible, there remained a widespread feeling that the divergences, although tamed, were indicative of something unsatisfactory in our approach to QFT, if not in QFT itself. This attitude has changed over the last twenty years, and the modern consensus is that the divergent nature of the radiative corrections to particle masses and coupling constants are reflections of the existence of an energy scale at which new degrees of freedom become excited, and not mere artefacts of a perturbative expansion. From this viewpoint, the quadratic divergences in renormalisable theories [1]-[10] in four dimensions become important. These divergences occur in the radiative corrections to scalar masses, and so in the specific case of the standard model, the Higgs mass term (which ultimately sets the scale for all the standard model particle masses) suffers quadratic divergences in general. This means that all the particle masses are very sensitive to unknown physics at higher energies. For this reason a great deal of interest has developed [11] in supersymmetric extensions of the standard model, since, generally speaking, supersymmetric theories are free of quadratic divergences. In a recent paper [7], we tried to argue that it might be the case that there exist non-supersymmetric theories free of quadratic divergences, and we identified a candidate for a class of such theories (unfortunately not one of phenomenological interest). From the point of view of real physics applications, it might well be argued in any case that the more important question is not whether a theory is free of quadratic divergences, but whether its parameters are insensitive to unknown physics at higher energies. As we already indicated in ref [8], freedom from quadratic divergences (to some loop order, say) is not in general sufficient for such insensitivity (to the same number of loops). This point is related to the fact that the form taken by the quadratic divergences in a given theory depends in a non-trivial way on the regulator employed. This happens even at one loop, as demonstrated in the pioneering work of Veltman [2], where (in gauge theories) one has to choose between regularisation by dimensional regularisation (DREG) and dimensional reduction (DRED) [9]. Although this specific issue does not arise in the four-dimensional ($d=4$) theories that we will consider, an analogous problem does arise in the second class we look at, namely two-dimensional sigma models. In spite, however, of the uncertainty with regard to regulator dependence and unknown higher energy physics, we believe it worthwhile pursuing the systematic analysis of quadratic divergences in the context of DREG that we began in refs. [7],[8]. In ref. [8] we demonstrated an unexpected relationship between the β -functions and the quadratic divergences in a renormalisable $d = 4$ theory with scalars and fermions. Here, in section 2, we investigate (in a theory with scalars only) whether this relationship persists at four

loops: disappointingly, it does not, and so the precise correspondence between quadratic divergences and scale invariance remains mysterious.

In section 3 we turn to $d = 2$ and examine non-linear sigma models. In the string context, the quadratic divergences in this case have a definite interpretation as a renormalisation of the tachyon background field; the fact that the tachyon can be decoupled from the spectrum in the supersymmetric case relates to the fact that the corresponding sigma model is free of quadratic divergences. We analyse up to the four-loop level, and find interesting similarities with the $d = 4$ calculation of section 2.

In section 4 we generalise the sigma model case with the inclusion of a torsion (antisymmetric tensor) potential. Because of uncertainties with regard to the treatment of the alternating tensor $\epsilon^{\mu\nu}$, our conclusions here are somewhat tentative, and we calculate only to the three-loop level. We contrast the structure of the quadratic divergences with the dilaton β -function and the string action.

2. Renormalisable scalar theories in $d = 4$

In this section we consider renormalisable theories of scalar fields in four space-time dimensions ($d=4$). In ref. [8] we gave a general analysis of how quadratic divergences manifest themselves in the DREG context as poles at unphysical values of d . In particular we showed how, at L loops, the simple pole at $d = 4 - 2/L$ [2] is the *leading* divergence, and we calculated this divergence up to $L = 3$ for a renormalisable scalar and fermion theory. Here we extend this analysis to the four-loop level in the case of a theory with scalar fields only. This is not an over-arduous calculation because of the feature mentioned above: namely that the *simple* pole in $g^{(L)} = 4 - \frac{2}{L} - d$ is what is required. As we shall see, some features occur at $L = 4$ that were not present in the lower order corrections discussed in ref. [8].

We begin with the basic Lagrangian:

$$L = \frac{1}{2}(\partial_\mu \phi^a)^2 + U(\phi), \quad (2.1)$$

where $a : 1, \dots, N$ and $U(\phi)$ is a polynomial in ϕ^a of degree four.

Straightforward power-counting shows that quadratic divergences occur in graphs with $E = 0, 1, 2$ where E is the number of external lines. Graphs corresponding to different values of E (for a given L) are calculated simultaneously if the background field method is employed, i.e. in eq(2.1) we let $\phi \rightarrow \phi + \phi_q$ and calculate graphs with internal ϕ_q (quantum) and external ϕ (classical) lines. Since we are interested only in the leading divergence (ie the pole in $\epsilon(L)$ at L loops) graphs with non-overlapping divergences or counter-term insertions can be ignored. Treatment of infra-red divergences is straightforward, with insertion of regulator masses or routing of external momenta as appropriate.

If we denote the coefficient of the leading quadratic divergence at L loops as Δ_L then for $L = 1, 2, 3$ the results are : [8]

$$\Delta_1 = U_{aa} \quad (2.2)$$

$$\Delta_2 = U_{abc} U_{abc} \quad (2.3)$$

$$\Delta_3 = U_{ab} U_{acde} U_{bcde} - \frac{9}{2} U_{abcd} U_{abc} U_{cde} \quad (2.4)$$

where $U_a = \frac{\partial U}{\partial \phi^a}$ etc. The normalisation of Δ_L is arbitrary for each value of L , that is, we have suppressed overall numerical coefficients, these being irrelevant to our purpose here. It is worth noting, however, that while the overall coefficients are not rational (involving, for example, $\Gamma(\frac{1}{2})$ in Δ_2 and $\Gamma(\frac{2}{3})$ in Δ_3) the ratio of the two contributions to Δ_3 is rational, although these contributions come from different graphs. It is this feature that changes at $L = 4$. For more details concerning the calculation of $\Delta_1 \dots \Delta_3$ and inclusion of fermions, see ref. [8].

It is straightforward to verify that the following relationships hold:

$$\Delta_2 = a_{11} \quad (2.5)$$

$$\Delta_3 = \frac{23}{36} a_{12} + \frac{1}{3} a_{21} \quad (2.6)$$

where

$$a_{LM} = \beta_L \cdot \frac{\partial}{\partial U} \Delta_M - \Delta_M \cdot \frac{\partial}{\partial U} \beta_L \quad (2.7)$$

and β_L is the L -loop contribution to the β -function, $\beta(U) = \mu \frac{\partial U}{\partial \mu}$.

To verify eq (2.5), (2.6) we need β_1, β_2 which are [12], [13].

$$\beta_1 = \frac{1}{2} U_{ab} U_{ab} \quad (2.8)$$

$$\beta_2 = -\frac{1}{2} U_{ab} U_{acd} U_{bcd} + \frac{1}{12} U_a U_{abcd} U_{abcd} \phi^a \quad (2.9)$$

Here we have suppressed a factor $(16\pi^2)^{-L}$.

The results eq (2.5), (2.6) clearly suggest a relationship between the quadratic divergence at L loops, Δ_L and the scale dependence of Δ_L , for $L' < L$. The significance of this result with regard to the conditions that a theory must satisfy so as to be free of quadratic divergences was pursued in refs. [7], [8]. Here we want to see whether something like these results, (which seem even more miraculous in the case with fermions) holds at $L = 4$. That is, we calculate Δ_4 and see if it can be written as a linear combination of a_3, a_{22} and a_{31} .

There are 5 graphs that contribute to Δ_4 (see fig 1). The result is:

$$\Delta_4 = \frac{2}{3} y A - \frac{32}{45} x B + \left(\frac{8}{3} y - \frac{64}{3} x \right) C + \frac{32}{3} x D - \frac{2}{3} y E \quad (2.10)$$

where

$$A = U_{abc} U_{bcde} U_{defg} U_{afg} \quad (2.11)$$

$$B = U_{abc} U_{bcd} U_{aefg} U_{defg} \quad (2.12)$$

$$C = U_{abc} U_{ade} U_{bdgf} U_{cefg} \quad (2.13)$$

$$D = U_{abc} U_{bcde} U_{adfg} U_{efg} \quad (2.14)$$

$$E = U_{ab} U_{acde} U_{defg} U_{bcfg} \quad (2.15)$$

and

$$x = \Gamma\left(\frac{3}{4}\right)^4, \quad (2.16)$$

$$y = \frac{\Gamma\left(\frac{3}{4}\right)^3 \Gamma\left(\frac{1}{4}\right)^3}{\Gamma\left(\frac{1}{2}\right)^3}. \quad (2.17)$$

The essential new feature of the $L = 4$ calculation is the presence in the results of x and y , above. Note that

$$\delta = \frac{y}{x} = \sqrt{\frac{2}{\pi}} \cdot \left[\Gamma\left(\frac{1}{4}\right) \right]^2 \quad (2.18)$$

This ratio cannot be expressed as a rational number, nor have we been able to discover any

relationship between it and $\zeta(3)$, which would have been interesting since $\zeta(3)$ occurs in β_3 , as we shall see below¹. It is difficult to imagine how δ would arise in a calculation of quadratic divergences using (for example) a cut-off. This highlights the scheme dependence of quadratic divergences alluded to in the introduction. It is, however, interesting that (as we shall see) δ arises at $L = 4$ also in the $d = 2$ sigma model calculations.

The crucial question with regard to the intriguing relationships eq.(2.5), (2.6) is whether we can now find $\alpha_1, \dots, \alpha_3$ such that:

$$\Delta_4 = \alpha_1 a_{13} + \alpha_2 a_{22} + \alpha_3 a_{31} . \quad (2.19)$$

We will need β_3 , which is [13]:

$$\begin{aligned} \beta_3 = & \frac{1}{4} U_{abcd} U_{efcd} U_{ae} U_{bf} \\ & - \frac{3}{16} U_{abcd} U_{ebcd} U_{af} U_{ef} \\ & + 2 U_{abcd} U_{ae} U_{bcd} U_{def} \\ & - \frac{1}{4} U_{abcd} U_{ef} U_{abe} U_{cdf} \\ & - \frac{1}{8} U_{abc} U_{dbc} U_{aef} U_{bef} \\ & + \frac{1}{2} \zeta(3) U_{abc} U_{ade} U_{dbf} U_{cef} \\ & - \frac{1}{16} U_a U_{abcd} U_{ebfg} U_{cdfg} \phi^e \end{aligned} \quad (2.20)$$

Using the results for Δ_L (eq. (2.2) - (2.4)) and β_L (eq. (2.8), (2.9) and (2.20)) it is straightforward to show that:

$$a_{13} = \frac{-27}{2} A + B - 9C - 18D + 15E \quad (2.21)$$

$$a_{22} = \frac{3}{2} B - 6C - 6D \quad (2.22)$$

$$a_{31} = \frac{-3}{2} A - \frac{7}{8} B + \left(\frac{9}{2} + 6\zeta(3)\right) C + \frac{7}{2} D + \frac{27}{8} E \quad (2.23)$$

Unfortunately it is then also straightforward to show that there exist no values of $\alpha_1, \dots, \alpha_3$ such that eq.(2.19) is true. It would thus appear that our previous conjecture, namely that absence of quadratic divergences at L loops is equivalent to demanding scale invariance of the naturalness

¹ It is interesting to note that the irrationality of $\zeta(3)$ was established by Apéry as recently as 1978. For an entertaining account of his proof, see ref [14].

conditions for all $L' < L$, is not true. Thus the precise relationship between scale invariance and quadratic divergences remains unclear. While this is disappointing, the simplicity of the results (including Eq.(2.10)) remains suggestive.

3. Quadratic divergences in the 2-d non-linear σ -model- the torsion-free case.

It is natural to inquire whether the intriguing features which we have discovered for quadratic divergences in four-dimensional theories are also displayed by theories in other dimensions. We elected to consider two-dimensional sigma models in view of their non-trivial structure and relevance for string theory and statistical mechanics.

In this section we examine the quadratic divergences for the torsion-free two-dimensional non-linear sigma model up to the four-loop order. The action, analytically continued to d dimensions, is

$$S^\sigma = \int d^d x \left[\frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j + T(\phi) \right] \quad (3.1)$$

where the scalar field $\phi^i(x)$, $i = 1, \dots, D$, may be regarded as a map from the d -dimensional worldsheet to the D -dimensional target space, and $g_{ij}(\phi)$ represents a metric on the target space. The scalar functional $T(\phi)$ will be necessary to absorb quadratic divergences; from the point of view of string theory it represents a background tachyon field. For simplicity we consider in Eq.(3.1) a sigma-model defined on a flat worldsheet - in fact this involves no loss of generality since, by simple power-counting, there are no worldsheet-curvature dependent quadratic divergences.

As is customary we use the background field method combined with a normal co-ordinate expansion of the sigma model action, so that the quantum field ζ^i is the tangent to the geodesic connecting the background field ϕ^i to a neighbouring field configuration. The field ϕ^i is itself dimensionless; hence graphs with two external $\partial_\mu \phi$ lines are logarithmically divergent, and provide corrections to the metric term in eq.(3.1), while graphs with no external $\partial_\mu \phi$ lines are quadratically divergent, and generate corrections to the tachyon term in eq.(3.1). (Such graphs may also have logarithmic divergences contributing to metric corrections). The logarithmic divergences of course appear as poles at $d = 2$, while the leading quadratic divergences in L -loop graphs appear as poles at $d = 2 - \frac{2}{L}$.

For our purposes, as explained above, we may omit the part of the normal co-ordinate expansion involving background $\partial_\mu \phi$ terms, and so we have [15]

$$S^\sigma(\phi, \zeta) = S^\sigma(\phi) + \int d^d x \left[\frac{1}{2} g_{ij} \partial_\mu \zeta^i \partial^\mu \zeta^j + \frac{1}{6} R_{iklj} \partial_\mu \zeta^i \partial^\mu \zeta^j \zeta^k \zeta^l + \frac{1}{12} D_m R_{iklj} \partial_\mu \zeta^i \partial^\mu \zeta^j \zeta^k \zeta^l \zeta^m \right] + \dots \quad (3.2)$$

The potentially quadratically divergent graphs up to four loops are depicted in Figs (2a - d). The background-dependent vertices are given by the quartic and quintic terms in eq.(3.2), while the propagator is derived from the quadratic term. It is straightforward to check that the two-loop diagram in Fig.(2a) in fact has no pole at $d = 1$ and hence does not produce a leading quadratic divergence according to the dimensional regularisation scheme. At $L = 3, 4$ we have drawn only those graphs which potentially contribute leading divergences.

The first non-zero quadratic divergence occurs at 3 loops in Fig.(2b), from which we find a correction to the tachyon

$$\Delta T^{(3)} = \frac{\mu^2}{\epsilon^{(3)}} \frac{\Gamma(2)^3}{(4\pi)^2} \frac{1}{12} R_{klmn} R^{klmn} \quad (3.3)$$

where

$$\epsilon^{(L)} = 2 - \frac{2}{L} - d \quad (3.4)$$

The poles at $d = \frac{3}{2}$ in the four-loop graphs of Figs (2c, d) may be evaluated by a straightforward but fairly tedious calculation. We find

$$\Delta T^{(4)} = \frac{\mu^2}{(4\pi)^3 \epsilon^{(4)}} \left[-\frac{1}{4} x \nabla_k R_{lmnp} \nabla_k R^{lmnp} + \frac{1}{32} y R_{klmn} R^{mnpq} R_{pq}^{kl} + \frac{1}{12} y R_{klmn} R^{k m}_{p q} R^{lpnq} \right] \quad (3.5)$$

where x and y are as given in eqs. (2.16), (2.17). It is remarkable that the same gamma function combinations arise at four loops in both four dimensions and two dimensions (as indeed occurs at three loops, as one sees by comparing Eq. (3. 3) with the results of ref. [8]).

We now wish to enquire whether there is any relation between the quadratic divergences at

different loop orders, such as we found in four dimensions. The natural analogue of a_{ij} in Eq. (2.7) is

$$c_{ij} = \beta^{g(i)} \cdot \frac{\partial}{\partial g} \Delta T^{(i)} - \Delta T^{(i)} \cdot \frac{\partial}{\partial T} \beta^{T(i)} \quad (3.6)$$

The lowest order results for β^g and β^T are [16]

$$\beta_{ij}^{g(1)} = R_{ij}, \quad \beta^{T(1)} = -\frac{1}{2} \nabla^2 T \quad (3.7)$$

whence we find, using eq. (3. 3), the lowest order c_{ij}

$$c_{13} = \text{const.} \left[\nabla_k R_{lmnp} \nabla^k R^{lmnp} - R_{klmn} R^{mn}_{pq} R^{pqkl} - 4 R_{klmn} R^{k m}_{p q} R^{lpnq} \right] \quad (3.8)$$

Tantalisingly, the potential terms involving the Ricci tensor cancel in eq.(3.6), leaving only terms which also appear in $\Delta T^{(4)}$ in eq.(3.5). However, as in section 2, the fact that y/x is transcendental clearly rules out any relation between c_{ij} and $\Delta T^{(4)}$.

Of course the lack of any relation between the four-loop leading quadratic divergence in four dimensions and the lower order results should probably have led us to be pessimistic in this case also. The problem is the lack of any non-trivial one- and two-loop results which would enable us to test for relationships at lower orders than $L = 4$. We can partially remedy this by adding a torsion term to the sigma model action, which generates non-trivial 2-loop quadratic divergences. This will be the subject of the next section.

4. Quadratic divergences in the 2-d non-linear σ -model — the case with torsion.

In this section we add a term containing an antisymmetric tensor field to the σ -model action. This may be regarded as representing torsion on the target manifold. We shall now find that the quadratic divergences start at two loops, and we shall seek a relationship between the three- and two-loop quadratic divergences in this case.

The action now becomes

$$S^\sigma = \int d^d x \left[\frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j + \frac{1}{2} b_{ij}(\phi) \epsilon^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j + T(\phi) \right] \quad (4.1)$$

where $b_{ij}(\phi)$ is antisymmetric and where $\epsilon^{\mu\nu}$ is the two-dimensional alternating symbol, appropriately extended to d dimensions. It is well-known that the definition of $\epsilon^{\mu\nu}$ away from 2 dimensions is fraught with difficulties, and some care is required [17],[20] to obtain correct results for the renormalisation group beta functions when dimensional regularisation is used in the

presence of torsion. The most satisfactory prescription [19],[21] seems to be to define $\epsilon^{\mu\nu}$ as an almost complex structure on the d -dimensional world-sheet, so that it is required to satisfy

$$\epsilon_\mu^{\rho} \epsilon_\rho^{\nu} = -\delta_\mu^{\nu}, \quad \epsilon^{\mu\nu} = -\epsilon^{\nu\mu}. \quad (4.2)$$

Nevertheless consistent results for the β -functions can also be obtained by taking in general [22]

$$\epsilon_\mu^{\rho} \epsilon_\rho^{\nu} = -(1+c\epsilon)\delta_\mu^{\nu} \quad (4.3)$$

where $\epsilon = 2 - d$, and c is arbitrary. The β -functions for different values of c are then related by field redefinitions [23]. Thus the treatment of $\epsilon^{\mu\nu}$ near $d = 2$ is, although requiring delicacy, reasonably well understood. In the context of quadratic divergences, we have the additional complication that the relevant expansion parameter becomes $\epsilon^{(L)}$ (see eq.(3.4)). Therefore, of course, ϵ as defined below eq.(4.3) is not "small". We will present our results in a form which avoids any assumption concerning properties of $\epsilon^{\mu\nu}$ (other than antisymmetry), and afterwards consider the effect of employing a specific natural generalisation of eq.(4.3). The normal coordinate expansion of the action eq.(4.1) is given by [15]

$$\begin{aligned} S^\sigma(\phi, \zeta) = S^\sigma(\phi) + \int d^d x \left[\frac{1}{2} g_{ij} \partial_\mu \zeta^i \partial^\mu \zeta^j \right. \\ \left. + \frac{1}{6} R_{ijkl} \partial_\mu \zeta^i \partial^\mu \zeta^j \zeta^k \zeta^l + \frac{1}{3} H_{ijk} \epsilon^{\mu\nu} \partial_\mu \zeta^i \partial_\nu \zeta^j \zeta^k \right. \\ \left. + \frac{1}{4} \nabla_l H_{ijk} \epsilon^{\mu\nu} \partial_\mu \zeta^i \partial_\nu \zeta^j \zeta^k \zeta^l + \dots \right] \end{aligned} \quad (4.4)$$

where $H_{ijk} = 3 \nabla_{[ij} \zeta_{k]}$, (again omitting terms involving $\partial_\mu \phi$ which do not contribute to quadratic divergences). The two- and three-loop diagrams contributing leading quadratic divergences constructed from the torsion-dependent interactions of Eq.(4.4) are displayed in Figs. 3 and 4. The two-loop quadratic divergence can be cancelled by the following correction to the tachyon field:

$$\Delta T^{(2)} = -\frac{\mu^2 \Gamma(\frac{1}{2})}{4\pi\epsilon^{(2)}} \frac{1}{6} \epsilon_l H_{klm} H^{klm} \quad (4.5)$$

where $\epsilon_l = \epsilon^{\mu\nu} \epsilon_{\mu\nu}$.

The fact that there is a quadratic divergence proportional to H^2 , but no corresponding term proportional to the Ricci scalar (the relevant graph being a "double tadpole", with no pole at $d = 1$) is reminiscent of the dilaton β -function, where the same thing happens. In particular, note that

$\Delta T^{(2)}$ does not vanish on a parallelised manifold, so that in general such manifolds are not free of quadratic divergences. This observation does not, of course, affect Mukhi's proof [24] that β_g and β^b (the torsion β -function) vanish to all orders on such a manifold. For finiteness, however, we should, strictly speaking, demand also absence of quadratic divergences.

At three loops we find, generalising Eq. (3. 3),

$$\begin{aligned} \Delta T^{(3)} = \frac{2}{3} \frac{\mu^2 \Gamma(\frac{3}{2})}{(4\pi)^2 \epsilon^{(3)}} \left[\frac{1}{8} R_{klmn} R^{klmn} + \frac{3}{8} \epsilon_1 \nabla_k H_{lmn} \nabla^k H^{lmn} \right. \\ \left. - \frac{27}{16} \epsilon_1 R_{klmn} H_p^{kl} H^{pmn} + \frac{81}{160} \epsilon_2 H_{klmn} H^{lmn} H^{kpq} H_{lpq} \right. \\ \left. + \frac{81}{1120} \epsilon_2 H_{klm} H^{kpq} H^l{}_{pq} H_q{}^{mn} \right] \end{aligned} \quad (4.6)$$

where $\epsilon_2 = \epsilon_1^2 + 2\epsilon^{\mu\nu} \epsilon_{\mu\sigma} \epsilon^\sigma{}_\nu$.

It is interesting that the particular combination ϵ_2 occurs in both the H^4 terms. The natural generalisation of eq. (4.3), noting that from eq. (3. 4) we have $\epsilon = \epsilon^{(L)} + 2/L$, is

$$\epsilon_\mu^{\rho} \epsilon_\rho^{\nu} = -\lambda^{(L)} \delta_\mu^{\nu}, \quad (4.7)$$

where

$$\lambda^{(L)} = 1 + \frac{2\epsilon}{L}. \quad (4.8)$$

We then have

$$\epsilon_1 = \frac{4}{3} \lambda^{(L)} \quad (4.9a)$$

and

$$\epsilon_2 = \frac{40}{9} \lambda^{(L)^2}. \quad (4.9b)$$

We now wish to investigate whether $\Delta T^{(3)}$ can be derived from $\Delta T^{(2)}$. The quantity c_{ij} , which we defined in eq. (3.6), now becomes in the presence of torsion

$$c_{ij} = \left(\beta^{g(i)} \cdot \frac{\partial}{\partial g} + \beta^{b(i)} \cdot \frac{\partial}{\partial b} \right) \Delta T^{(2)} \cdot \frac{\partial}{\partial T} \beta^{T(i)} \quad (4.10)$$

where now

$$\beta_{ij}^{g(1)} = -\frac{1}{16\pi^2} (R_{ij} - H_{ikl} H_j{}^{kl}) \quad (4.11a)$$

$$\beta_{ij}^{b(1)} = -\frac{1}{16\pi^2} \nabla^k H_{kij} \quad (4.11b)$$

and we find the lowest order c_{ij} :

$$c_{12} = \text{const.} [3H_{pq} H^{lpq} H^{krs} H_{rs} - 2R_{pqkl} H^{klm} H_m^{pq} + \nabla_k H_{lmn} \nabla^k H^{lmn}] \quad (4.12)$$

As in section 3, it is intriguing that terms involving R_{ij} and $\nabla^k H_{kij}$ cancel out in eq.(4.10), leaving only a subset of the terms present in eq.(4.6). However it is immediately apparent that c_{12} bears no relation to $\Delta T^{(3)}$, for any value of $\lambda^{(3)}$, since c_{12} does not contain $R_{klmn} R^{klmn}$.

It is also tempting to speculate about possible relations between the quadratic divergences and the dilaton β -function, since both are given by the diagrams of figs. 2 - 4. Indeed, if we follow the prescription for ϵ_1 and ϵ_2 given by Eq.(4.7), and take $\lambda^{(3)} = \frac{1}{3}$, corresponding to $c = -1$, in Eqs.(4.3) and (4.8) we find that the first four terms in $\Delta T^{(3)}$ are in the same ratio as they appear in the string effective action corresponding to the β -functions derived from the $c = -1$ prescription in eq.(4.3) [23]. The string effective action is well-known to be closely related to the dilaton β -function. The last term in eq.(4.6) is apparently somewhat anomalous, not least because of its unusual coefficient. This coefficient arises from performing momentum integrals with free Lorentz indices in $d = \frac{4}{3}$ dimensions. It is hence very specific to our dimensional regularisation scheme for extracting quadratic divergences. It would be interesting to perform the calculation using a purely 2-dimensional regularisation scheme to see if complete agreement with the string effective action could be obtained.

The fact that this near agreement of $\Delta T^{(3)}$ with the string effective action occurred only for $c = -1$ may not be fortuitous, since $c = -1$ corresponds to conventional dimensional reduction [25]. In dealing with gauge theories in four dimensions, we similarly found that relationships between quadratic divergences at different loop orders only occurred when dimensional reduction was used.

5. Discussion.

In this paper we have calculated the "leading" quadratic divergence (as manifested in dimensional regularisation) through four loops in renormalisable scalar theories in four dimensions and sigma models in two dimensions. Our main purpose in this was to explore the apparent connection between quadratic divergences and logarithmic divergences we discovered in refs. [7], [8]. This connection (between scale invariance of the quadratic divergence at L loops and the quadratic divergence at L + 1 loops) appears to fail at the four-loop level. In the $d = 4$ case, this conclusion depends crucially on the correctness of the three-loop β -function results of

ref. [13]; but we have no real reason to doubt this calculation. A similar disappointment was reached in the $d = 2$ case, where the ambiguities associated with $\epsilon^{\mu\nu}$ led to further problems.

Our original [7] goal in this investigation was to demonstrate that there might exist non-supersymmetric theories free of quadratic divergences. While this question remains open with regard to the class of theories (inspired by dimensional reduction) proposed in ref. [7], it now appears that vanishing of the quadratic divergence at L loops is not guaranteed by requiring scale invariance (to L loops) of the quadratic divergence conditions for $L' < L$. This would suggest that there is in general more than one condition that must be satisfied to ensure quadratic divergence cancellation at a given loop order.

We have throughout employed minimal subtraction, and so in particular have not explored the freedom available to change the β -functions by making coupling constant redefinitions. Indeed, quadratic divergences are themselves sensitive to the choice of regulator, as is particularly clear in the gauge theory case with the DREG/DRED dilemma [21],[9]. This regulator dependence is related to the whole question of the physical relevance of quadratic divergences. In supersymmetric theories it is the no-renormalisation theorem [26], rather than the absence of quadratic divergences, that protects the low energy theory from "large" radiative corrections due to heavy fields. Nevertheless, we still believe that one is entitled to speculate that there might exist non-supersymmetric theories free of quadratic divergences, even if such theories did not lend themselves easily to embedding in a grand unified structure.

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Figure Captions

- Fig 1. The graphs contributing to Δ_4 . Three- and four-point vertices represent U_{abc} and U_{abcd} respectively. The x represents U .
- Fig 2. Potentially quadratically divergent graphs in the torsion-free sigma model.
- Fig 3. The graph contributing an H -dependent piece to $\Delta T^{(2)}$.
- Fig 4. The graphs contributing H -dependent pieces to $\Delta T^{(3)}$.

Fig. 1

Fig. 2

Fig. 3

Fig. 4