An attempt to relate area-preserving diffeomorphisms to Kac-Moody algebras

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Abstract

In the same way the Virasoro algebra can be connected with Kac-Moody algebras defined on the S^1 circle, the area preserving diffeomorphism algebra $\text{SDiff}(\mathcal{M})$, where $\mathcal{M}$ is a two-dimensional surface, acts as a derivation algebra on super Kac-Moody algebras with one or two supersymmetries. Then a Sugawara-like construction with fermions of the non-extended $\text{SDiff}(\mathcal{M})$ algebra is discussed.

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1 Introduction

The algebra of volume preserving diffeomorphisms on a surface $\mathcal{M}$, usually denoted by $\text{SDiff}(\mathcal{M})$, directly shows up in many different areas of theoretical physics : theory of a perfect incompressible fluid on $\mathcal{M}$ [1], “restricted gravity” in the context of the cosmological constant [2,3], self dual Einstein equations in relativity [4], theory of relativistic surfaces [5], and so on... Several papers have been recently proposed, describing mathematical properties of $\text{SDiff}(\mathcal{M})$. We can note in particular the relationship between the $\text{SU}(\infty)$ algebras and $\text{SDiff}(\mathcal{M})$ when $\mathcal{M}$ is either the two-dimensional torus or the sphere S^2 [6], as well as the connection between w_∞ (a limit when $N \rightarrow \infty$ of w_N algebras) and the algebra of area preserving diffeomorphisms on a cylinder [7].

One may be tempted to try a correspondence, at least formally, between the symmetry algebra of the string action, i.e. the Virasoro algebra seen as the (central extension of) diffeomorphism algebra on the world sheet, and the residual symmetry in the theory of more extended objects $\mathcal{M}$, with $\dim \mathcal{M} > 1$, that is $\text{SDiff}(\mathcal{M})$. At this point, let us remember that a very important characteristic of the Virasoro algebra $\hat{\mathcal{G}}(S^1)$ is its property to act as a derivation algebra on any Kac-Moody (KM) algebra $\hat{\mathcal{G}}(S^1)$ where S^1 is the one-dimensional circle. The existence of the semi-direct sum $\hat{\mathcal{G}}(S^1) \square V$ plays a fundamental role not only on a technical point of view (i.e. construction of Virasoro representations from KM ones), but also because it relates current algebra and conformal symmetry (i.e. W.Z.W. models).

In a previous publication [8], we have studied explicitly all the central extensions which can be defined for a surface algebra $\mathcal{G}(\mathcal{M})$ -i.e. direct generalization of the loop algebra $\mathcal{G}(S^1)$ when S^1 is replaced by a closed surface of higher dimension. Given a surface $\mathcal{M}$ with $\dim \mathcal{M} > 1$, and contrarily to the usual KM case, the number of central extensions is no more one, up to a multiplicative factor, but infinite. Denoting ω the cocycle associated to a central extension, the corresponding algebra will be denoted by $\tilde{\mathcal{G}}_\omega(\mathcal{M})$. Then we have determined, for each ω , the subalgebra of the diffeomorphism algebra $\text{Diff}(\mathcal{M})$ compatible with $\tilde{\mathcal{G}}_\omega(\mathcal{M})$, that is the set of elements in $\text{Diff}(\mathcal{M})$ which act as derivations on $\tilde{\mathcal{G}}_\omega(\mathcal{M})$. It happens that it does not exist any ω such that the semi-direct sum of $\tilde{\mathcal{G}}_\omega(\mathcal{M})$ by $\text{SDiff}(\mathcal{M})$ is possible. However, some geometrical properties concerning $\text{SDiff}(\mathcal{M})$ have been deduced, in particular the one-to-one correspondence between the differential operators specifying the central extensions and the elements of $\text{SDiff}(\mathcal{M})$.

In this note, we consider in more detail the case of a two-dimensional compact surface $\mathcal{M}$. Then enlarging $\mathcal{M}$ to the supermanifold $\tilde{\mathcal{M}}_1 = \mathcal{M} \times Gr(1)$ where $Gr(1)$ denotes the Grassmann algebra with one θ generator, and looking at the super KM algebras defined on $\tilde{\mathcal{M}}_1$, we will show that it does exist a cocycle ω such that the $\tilde{\mathcal{G}}_\omega(\tilde{\mathcal{M}}_1)$ compatible algebra in $\text{Diff}(\tilde{\mathcal{M}}_1)$ contains $\text{SDiff}(\mathcal{M})$ as a subalgebra. As can be seen in the following, this particular central extension shows up among fermionic generators of $\tilde{\mathcal{G}}(\tilde{\mathcal{M}}_1)$. This will allow to suggest a “Sugawara-like” construction of $\text{SDiff}(\mathcal{M})$ in terms of fermions. In

a second step, we will consider the $N = 2$ supermanifold $\tilde{\mathcal{M}}_2 = \mathcal{M} \times Gr(2)$ with two supersymmetries : then it will be possible to select a super KM algebra $\tilde{\mathcal{G}}_\omega(\tilde{\mathcal{M}}_2)$ such that its compatible algebra contains the supersymmetric extension of $\text{SDiff}(\mathcal{M})$ proposed by membrane theorists [9]. Finally, relaxing the condition of closure for $\mathcal{M}$, and considering the cylinder, we will see that it is still possible to associate a super Kac Moody algebra to $\text{SDiff}(S^1 \times R)$.

We start by remembering in Section 2 the two main results we need in order to achieve the above mentioned program, namely the explicit expression of the most general central extension -or cocycle ω - for a KM algebra defined on a supermanifold $\mathcal{M} \times Gr(N)$, and the algebraic condition to be satisfied by a supervector field to act as a derivation on $\tilde{\mathcal{G}}_\omega(\mathcal{M} \times Gr(N))$. Then Sections 3 and 4 are respectively devoted to the cases $\dim \mathcal{M} = 2$ with one and two supersymmetries. Comments on the obtained results will constitute the conclusion.

2 Central extensions and compatible condition :

We start by considering the supersurface algebra $\mathcal{G}(\mathcal{M} \times Gr(N))$ direct generalization of the loop algebra $\mathcal{G}(S^1)$ in which the circle S^1 has been replaced by the supersurface $\mathcal{M} \times Gr(N)$, $\mathcal{M}$ being a compact closed surface of dimension m and $Gr(N)$ the Grassmann algebra generated by N anticommuting fermions θ_α ($\alpha = 1, \dots, N; \theta_\alpha^2 = 0$). There is a one to one correspondence between the possible central extensions of $\mathcal{G}(\mathcal{M} \times Gr(N))$ and the divergence free superfields on $\mathcal{M} \times Gr(N)$. More explicitly, let us denote by ω the antisymmetric two-form associated with a particular central extension, and by $\tilde{\mathcal{G}}_\omega(\mathcal{M} \times Gr(N))$ the corresponding obtained KM algebra, in which the Lie bracket of any couple of elements X, Y reads :

$$[\tilde{X}, \tilde{Y}] = [X, Y] + \omega(X, Y)e, \quad (2.1)$$

e being the central element. Then, one has :

$$\omega(X, Y) = (-1)^{\theta(X)\theta(Y)} \int \langle \mathcal{D}Y, X \rangle du_1 \dots du_m d\theta_1 \dots d\theta_N \quad (2.2)$$

where $u_1, \dots, u_m$ constitute a set of variables on $\mathcal{M}$, and the differential operator $\mathcal{D}$ on $\mathcal{G}^\infty(\mathcal{M} \times Gr(N))$:

$$\mathcal{D} = \sum_{\alpha=1}^N Q^\alpha(u_1, \dots, u_m; \theta_1, \dots, \theta_N) \frac{\partial}{\partial \theta_\alpha} + \sum_{i=1}^m P^i(u_1, \dots, u_m; \theta_1, \dots, \theta_N) \frac{\partial}{\partial u_i}, \quad (2.3)$$

is divergence free :

$$\text{div } \mathcal{D} = \sum_{\alpha=1}^N \frac{\partial Q^\alpha}{\partial \theta_\alpha} + \sum_{i=1}^m \frac{\partial P^i}{\partial u_i} = 0 \quad (2.4)$$

$Q^{\alpha*}$ being deduced from Q^α by changing the sign in front of the odd parts (in θ_α variables) of θ^α . The numerical factor in front of the integral refers to the grading of the elements :

$g(X) = 1$ (-1) if X is even (odd) in θ_a . Finally the symbol $<, >$ inside the integral refers to the scalar product relative to the finite dimensional Lie algebra $\mathcal{G}$ (i.e. Killing form if $\mathcal{G}$ is semi-simple).

The proof of this result, based on cyclic cohomology, in the case of an algebra defined on a supercircle $S^1 \times Gr(N)$ can be found in ref.[10]. Its generalization to the case $\mathcal{M} \times Gr(N)$ (with $\dim \mathcal{M} > 1$) is straightforward up to some technical difficulties and is presented in ref.[11]. Note also that a detailed study of the case $\tilde{\mathcal{G}}_\omega(\mathcal{M})$ (with $\dim \mathcal{M} > 1$) appears in ref.[8].

It may be worthwhile to remark that the divergence free operators $\mathcal{D}$ on $\mathcal{M} \times Gr(N)$ form a Lie algebra, $\mathcal{A}(\mathcal{M}, N)$, the commutator of two divergence free operators being divergence free. Such a property induces a structure of Lie algebra on the set of cyclic 1-cocycles ω on $C^\infty(\mathcal{M}) \times Gr(N)$. Actually, the algebra of divergence free differential operators on a supermanifold reduces in Kac's notations [12] to the algebra $S(m, N)$ in the case where $\mathcal{M} \cong R^m$ (C^m) and the functions Q^a and P^i are polynomials in the u 's and θ 's variables. We follow ref.[12] for the definition of the set S of generators :

$$\begin{aligned} D_{i,j}^f &= \partial_{u_i} f \partial_{u_j} - \partial_{u_j} f \partial_{u_i} \quad \text{for } 1 \leq i < j \leq m \\ D_{(\alpha,\beta)}^f &= \partial_{\theta_\alpha} f^* \partial_{u_\beta} - \partial_{u_\beta} f \partial_{\theta_\alpha} \quad \text{for } 1 \leq i \leq p \text{ and } 1 \leq \alpha \leq N \\ D_{(\alpha,\beta)}^f &= \partial_{\theta_\alpha} f^* \partial_{\theta_\beta} + \partial_{\theta_\beta} f^* \partial_{\theta_\alpha} \quad \text{for } 1 \leq \alpha < \beta \leq N. \end{aligned} \quad (2.5)$$

for any function $f \in C^\infty(\mathcal{M}) \times Gr(N)$.

Note that S is a complete set of generators for $\mathcal{A}(\mathcal{M}, N)$ when the de Rham cohomology of the manifold $\mathcal{M}$ is trivial -i.e. any closed one-form is an exact one- as in the case of the S^2 sphere. In the torus $\mathcal{M} = T^m$ ($N = 0$) case, we have to add the m translations :

$$P_j = \partial_{u_j} \quad (j = 1, \dots, m) \quad (2.6)$$

the functions f in eq.(2.5) being periodic in the u_j variables. Let us add that in this last case, the divergence free operators $D_{i,j}^f$ and the P_j generate the volume preserving algebra $S\text{Diff}(\mathcal{M})$.

Now an element $\mathcal{L}$ of the diffeomorphism algebra $\text{Diff}(\mathcal{M} \times Gr(N))$ will be called ω -compatible [10], if it acts as a (graded) derivation on $\tilde{\mathcal{G}}_\omega(\mathcal{M} \times Gr(N))$, that is, for any X, Y in the KM algebra :

$$\mathcal{L}([\widetilde{X}, \widetilde{Y}]) = [\mathcal{L}\widetilde{X}, \widetilde{Y}] + (-1)^{\vartheta(X)\vartheta(Y)}[\widetilde{X}, \mathcal{L}Y] \quad (2.7)$$

From eq.(2.1), the above condition reduces to :

$$\omega(\mathcal{L}X, Y) + (-1)^{\vartheta(X)\vartheta(Y)}\omega(X, \mathcal{L}Y) = 0. \quad (2.8)$$

Using (2.2), an integration by parts and the (graded) antisymmetry of ω , one finally obtains :

$$[\mathcal{L}, \mathcal{D}] + (\text{div} \mathcal{L})\mathcal{D} = 0 \quad (2.9)$$

One can easily check that, $\mathcal{L}$ and $\mathcal{L}'$ being of same parity :

$$[[\mathcal{L}, \mathcal{L}'], \mathcal{D}] + (\text{div}[\mathcal{L}, \mathcal{L}'])\mathcal{D} = 0 \quad (2.10)$$

such a property conferring to the set of ω -compatible (or $\mathcal{D}$ -compatible) diffeomorphisms a structure of Lie algebra : we will call it $\text{Der}_\mathcal{D}(C^\infty(\mathcal{M}) \times Gr(N))$. Let us remember that in the case where the supermanifold reduces to the simple circle S^1 , the only possible extension (up to a multiplicative constant) for a loop algebra $\mathcal{G}(S^1)$ is related to the differential operator ∂_z if φ is the angle specifying a point on S^1 . Then $\text{Der}_\mathcal{D}(S^1)$ is exactly $\text{Diff}(S^1)$ and its central extension the Virasoro algebra.

We can also note that, since $\text{div} \mathcal{D} = 0$, the differential operator $\mathcal{D}$ defining the cocycle $\omega_\mathcal{D}$ is itself a $\mathcal{D}$ -compatible super derivation if $\mathcal{D}$ is even; if $\mathcal{D}$ is odd, $[\mathcal{D}, \mathcal{D}]$ represents an anticommutator which does not vanish necessarily. Finally, we notice that the compatibility equation (2.9) represents actually two sets of equations since we demand the super derivation $\mathcal{L}$ to be of definite Z_2 -grading.

3 The $N = 1$ case.

Let u_1 and u_2 specify a point on the two-dimensional closed surface $\mathcal{M}$. The supersurface $\mathcal{G}(\mathcal{M} \times Gr(1))$ admits two kinds of generators : the bosonic ones $T_j^a = f(u_1, u_2) \otimes T^a$ and the fermionic ones $\psi_j^a = \theta f(u_1, u_2) \otimes T^a$ if $f \in C^\infty(\mathcal{M})$ and the T^a ($a = 1, \dots, \dim \mathcal{G}$) form a basis for the finite dimensional Lie algebra $\mathcal{G}$. If the de Rham cohomology of $\mathcal{M}$ is trivial, a basis for the divergence free operators is given by :

$$\begin{aligned} \mathcal{D}_{u_i, u_j}^* &= \partial_1 k \partial_2 - \partial_2 k \partial_1 \\ \mathcal{D}_{\theta_i, u_i}^* &= \partial_\theta k \partial_i - \partial_i k \partial_\theta \end{aligned} \quad (3.1)$$

where $k(u_1, u_2; \theta) = k_1(u_1, u_2) + \theta k_2(u_1, u_2)$ with k_1, k_2 smooth functions of u_1, u_2 and ∂_i stands for $\partial_{u_i} = \frac{\partial}{\partial u_i}$ ($i = 1, 2$). If the genus of the surface is not zero, "discrete" operators must be added : for example, in the case of the $S^1 \times S^1$ torus :

$$P_i = \partial_i \quad i = 1, 2. \quad (3.2)$$

It is easy to show that there exist cocycles $\mathcal{D} = R_1 \partial_1 + R_2 \partial_2 + R_\theta \partial_\theta$ leading to a compatible algebra containing the area preserving diffeomorphism on $\mathcal{M}$, that is the algebra generated by :

$$L_h = (\partial_1 h) \partial_2 - (\partial_2 h) \partial_1 \quad h \in C^\infty(\mathcal{M}) \quad (3.3)$$

plus possibly, “translation” operators, i.e. for $\mathcal{M} = S^1 \times S^1$:

$$P_i = \partial_i \quad (i = 1, 2) \quad (3.4)$$

Indeed, condition (2.9) reduces to $[L_h, \mathcal{D}] = 0$, i.e. :

$$\begin{aligned} \{h, R_1\}_{1,2} + (\partial_1 \partial_2 h) R_1 + (\partial_2^2 h) R_2 &= 0 \\ \{h, R_2\}_{1,2} - (\partial_1 \partial_2 h) R_2 - (\partial_1^2 h) R_1 &= 0 \\ \{h, R_\theta\}_{1,2} &= 0 \end{aligned} \quad (3.5)$$

where $\{h, R\}_{1,2} = \partial_1 h \partial_2 R - \partial_2 h \partial_1 R$ (usual Poisson bracket). Conditions (3.5) imply :

$$R_1 = R_2 = 0 \quad R_\theta = \alpha + \beta \theta \quad \alpha, \beta \in \mathbb{C} \quad (3.6)$$

while the divergence free condition on $\mathcal{D}$ imposes $\beta = 0$. One remarks that condition (2.9) on the P_i does not reduce the set of solutions (3.6).

Thus, the only ($N = 1$) cocycle which leads to a compatible algebra containing the area-preserving diffeomorphisms is

$$\mathcal{D} = \alpha \partial_\theta \quad \alpha \in \mathbb{C} \quad (3.7)$$

We now compute the complete compatible algebra associated to this cocycle. The compatibility equations reads for a vector field $\mathcal{L} = L_1 \partial_1 + L_2 \partial_2 + L_\theta \partial_\theta$:

$$\partial_\theta L_1 = 0 \quad (3.8)$$

$$\partial_\theta L_2 = 0 \quad (3.9)$$

$$\partial_\theta L_\theta = -\frac{1}{2} (\partial_1 L_1 + \partial_2 L_2) \quad (3.10)$$

To make the area-preserving diffeomorphisms appear, we take the basis introduced in [8] for vector fields on $\mathcal{M}$:

$$T L_h = T(u_1, u_2) ((\partial_1 h) \partial_2 - (\partial_2 h) \partial_1) \quad (3.11)$$

Then, the derivation algebra is

$$L_h^T = T(u_1, u_2) L_h + \frac{1}{2} \{h, T\}_{1,2} \partial_\theta \quad (3.12)$$

$$G_V = V(u_1, u_2) \partial_\theta \quad (3.13)$$

and, when they exist, the translations.

The area-preserving diffeomorphisms L_h correspond to the special value $T(u_1, u_2) = 1$:

$$L_h = L_h^{(T=1)} = (\partial_1 h(u_1, u_2)) \partial_2 - (\partial_2 h(u_1, u_2)) \partial_1 \quad (3.14)$$

We list the (non-vanishing) commutation relations of this algebra (with L_h explicitly written)^{1, 2} :

$$\begin{aligned} [L_h, L_k] &= L_{(h,k),1,2} & [P_i, L_h] &= L_{\partial_{u_i} h} \quad (i = 1, 2) \\ [L_h^T, L_k^T] &= L_{(h,k),1,2}^{TU} + L_k^{T(h,U),1,2} - L_h^{U(k,T),1,2} & [P_i, L_h^T] &= L_{\partial_{u_i} h}^T + L_h^{\partial_{u_i}, T} \\ [L_h^T, G_U] &= G_{T(h,U),1,2} - \frac{1}{2} G_{U(h,T),1,2} & [P_i, G_T] &= G_{\partial_{u_i} T} \\ [L_h, G_U] &= G_{(h,U),1,2} \end{aligned} \quad (3.15)$$

As far as the KM is concerned, its elements satisfy the following commutation relations (we have used the notation $T_f^a = f(u_1, u_2) \otimes T^a$ and $\psi_f^a = \theta \otimes f(u_1, u_2) \otimes T^a$):

$$\begin{aligned} [T_f^a, T_g^b] &= i f^{abc} T_{fg}^c \\ [T_f^a, \psi_g^b] &= i f^{abc} \psi_{fg}^c \\ \{\psi_f^a, \psi_g^b\} &= -\alpha \omega(f, g) \delta^{ab} \end{aligned} \quad (3.16)$$

with $\omega(f, g) = \int_{\mathcal{M}} f(u_1, u_2) g(u_1, u_2) du_1 du_2$.

Finally, the action of the diffeomorphism algebra on the KM one reads

$$\begin{aligned} [L_h^U, T_k^a] &= T_{U(h,k),1,2}^a & [L_h^U, \psi_k^a] &= \psi_{U(h,k),1,2}^a - \frac{1}{2} \psi_k^a (U, h)_{1,2} \\ [P_i, T_h^a] &= T_{\partial_{u_i} h}^a & [P_i, \psi_h^a] &= \psi_{\partial_{u_i} h}^a \quad (i = 1, 2) \\ [G_U, T_k^a] &= 0 & \{G_U, \psi_k^a\} &= T_{Uk}^a \end{aligned} \quad (3.17)$$

4 The $N = 2$ case

The loop algebra is defined through its generators $T_f^a = f(u_1, u_2) \otimes T^a$, $\psi_f^a = \theta, f(u_1, u_2) \otimes T^a$, $\phi_f^a = \theta, f(u_1, u_2) \otimes T^a$; $X_f^a = \theta, \partial_2 f(u_1, u_2) \otimes T^a$.

Mimicking the $N = 1$ case, we obtain the constraint equations for the cocycles :

$$\begin{aligned} \{h, R_{u_1}\}_{u_1, u_2} + (\partial_{u_1} \partial_{u_2} h) R_{u_1} + (\partial_{u_2}^2 h) R_{u_2} &= 0 \\ \{h, R_{u_2}\}_{u_1, u_2} - (\partial_{u_1} \partial_{u_2} h) R_{u_2} - (\partial_{u_1}^2 h) R_{u_1} &= 0 \\ \{h, R_{\theta_1}\}_{u_1, u_2} &= 0 \\ \{h, R_{\theta_2}\}_{u_1, u_2} &= 0 \end{aligned} \quad (4.18)$$

which lead to the result

¹From now on, we write the translations P_i , with the convention that they are null when the cohomology of $\mathcal{M}$ is trivial.

²the reader should be aware that $\{, \}$ stands for an anticommutator whereas $\{, \}_{1,2}$ is a Poisson bracket with respect to the variables u_1 and u_2

The most general ($N = 2$) cocycle which leads to a compatible algebra containing the area-preserving diffeomorphisms takes the form (up to a "rotation" in the θ^i 's)

$$\mathcal{D} = (\alpha + \beta\partial_2)\partial_1 + (\gamma + \lambda\partial_1)\partial_2 + \mu(\theta_1\partial_1 - \theta_2\partial_2) \quad \text{with } \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{C} \quad (4.19)$$

A priori, we have $2^5 = 32$ different cocycles, corresponding to each possibility of annulation of the constants $\alpha, \beta, \gamma, \lambda$ and μ . We present here a study of the cocycles we think to be relevant, namely :

$$\begin{aligned} \mathcal{D}_{\alpha\gamma} &= \alpha\partial_1 + \gamma\partial_2 \\ \mathcal{D}_{\beta\lambda} &= \beta\partial_2\partial_1 + \lambda\partial_1\partial_2 \\ \mathcal{D}_\mu &= \mu(\theta_1\partial_1 - \theta_2\partial_2) \end{aligned}$$

Note that the first operator is odd whereas the last two are even. In the case where the operators $\mathcal{D}$ are even, we already know [13] that $\mathcal{D}_F = F(u_1, u_2, \theta_1, \theta_2)\mathcal{D}$ ("super-dilatations" of $\mathcal{D}$) are compatible diffeomorphisms, but we will see that they are not the only ones. We start by giving the C.R. of the most general KM algebra whose compatible algebra contains the area-preserving one.

$$\begin{aligned} [T_f^a, T_g^b] &= if^{abc}T_{fg}^c & [T_f^a, X_g^b] &= if^{abc}X_{fg}^c \\ [T_f^a, \phi_g^b] &= if^{abc}\phi_{fg}^c & [T_f^a, \psi_g^b] &= if^{abc}\psi_{fg}^c \\ \{\psi_f^a, \phi_g^b\} &= if^{abc}X_{fg}^c + \mu\omega(f, g)\delta^{ab} & \{\psi_f^a, \psi_g^b\} &= \gamma\delta^{ab}\omega(f, g) \\ [X_f^a, \psi_g^b] &= \alpha\delta^{ab}\omega(f, g) & \{\phi_f^a, \phi_g^b\} &= -\lambda\delta^{ab}\omega(f, g) \\ \{\psi_f^a, \psi_g^b\} &= \beta\delta^{ab}\omega(f, g) \end{aligned} \quad (4.20)$$

• We first study the cocycle associated to $\mathcal{D}_\mu$.
The compatible algebra is then

$$\begin{aligned} L_h^T &= TL_h + \frac{1}{2}\{h, T\}_{12}(\theta_1\partial_1 + \theta_2\partial_2) & P_i &= \partial_{u_i} \quad (i = 1, 2) \\ D_h^T &= T(u_1, u_2)\theta_1\theta_2((\partial_{u_1}h)\partial_{u_2} - (\partial_{u_2}h)\partial_{u_1}) & H_T &= T(u_1, u_2)(\theta_1\partial_1 - \theta_2\partial_2) \\ Q_T^1 &= T(u_1, u_2)\theta_1\partial_1 & Q_T^2 &= T(u_1, u_2)\theta_1\theta_2\partial_2 \end{aligned} \quad (4.21)$$

H_T, Q_T^1 and Q_T^2 correspond to the superdilations of $\mathcal{D}_\mu$ (they are solutions, as expected). Again, the area-preserving diffeomorphisms appear for $T = 1$: $L_h = L_h^{(T=1)}$. The (non-vanishing) commutation relations for this algebra are

$$[L_h^T, L_k^U] = L_{\{h,k\}12}^{TU} + L_k^{T(h,U)12} - L_h^{U(k,T)12} \quad (4.22)$$

$$[P_i, L_k^U] = L_{\partial_{u_i}h}^T + L_h^{\partial_{u_i}T} \quad (4.23)$$

$$\begin{aligned} [L_h^T, K_U] &= K_{T(h,U)12} & (4.24) \\ [L_h^T, D_k^U] &= D_{\{h,k\}12}^{TU} + D_k^{T(h,U)12} - D_h^{U(k,T)12} & (4.25) \\ [L_h^T, Q_U^i] &= Q_{T(h,U)12}^i - Q_{U(h,T)12}^i \quad \text{for } i = 1, 2 & (4.26) \\ [P_i, D_h^T] &= D_{\partial_{u_i}h}^T + D_h^{\partial_{u_i}T} & (4.27) \\ [P_i, Q_h^j] &= Q_{\partial_{u_i}h}^j & [P_i, K_h] &= K_{\partial_{u_i}h} & (4.28) \\ [Q_h^i, H_U] &= H_{hU} & [Q_h^2, H_U] &= -H_{hU} & (4.29) \end{aligned}$$

• For $\mathcal{D}_{\alpha\gamma}$, we obtain

$$\begin{aligned} L_h^T &= T(u_1, u_2)((\partial_{u_1}h)\partial_{u_2} - (\partial_{u_2}h)\partial_{u_1}) + \frac{1}{2}\{h, T\}_{12}(\theta_+ \partial_+) & (4.30) \\ P_i &= \partial_{u_i} \quad (i = 1, 2) & (4.31) \\ G_h^T &= T(u_1, u_2)\theta_-((\partial_{u_1}h)\partial_{u_2} - (\partial_{u_2}h)\partial_{u_1}) - \frac{1}{2}\{h, T\}_{12}\theta_+\theta_-\partial_+ & (4.32) \\ H_T^+ &= T(u_1, u_2)\partial_+ & (4.33) \\ H_T^- &= T(u_1, u_2)\partial_- & (4.34) \\ K_T &= T(u_1, u_2)\theta_-\partial_+ & (4.35) \\ S_T &= T(u_1, u_2)(\theta_+\partial_+ - 2\theta_-\partial_-) & (4.36) \end{aligned}$$

where we have defined

$$\theta_\pm = \frac{1}{2\alpha\gamma}(\gamma\theta_1 \pm \alpha\theta_2) \quad (4.37)$$

$$\partial_\pm = \alpha\partial_1 \pm \gamma\partial_2 \quad (4.38)$$

With these simplifications, the commutation relations read

$$[L_h^T, L_k^U] = L_{\{h,k\}12}^{TU} + L_k^{T(h,U)12} - L_h^{U(k,T)12} \quad (4.39)$$

$$[P_i, L_h^T] = L_{\partial_{u_i}h}^T + L_h^{\partial_{u_i}T} \quad (4.40)$$

$$[L_h^T, G_k^U] = G_{\{h,k\}12}^{TU} + G_k^{T(h,U)12} - G_h^{U(k,T)12} - \frac{1}{2}G_k^{U(h,T)12} \quad (4.41)$$

$$[L_h^T, H_T^+] = H_{T(h,U)12}^+ - \frac{1}{2}H_{U(h,T)12}^+ \quad (4.42)$$

$$[L_h^T, H_T^-] = H_{T(h,U)12}^- \quad (4.43)$$

$$[L_h^T, K_U] = \frac{3}{2}K_{T(h,U)12} \quad (4.44)$$

$$[L_h^T, S_U] = S_{T(h,U)12} \quad (4.45)$$

$$[P_i, G_h^T] = G_{\partial_{u_i}h}^T + G_h^{\partial_{u_i}T} \quad (4.46)$$

$$[P_i, H_h^\pm] = H_{\partial_{u_i}h}^\pm \quad (4.47)$$

$$[P_i, K^T] = K^{\partial_{u_i}T} \quad [P_i, S^T] = S^{\partial_{u_i}T} \quad (4.48)$$

5 Fermionic construction of $\text{SDiff}(\mathcal{M})$:

As a first application of the above result, one may think at a “Sugawara like” construction for $\text{SDiff}(\mathcal{M})$. Although we did not succeed in solving completely this problem, we can propose a formal construction, and make the difficulties precise.

Thus, let us start by treating explicitly the torus case $\mathcal{M} = S^1 \times S^1$. A natural basis of generators for the KM algebra associated with the cocycle relative to $\mathcal{D} = \partial_\theta$ is given by

$$T_{\vec{m}}^a = e^{i(\vec{m}_1 u_1 + \vec{m}_2 u_2)} T^a, \quad \psi_{\vec{m}}^a = \theta e^{i(\vec{m}_1 u_1 + \vec{m}_2 u_2)} T^a \quad (5.1)$$

with $\vec{m} = (m_1, m_2)$ and the commutation relations are :

$$[T_{\vec{m}}^a, T_{\vec{n}}^b] = i f_c^{ab} T_{\vec{m}+\vec{n}}^c \quad (5.2)$$

$$[T_{\vec{m}}^a, \psi_{\vec{n}}^b] = i f_c^{ab} \psi_{\vec{m}+\vec{n}}^c \quad (5.3)$$

$$\{\psi_{\vec{m}}^a, \psi_{\vec{n}}^b\} = -\delta_{\vec{m}+\vec{n},0} \delta^{ab} \quad (5.4)$$

We also recall the usual basis of $\text{SDiff}(S^1 \times S^1)$

$$L_{\vec{n}} = i h_{\vec{n}} (n_1 \partial_1 - n_2 \partial_2) \quad \text{with} \quad h_{\vec{n}} = e^{i(n_1 u_1 + n_2 u_2)} \quad (5.5)$$

$$P_1 = \partial_1 ; \quad P_2 = \partial_2 \quad (5.6)$$

with the commutation relations $(\vec{m} \times \vec{n} = m_1 n_2 - m_2 n_1)$:

$$[L_{\vec{m}}, L_{\vec{n}}] = (\vec{m} \times \vec{n}) L_{\vec{m}+\vec{n}} \quad (5.7)$$

$$[P_i, L_{\vec{m}}] = m_i L_{\vec{m}} \quad i = 1, 2 \quad (5.8)$$

$$[P_1, P_2] = 0 \quad (5.9)$$

Now, let us formally set :

$$\mathcal{L}_{\vec{m}} = \frac{1}{2} \sum_{\vec{p}} \sum_{\vec{q}}^{dim \mathcal{Q}} (\vec{m} \times \vec{p}) \psi_{\vec{p}}^a \psi_{\vec{m}-\vec{p}}^a \quad (5.10)$$

and :

$$P_j = \sum_{\vec{r}}^{dim \mathcal{Q}} \sum_{\vec{r}} r_j \psi_{\vec{r}}^a \psi_{-\vec{r}}^a \quad j = 1, 2 \quad (5.11)$$

then, one can directly check :

$$[\mathcal{L}_{\vec{m}}, \psi_{\vec{n}}^a] = (\vec{m} \times \vec{n}) \psi_{\vec{m}+\vec{n}}^a \quad (5.12)$$

$$[P_j, \psi_{\vec{n}}^a] = 2 n_j \psi_{\vec{n}}^a \quad (5.13)$$

$$[H_T^+, S_U] = H_{TU}^+ \quad [H_T^-, S_U] = -2 H_{TU}^- \quad (4.49)$$

$$[K_T, S_U] = -3 K_{TU} \quad [K_U, G_T^+] = 0 \quad (4.50)$$

$$\{H_U^+, G_T^+\} = K_{T(h,U)12} - \frac{1}{2} K_{U(h,T)12} \quad (4.51)$$

$$\{H_U^-, G_T^+\} = L_h^{TU} - \frac{1}{2} S_{T(h,U)12} \quad (4.52)$$

$$[S_U, G_T^+] = -G_h^{TU} \quad (4.53)$$

Let us stress that, assembling (4.32) with (4.34), we obtain together with (4.30), the analogue of the Kac's $H(2,1)$ algebra³ :

$$L_h = L_h^{(T=1)} = (\partial_{u_1} h) \partial_{u_2} - (\partial_{u_2} h) \partial_{u_1} \quad (4.54)$$

$$G_h = G_h^{(T=1)} + H_h^- = \theta_- ((\partial_{u_1} h) \partial_{u_2} - (\partial_{u_2} h) \partial_{u_1}) + h(u_1, u_2) \partial_- \quad (4.55)$$

This algebra has been remarked upon in [9] as a candidate for super-membrane symmetry algebra. Note that, because of the redefinition in the Grassman variables, it is easy to guess that this algebra also appears in $\mathcal{D}_\alpha$ and in $\mathcal{D}_\gamma$. This can be checked explicitly. In fact, $\mathcal{D} = \alpha \partial_1$ is (up to a “rotation” in θ)’s which gives any combination $\alpha \theta_1 + \gamma \theta_2$ the only $(N=2)$ cocycle whose compatible algebra contains $H(2,1)$.

• Finally, we study $\mathcal{D}_{3\lambda}$:

$$L_h^T = T(u_1, u_2) ((\partial_{u_1} h) \partial_{u_2} - (\partial_{u_2} h) \partial_{u_1}) \quad (4.56)$$

$$P_i = \partial_{u_i} \quad (4.57)$$

$$D_T = T(u_1, u_2) \theta_1 \theta_2 ((\partial_{u_1} h) \partial_{u_2} - (\partial_{u_2} h) \partial_{u_1}) \quad (4.58)$$

$$K_T = T(\beta \theta_2 \partial_1 + \lambda \theta_1 \partial_2) \quad (4.59)$$

$$Q_T^i = T \theta_1 \theta_2 \partial_i \quad \text{for} \quad i = 1, 2 \quad (4.60)$$

Since $\mathcal{D}_{3\lambda}$ is even, the super-dilatations again appear. The non-vanishing commutation relations are:

$$[L_h^T, L_k^U] = L_k^{T(h,U)12} - L_h^{U(k,T)12} + L_{(h,k)12}^{TU} \quad (4.61)$$

$$[L_h^T, D_k^U] = D_k^{T(h,U)12} - D_h^{U(k,T)12} + D_{(h,k)12}^{TU} \quad (4.62)$$

$$[L_h^T, K_U] = K_{T(h,U)12} \quad (4.63)$$

$$[L_h^T, Q_U^i] = Q_{T(h,U)12}^i \quad \text{for} \quad i = 1, 2 \quad (4.64)$$

$$[P_i, L_h^T] = L_{\partial_{u_i} h}^T + L_h^{\partial_{u_i} T} \quad (4.65)$$

$$[P_i, D_h^T] = D_{\partial_{u_i} h}^T + D_h^{\partial_{u_i} T} \quad (4.66)$$

$$[P_i, K^T] = K^{\partial_{u_i} T} \quad [P_i, Q_T^j] = Q_{\partial_{u_i} T}^j \quad (4.67)$$

$$[K_T, Q_U^i] = -\lambda Q_{TU}^2 \quad [K_T, Q_U^j] = -\alpha Q_{TU}^1 \quad (4.68)$$

³the functions $h(u_1, u_2)$ showing up in the $H(2,1)$ Kac's algebra are polynomials in the u_1, u_2 variables defined in the plane.

and also :

$$[\mathcal{L}_{\vec{m}}, \mathcal{L}_{\vec{n}}] = (\vec{m} \times \vec{n}) \mathcal{L}_{\vec{m} + \vec{n}} \quad (5.14)$$

$$[\mathcal{P}_j, \mathcal{L}_{\vec{m}}] = m_j \mathcal{L}_{\vec{m}} \quad (5.15)$$

$$[\mathcal{P}_i, \mathcal{P}_j] = 0 \quad (5.16)$$

Such a construction can be extended to the bosonic KM generators :

$$\mathcal{T}_{\vec{m}}^a = -\frac{i}{2} f_{bc}^a \sum_{\vec{p}} \psi_{\vec{p}}^b \psi_{\vec{m} - \vec{p}}^c \quad (5.17)$$

as well as to all the fermionic generators of the $\mathcal{D}$ -compatible algebra :

$$\mathcal{G}_{\vec{m}} = \frac{i}{6} f_{abc} \sum_{\vec{p}, \vec{q}, \vec{r}} \delta_{\vec{p} + \vec{q} + \vec{r}, \vec{m}} \psi_{\vec{p}}^a \psi_{\vec{q}}^b \psi_{\vec{r}}^c \quad (5.18)$$

However we have not been able, up to now, to realize the above construction in terms of a Fock space, and to define a normal ordered product. As a consequence, central extensions of $\text{SDiff}(S^1 \times S^1)$ cannot be provided.

Let us remark that the above "formal" construction can be done for any closed compact two-dimensional surface $\mathcal{M}$. Indeed, denoting by $\{A_\alpha\}_{\alpha \in \Lambda}$ a basis of smooth functions defined on $\mathcal{M}$, we define :

$$\{A_\alpha, A_\beta\}_{1,2} = (\partial_1 A_\alpha)(\partial_2 A_\beta) - (\partial_2 A_\alpha)(\partial_1 A_\beta) = g_{\alpha\beta}^{\gamma} A_\gamma \quad (5.19)$$

and :

$$\int_{\mathcal{M}} A_\alpha A_\beta du_1 du_2 = \eta_{\alpha\beta} = \eta_{\beta\alpha} \quad (5.20)$$

The generators of the super KM algebra $\tilde{\mathcal{G}}_w(\mathcal{M} \times Gr(1))$ with

$$\omega(X, Y) = \int \det d^2 u X \partial_\theta Y \quad (5.21)$$

will be written as

$$\begin{cases} T_\alpha^a = A_\alpha(u_1, u_2) \otimes T^a \\ \psi_\beta^a = \theta A_\beta(u_1, u_2) \otimes T^a \end{cases} \quad (5.22)$$

and they satisfy the CR :

$$\begin{aligned} [T_\alpha^a, T_\beta^b] &= i f_c^{ab} F_{\alpha\beta}^c T_\gamma^c \quad \text{with} \quad A_\alpha A_\beta = F_{\alpha\beta}^\gamma A_\gamma \\ [T_\alpha^a, \psi_\beta^b] &= i f_c^{ab} F_{\alpha\beta}^c \psi_\gamma^c \\ \{\psi_\alpha^a, \psi_\beta^b\} &= \delta^{ab} \eta_{\alpha\beta} \end{aligned} \quad (5.23)$$

The "exact part" of area-preserving diffeomorphism will be generated by :

$$L_\alpha = (\partial_1 A_\alpha) \partial_2 - (\partial_2 A_\alpha) \partial_1 \quad (5.24)$$

with

$$[L_\alpha, L_\beta] = g_{\alpha\beta}^\gamma L_\gamma \quad (5.25)$$

Setting :

$$g_{\alpha\beta\gamma} = g_{\alpha\beta}^\mu \eta_{\mu\gamma} = \int du_1 du_2 \{A_\alpha, A_\beta\}_{1,2} A_\gamma \quad (5.26)$$

one can check that these quantities are completely antisymmetric ($g_{\alpha\beta\gamma} = -g_{\beta\alpha\gamma}$ from the definition, and $g_{\alpha\beta\gamma} = -g_{\gamma\beta\alpha}$ after an integration by parts).

Then, defining :

$$\mathcal{L}_\alpha = -\frac{1}{2} \sum_{\beta, \gamma}^{dim \mathcal{G}} g_{\alpha\beta}^\gamma \psi_\beta^a \psi_\gamma^a \quad (5.27)$$

we easily check :

$$[\mathcal{L}_\alpha, \psi_\beta^a] = g_{\alpha\beta}^\gamma \psi_\gamma^a \quad (5.28)$$

and

$$[\mathcal{L}_\alpha, \mathcal{L}_\beta] = g_{\alpha\beta}^\gamma \mathcal{L}_\gamma \quad (5.29)$$

Note that this general framework can be immediately applied to the case $\mathcal{M} = S^2$, the natural basis being the spherical harmonics $Y_m^l(\theta, Y)$ [5].

6 Conclusion :

We have seen that, although $\text{SDiff}(\mathcal{M})$ cannot appear as a diffeomorphism subalgebra acting as a derivation algebra on a KM $\tilde{\mathcal{G}}_w(\mathcal{M})$, the situation changes when supersymmetry is considered. Indeed, introducing one supersymmetry, $\text{SDiff}(\mathcal{M})$ acts as a derivation algebra on the super KM $\tilde{\mathcal{G}}_w(\mathcal{M} \times Gr(1))$, the central extension associated to ω involving only fermionic generators. Enlarging the supermanifold $\mathcal{M} \times Gr(1)$ with a second supersymmetry, it is a supersymmetric extension of $\text{SDiff}(\mathcal{M})$ -analogue of the $H(2,1)$ Kac's algebra in the sense of ref.[12]- which can be obtained. Our approach induces a fermionic construction, à la Sugawara, of $\text{SDiff}(\mathcal{M})$. However, we have only been able to propose a formal construction of it. We address the question of a normal ordered product to completely solve this problem, and in particular to get the central extensions of $\text{SDiff}(\mathcal{M})$ when they exist. At this point, one can add the following comment. Rewriting the bosonic generators (3.12) acting on $\tilde{\mathcal{G}}_w(\mathcal{M} \times Gr(1))$ as :

$$L_h^T = T.L_h + \frac{1}{2}[L_h, T]\theta\partial_\theta \quad (6.1)$$

with $L_h = \partial_1 h \partial_2 - \partial_2 h \partial_1$ one remarks that they appear as the supersymmetric generalization of $\text{Diff}(\mathcal{M})$ generated by :

$$L_h^T = T.L_h \quad (6.2)$$

in the same way the bosonic generators of the N.R.S. ($N=1$) super-conformal algebra :

$$L_f = f(z)\partial_z + \frac{1}{2}[\partial_z, f(z)]\theta\partial_\theta \quad (6.3)$$

appear as the generalization of the $\text{Diff}(S^1)$ algebra :

$$L_f = f(z)\partial_z. \quad (6.4)$$

In this spirit, a natural supersymmetric extension of $\text{SDiff}(\mathcal{M})$ would simply be the algebra generated by L_h .

The above investigations have been performed in the case of a two-dimensional smooth closed compact surface $\mathcal{M}$. Let us add that almost the same kind of results can be obtained by relaxing the condition of closure on $\mathcal{M}$.

More precisely, let us consider the case of the cylinder which deserves some interest in the context of the w_∞ algebra [7]. Then let $f(e^{iu_1}, u_2)$ be an infinitely differentiable function on $\mathcal{M} = S^1 \times R$. Restricting the functions f to be in the u_2 variable, such that :

$$\lim_{u_2 \rightarrow \pm\infty} (f(u_2)u_2^m) = 0 \quad \forall m \in \mathbb{Z}, \quad (6.5)$$

one can consider the cocycle ω associated to $\mathcal{D} = \partial_\theta$. With the choice of generators for $\tilde{\mathcal{G}}_\omega(S^1 \times R)$

$$\begin{aligned} T_{\alpha,n}^a &= e^{inu_1} A_\alpha(u_2) \otimes T^a \\ \psi_{\alpha,n}^a &= \theta e^{inu_1} A_\alpha(u_2) \otimes T^a \end{aligned} \quad (6.6)$$

where the $\{A_\alpha\}_{\alpha \in I}$ form a basis of functions satisfying (6.5), we write the non-trivial action of ω on $\tilde{\mathcal{G}}_\omega \times \tilde{\mathcal{G}}_\omega$:

$$\omega(\psi_{\alpha,n}^a, \psi_{\beta,m}^b) = \delta^{ab} \delta_{n,m} \eta_{\alpha\beta} \quad \text{with} \quad : \eta_{\alpha\beta} = \int_{-\infty}^{+\infty} du_2 A_\alpha \cdot A_\beta \quad (6.7)$$

and the corresponding derivation algebra :

$$\begin{aligned} L_h^I &= T(\partial_1 h \partial_2 - \partial_2 h \partial_1) + \frac{1}{2} \{h, T\}_{12} \theta \partial_\theta \\ P_{u_1} &= \partial_1 \\ G_k &= k \partial_\theta \end{aligned} \quad (6.8)$$

in which $\text{SDiff}(S^1 \times R)$ is generated by $L_h^{I=1}$ and P_{u_1} . Of course the method explicated in Section 5 can be used to realize $\text{SDiff}(S^1 \times R)$ from the $\psi_{\alpha,n}^a$, or more precisely, a subset of the area preserving diffeomorphisms algebra, the functions $A_\alpha(u_2)$ being constrained to the condition (6.5).

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