CERN-TH.5816/90
HD-TVP-90-9
August 1990

Searching for an Optimal Lattice Colour Dielectric Theory

B. Großmann¹, H.J. Pirner, A.I. Signal

Institute for Theoretical Physics
Philosophenweg 19
D-6900 Heidelberg, F. R. Germany ²

R. Baier³

Department of Theoretical Physics
University of Bielefeld
D-4800 Bielefeld 1, F. R. Germany

J. Wroldsen⁴

Division TH, CERN
CH-1211 Geneva, Switzerland

Abstract

We investigate the use of renormalization group transformations (RGT) to define an effective colour dielectric theory on the lattice. Starting from the pure gauge sector of $SU(2)$, we try to find an optimal RGT which leads to an effective action with few parameters. We test this effective action by simulating the colour dielectric theory on the lattice and checking for operator matching.

¹Supported by Studienstiftung des deutschen volkes

²Supported by the BMFT under contract number 06-HD-756

³Supported in part by Deutsche Forschungsgemeinschaft.

⁴Permanent Address: Gjøvik College of Engineering, Box 191, N-2801 Gjøvik, Norway

1 Introduction

Colour dielectric theory is an approach to the strong interaction based on regarding the QCD vacuum as a medium characterised by a dielectric field [1,2]. The attraction of this theory is its promise to construct an effective action for hadronic structure from the QCD Lagrangian. In this paper we present work in progress towards this ultimate goal.

The major aim of this work is to connect distance scales of < 0.1 fm, where QCD is asymptotically free, with hadronic distance scales of ~ 1 fm, where confinement is important. The Monte Carlo renormalization group (MCRG) [3] is a good tool for this approach, as it enables us to relate theories with different distance scales via renormalization group transformations (RGT's). In moving from small to large distance scales via a succession of RGT's, the high frequency components of the gluon field are integrated out to give a polarizable medium, which eventually gives rise to confinement on hadronic distance scales.

2 Colour Dielectrics on the Lattice

For our starting point we take pure $SU(2)$ gauge theory with the Wilson action on a lattice with spacing a in Euclidean space-time. The gluon fields are represented as elements of $SU(2)$, $A_\lambda(x) = \frac{1}{2} \sum_{i=1}^3 A_\lambda^i(x) \sigma^i$, and the link variable from site x in direction μ is $U_{x,\mu} = \exp\{ia g A_\mu\}$. The partition function of the theory is $Z = \int [\mathcal{D}U] \exp\{-S(U)\}$, where $[\mathcal{D}U]$ is the Haar measure for the unitary fields, and the action is $S(U) = -\frac{1}{2} \beta \sum_{\square} \text{Tr} U_{\square}$, with $\beta = 4/g^2$ and $U_{\square}$ being the product of four U -matrices going around a fundamental plaquette. In the naive continuum limit ($a \rightarrow 0$) we recover the well-known action for pure gauge theory $S = \int d^4x \frac{1}{2} \text{Tr} F_{\mu\nu} F_{\mu\nu}$.

To define a colour dielectric action we use a block spin transformation (BST). The block

spin variable ϕ is a variable on a lattice with spacing $2a$, and is determined by taking a weighted sum over paths on the original lattice. The action of the blocked theory on the coarse lattice is given by

$$\exp\{-S_{\text{block}}(\phi)\} = \int [\mathcal{D}U] \exp\{-S(U)\} \prod_{x',\mu} \delta(\phi_{x',\mu} - \tilde{\phi}_{x',\mu}(U)), \quad (1)$$

where $\tilde{\phi}_{x',\mu}(U)$ is given by the BST (see Fig. 1):

$$\begin{aligned} \tilde{\phi}_{x',\mu}(U) = & \frac{1}{|\rho_1| + 6|\rho_2| + 12|\rho_3|} \left\{ \frac{\rho_1}{18} \sum_{\nu,1,\mu} (U_{x,\nu} U_{x+\nu,\mu}^\dagger U_{x,\mu} U_{x+\mu,\mu}) \right. \\ & + U_{x,\mu} U_{x+\mu,\nu} U_{x+\mu,\nu}^\dagger U_{x+\mu,\mu} \\ & + U_{x,\mu} U_{x+\mu,\mu} U_{x+2\mu,\nu} U_{x+2\mu,\nu}^\dagger \\ & + \rho_2 \sum_{\nu,1,\mu} U_{x,\nu} U_{x+\nu,\mu} U_{x+\nu+\mu,\mu}^\dagger U_{x+2\mu,\nu} \\ & + \rho_3 \sum_{\nu,1,\mu} (U_{x,\nu} U_{x+\nu,\mu} U_{x+\mu,\nu}^\dagger U_{x+\mu,\mu} \\ & \left. + U_{x,\mu} U_{x+\mu,\nu} U_{x+\mu+\nu,\mu}^\dagger U_{x+2\mu,\nu}) \right\}. \quad (2) \end{aligned}$$

Each term of this BST involves four link variables, so if we replace the U -matrices by ϕ -matrices, we can use the same BST to transform from the lattice with spacing $2a$ to a lattice with spacing $4a$. This process can be repeated, and so we form a chain of theories on lattices with spacing $2^n a$ where n is a non-negative integer. The variables ϕ of these theories are formed from a weighted sum of elements of $SU(2)$, and so they have a ray decomposition into the product of a scalar, χ , and an element of $SU(2)$, V : $\phi = \chi V$ [2]. From eqn. (2) we have that $0 \leq \chi \leq 1$. The positive parity field χ is identified with the colour dielectric field.

We now want to calculate the blocked action $S_{\text{block}}(\phi)$ in terms of the new variables ϕ . In general there will be an infinite number of terms in this new action and for practical purposes we can only approximate $S_{\text{block}}(\phi)$ by using a finite number of these. Even the

approximation to the blocked action, which we shall call the effective action $S_{\text{eff}}(\phi)$, will be more complicated than the Wilson action. However if we include only a few terms in $S_{\text{eff}}(\phi)$, then the increase in the complexity of our theory should be more than offset by our ability to study the long-range properties of QCD.

In this study we use an effective action containing terms up to and including fourth order in the ϕ field:

$$S_{\text{eff}} = -\beta' O(1) + \mu^2 O(2) + \lambda O(3) + \lambda' O(4) \quad (3)$$

with the operators given by

$$\begin{aligned} O(1) &= \sum_{\mathbf{a}} \frac{1}{2} \text{Tr} \phi_{\mathbf{a}} \\ O(2) &= \sum_{\mathbf{x}, \mu} \chi_{\mathbf{x}, \mu}^2 \\ O(3) &= \sum_{\mathbf{x}, \mu} \chi_{\mathbf{x}, \mu}^4 \\ O(4) &= \sum_{\mathbf{x}, \mu} \chi_{\mathbf{x}, \mu}^2 \cdot \frac{1}{12} \sum_{\nu, \mathbf{1}\mu} (\chi_{\mathbf{x}, \nu}^2 + \chi_{\mathbf{x}+\mu, \nu}^2), \end{aligned} \quad (4)$$

where the subscript μ varies over the positive directions, and the subscript ν varies over positive and negative directions. Path integrals in the colour dielectric theory are evaluated with the measure

$$\int [\mathcal{D}\phi] = \int [\mathcal{D}V] \int [\mathcal{D}\chi] = \int [\mathcal{D}V] \int_0^1 d\chi \chi^3. \quad (5)$$

The first term of the colour dielectric action is the same as that of pure gauge theory, except for the function $\varepsilon = \chi^4$ which modifies the gluon energy density to the same form as is used in the electrodynamics of macroscopic media. The high frequency dynamics of the gauge fields is absorbed into the χ -field, and the remaining gauge fields V contain long wavelength gluon modes. We note that Mack [2] has shown that for μ^2, λ, λ' all positive

the colour dielectric theory is confining; however, we should remember that it may well be possible for one or more of these couplings to be negative and the theory to still be confining, ie. Mack's condition of positivity of the couplings is sufficient but not necessary to ensure that the theory is confining.

We can study the effects of including only a finite number of terms in our effective action by simulating S_{eff} on a lattice also of spacing $2a$. For this simulation we calculate various operators, and check how well they match between the simulated colour dielectric theory, with S_{eff} , and the theory obtained by blocking from $SU(2)$, S_{block} . By a prudent choice of the block spin parameters $\{\rho_i\}$ we hope to obtain approximate matching for this lattice size, thus reducing the effects of irrelevant operators in the blocked action. Once we obtain satisfactory matching on the lattice with spacing $2a$, we will then be able to make further BST's to lattices with spacings $4a, 8a$ etc. and compare matching between the blocked theory (which originated with pure gauge $SU(2)$) and the simulated theory. If the matching is satisfactory then we can study further the properties of the colour dielectric theory defined by our effective action, eqn. (3).

3 Schwinger-Dyson Equations for Colour Dielectric Theory

In order to find the values of the couplings of the effective theory S_{eff} , we solve a set of a finite number of Schwinger-Dyson equations in the four couplings $\{\beta', \mu^2, \lambda, \lambda'\}$ for each of the simulations [4,5]. Using the fact that the measure is invariant under left (or right) $SU(2)$ multiplications $d\phi_i \rightarrow d(e^{i\theta\sigma} \phi_i)$, one can derive [6] an equation for the gauge coupling

β' which is analogous to the well-known equation found by Makeenko and Migdal [7]:

$$\frac{1}{2}\beta' \left\langle \sum_{\square} \text{Tr} \phi_1^\dagger \phi_1^\dagger \phi_6^\dagger \phi_3^\dagger \phi_4^\dagger \phi_3 \phi_2 \phi_1 - \text{Tr} \phi_5 \phi_6 \phi_7 \phi_1 \phi_4 \phi_3 \phi_2 \phi_1 \right\rangle = 3 \langle \text{Tr} \phi_4 \phi_3 \phi_2 \phi_1 \rangle \quad (6)$$

where the summation is over all plaquettes $\square$ containing the links ϕ_1, ϕ_5, ϕ_6 and ϕ_7 . The first combination of eight links on the left is arranged as a straight window, while the second combination corresponds to a crossed window, see Fig. 2.

To derive equations involving the other three couplings, we make use of the transformation properties of the integral defining the action. If we consider the expectation value obtained after blocking of some arbitrary operator, $O(\chi, V)$, which is differentiable in χ , we have

$$\int [DV] \int [\mathcal{D}\chi] e^{-S_{\text{block}}(\chi, V)} \frac{\partial}{\partial \chi_i} O(\chi_i, V) = \left\langle \frac{\partial}{\partial \chi_i} O(\chi_i, V) \right\rangle. \quad (7)$$

From eqn. (5) we have the measure for the variables χ

$$\int [\mathcal{D}\chi] = \prod_j \int_0^1 d\chi_j P(\chi_j), \quad (8)$$

where the phase space function for the χ 's is $P(\chi) = \chi^3$. The integral over χ_i on the left of eqn. (7) can be done by parts:

$$\begin{aligned} \left\langle \frac{\partial}{\partial \chi_i} O(\chi_i, V) \right\rangle &= \int [DV] \prod_{j \neq i} \int_0^1 d\chi_j P(\chi_j) \times \\ &\times \left\{ P(\chi_i) e^{-S_{\text{block}}(\chi, V)} O(\chi_i, V) \right\}_{\chi_i=0}^{\chi_i=1} \\ &- \int_0^1 d\chi_i O(\chi_i, V) \frac{\partial}{\partial \chi_i} \left[P(\chi_i) e^{-S_{\text{block}}(\chi, V)} \right]. \end{aligned} \quad (9)$$

The first term on the right of eqn. (9) will vanish provided that the product $P(\chi)O(\chi)$ also vanishes at $\chi = 0$ and $\chi = 1$, thus we will choose the operators $O(\chi)$ to vanish at $\chi = 1$ and to be less divergent than χ^{-3} at $\chi = 0$. The operators we use in this work are

(where we now set $\chi_i = \chi_{x,\mu}$):

$$\begin{aligned} (1 - \chi_{x,\mu}) \chi_{x,\mu}^n, \quad n = 1 \dots 4, \\ (1 - \chi_{x,\mu}) \sum_{\square} \text{Tr} \phi_{\square}, \quad \phi_{x,\mu} \in \phi_{\square}, \\ (1 - \chi_{x,\mu}) \chi_{x,\mu} \frac{1}{12} \sum_{\nu \perp \mu} (\chi_{x,\nu}^2 + \chi_{x+\mu,\nu}^2), \\ (1 - \chi_{x,\mu}) \chi_{x,\mu} \frac{1}{2} \sum_{\nu \perp \mu} (\chi_{x,\nu}^2 \chi_{x+\nu,\mu}^2 + \chi_{x,\mu}^2 \chi_{x+\mu,\nu}^2). \end{aligned} \quad (10)$$

After partial integration we obtain a Schwinger-Dyson equation for the operator $O(\chi, V)$, which is an equation in the couplings of the blocked theory:

$$\left\langle \left\{ \frac{\partial}{\partial \chi_{x,\mu}} S_{\text{block}}(\chi, V) - \frac{3}{\chi_{x,\mu}} \right\} O(\chi_{x,\mu}, V) \right\rangle = \left\langle \frac{\partial}{\partial \chi_{x,\mu}} O(\chi_{x,\mu}, V) \right\rangle. \quad (11)$$

Because we cannot calculate the infinity of couplings of $S_{\text{block}}(\chi, V)$, we have to use the effective action, $S_{\text{eff}}(\chi, V)$, in the Schwinger-Dyson equations. Using this procedure amounts to minimizing the 'distance', in the sense of reference [5], between our effective action, S_{eff} , and the optimal (but uncalculable) S_{block} . However this does not amount exactly to minimizing the differences between operator expectation values calculated for the blocked and effective theories. The effects of this will be considered in the next section.

From eqns. (3) and (4), we have

$$\begin{aligned} \frac{\partial}{\partial \chi_{x,\mu}} S_{\text{eff}}(\chi, V) &= -\frac{1}{2} \beta' \sum_{\phi_{x,\mu} \in \phi_{\square}} \text{Tr} \phi_{\square} / \chi_{x,\mu} \\ &+ 2\mu^2 \chi_{x,\mu} + 4\lambda \chi_{x,\mu}^3 + 4\lambda' \chi_{x,\mu} \frac{1}{12} \sum_{\nu \perp \mu} (\chi_{x,\nu}^2 + \chi_{x+\mu,\nu}^2). \end{aligned} \quad (12)$$

The factor of four multiplying the λ' term arises because we must consider both the μ and the ν directions when we take the derivative. Putting this expression into the Schwinger-Dyson equation, eqn. (11), we obtain a system of linear equations for the four couplings of S_{eff} .

Our set of eight equations (including eqn. (6)) in four unknowns is overdetermined, so we use a least squares error routine to determine the best solution set for the couplings $\{\beta', \mu^2, \lambda, \lambda'\}$. We have also checked that omitting one of the equations does not change our solution to any extent.

4 Numerical Results

We first present results for various choices of the block spin parameters (ρ_1, ρ_2, ρ_3) in table 1.

These choices were motivated by the work of Lang and Salmhofer [8] in the case of pure SU(2) theory. We can set ρ_1 to unity without any loss of generality. First we simulate pure SU(2) theory on a 16^4 lattice with $\beta = 2.35$ and apply the BST, eqn. (2), to obtain a blocked theory on an 8^4 lattice. Typically we discard the first 5000 configurations on the 16^4 lattice, then apply the BST to every 10th configuration, and take 500 measurements on the 8^4 lattice. Using the set of Schwinger-Dyson equations above, we determine the four couplings $\{\beta', \mu^2, \lambda, \lambda'\}$ of the blocked theory. We also measure $\overline{O}(i)$, defined as $O(i)$ in eqn. (4) calculated with the weight $[DV\mathcal{D}\chi] \exp\{-S_{\text{block}}(\chi, V)\}$, divided by either the number of plaquettes, in the case of the plaquette operator $O(1)$, or the number of lattice links for the other three operators. These are displayed in the columns of table 1 under the headings *Block* for four choices of $\{\rho_i\}$.

For each choice of $\{\rho_i\}$ we take the four couplings determined from the 1st blocking step and use these as inputs to our simulation of the effective action S_{eff} on an 8^4 lattice. In a typical simulation of the effective action, we discard the first 1500 configurations, then take 500 measurements, each measurement taken every 10 configurations. The results of these simulations, with the operator expectation values now calculated with the weight

Table 1: Values of the coupling constants and operator expectation values as a function of the BST parameters ($\beta = 2.35$).

$\{\rho_i\}$	(1, 1, 1)		(1, -6, -0.18)		(1, 1, -0.09)		(1, -0.7, -0.18)	
	Block	Eff.	Block	Eff.	Block	Eff.	Block	Eff.
β'	14.57(1)	14.56(2)	17.99(1)	17.97(1)	50.84(3)	50.89(6)	57.80(4)	57.79(8)
μ^2	3.72(6)	3.70(8)	-10.97(7)	-10.95(6)	-18.4(1)	-18.5(1)	-16.23(8)	-16.1(2)
λ	39.21(3)	39.23(4)	15.54(2)	15.51(2)	40.31(4)	40.38(6)	55.07(9)	55.2(2)
λ'	-28.28(9)	-28.2(2)	21.6(1)	21.6(1)	60.3(3)	60.4(3)	55.2(3)	54.7(6)
$\overline{O}(1)$	0.0949(1)	0.00638(2)	0.05158(8)	0.02649(2)	0.07282(3)	0.01300(2)	0.01626(3)	0.00743(6)
$\overline{O}(2)$	0.4135(1)	0.2032(1)	0.3223(1)	0.28451(7)	0.2082(1)	0.1840(1)	0.17980(7)	0.1540(2)
$\overline{O}(3)$	0.1827(1)	0.04955(4)	0.1211(1)	0.09870(5)	0.05018(3)	0.04088(3)	0.03749(3)	0.02879(7)
$\overline{O}(4)$	0.1724(1)	0.04181(4)	0.1048(1)	0.08068(4)	0.04369(3)	0.03382(3)	0.03264(3)	0.02369(6)
D	0.295(1)		0.056(1)		0.029(1)		0.030(1)	

$[DV\mathcal{D}\chi] \exp\{-S_{\text{eff}}(\chi, V)\}$, are displayed in table 1 under the headings *Eff.* We also show the ‘distance’ in operator space, defined by $D = \left[\sum_i (\overline{O}_{\text{block}}(i) - \overline{O}_{\text{eff}}(i))^2 \right]^{1/2}$, between the theory resulting from blocking of SU(2) theory and the theory resulting from simulating the effective action. The errors in our measurements of the couplings and the operators are determined using the method of reference [9].

We note that the values for the coupling constants in table 1 under *Eff.* are determined from the Schwinger-Dyson-equations using the configurations generated with the input couplings listed under *Block*: The comparison shows that we are indeed able to measure reasonably well the values of the couplings in the simulations of the effective action, which gives us confidence in our set of Schwinger-Dyson equations. However we are not always able to come close to reproducing the expectation values of the four operators, especially the plaquette operator $\overline{O}(1)$. The best choice for the blocking parameters agrees with the results of Lang and Salmhofer [8] in pure SU(2) theory. They found the optimal choice to be $(1, -0.7, -0.18)$, which also appears to be a reasonable choice for our colour dielectric theory. In the following we will use this choice of blocking parameters.

Now the theory obtained by blocking from SU(2) will be affected by the terms of order

Table 2: Values of the operator expectation values in the effective theory as a function of the input couplings ($\beta = 2.35$)¹

	Block	Sim.1	Sim.2	Sim.3	Sim.4	MF
β'	57.80(4)	57.80	60.57	60.59	60.59	57.80
μ^2	-16.23(8)	-16.23	-17.15	-17.14	-17.15	-16.23
λ	55.07(9)	55.07	54.50	54.50	54.90	55.07
χ	55.2(3)	55.19	54.50	54.50	54.50	55.19
$O(1)$	0.01626(3)	0.00743(6)	0.01468(5)	0.01331(3)	0.01334(2)	0.008
$O(2)$	0.17980(7)	0.1540(2)	0.1805(1)	0.1807(1)	0.1764(1)	0.157
$O(3)$	0.03749(3)	0.02879(7)	0.03836(6)	0.03843(6)	0.03678(3)	0.024
$O(4)$	0.03264(3)	0.02369(6)	0.03276(6)	0.03282(6)	0.03125(3)	
D		0.030(1)	0.0020(1)	0.0048(1)	0.0047(1)	

higher than χ^4 or $\text{Tr}\phi_n$, which we have neglected in the effective action. In particular the expectation values even of low order operators do not match particularly well between the two theories. One possibility to account for the effects of ignoring the higher order terms in S_{block} is to vary slightly, or ‘renormalize’, the input couplings for the simulations of the effective theory. In table 2 we investigate the variation in operator matching between the blocked and the effective theories starting from an initial value of $\beta = 2.35$ on the 16^4 lattice. We vary the inputs to the effective theory about the values measured from the blocked theory, and check whether we obtain better matching of the operators $\overline{O(i)}$. We see that small variations ($\sim 5\%$) in the input values of the couplings can have a marked effect on operator matching between the blocked and effective theories. By varying the couplings in order to increase $\overline{O(1)}$ and $\overline{O(2)}$, we found that small increases in the magnitudes of β' and μ^2 , and small decreases in the magnitudes of λ and χ will improve the matching.

In the final column of table 2 we display the results of an approximation to the colour dielectric effective action, eqn. (3), obtained by making a strong coupling expansion for

¹The values of the couplings displayed in table 2 are the inputs to the simulations, rather than the values obtained from Schwinger-Dyson equations during the simulations. Thus there is a slight discrepancy between the couplings shown in column 8 of table 1 and column 2 of table 2.

the unitary fields in the plaquette term [10], and neglecting the variations and correlations of the χ fields. The resulting mean field partition function for the χ field has the form

$$Z_{\text{MF}} = \int_0^1 d\chi \exp\{-S_{\text{MF}}(\chi)\} \quad (13)$$

with

$$S_{\text{MF}} = -3 \ln \chi + \mu^2 \chi^2 + (\lambda + \chi') \chi^4 - \frac{3}{16} \beta'^2 \chi^8 + \frac{1}{256} \beta'^4 \chi^{16} \dots \quad (14)$$

The term $-3 \ln \chi$ comes from the phase space factor. By finding the saddle-point of this approximation to the effective action, we can calculate the expectation values in table 2. We see that this approximation reproduces the result of the simulation of the effective theory (Sim.1) with reasonable accuracy ($\sim 10\%$ error). We conclude that we can understand the behaviour of the effective theory, in the region of coupling constant space near the point given by the blocked theory, in terms of strong coupling theory for the unitary fields in the action, and a mean field type approximation for the χ fields. This approximation can be very useful in investigating the behaviour of the effective action, particularly the effects of changing one or more of the couplings, or adding higher order terms to the action, as well as in the application of the colour dielectric theory to physical problems.

Acknowledgements

We would like to thank P.H. Damgaard, R. Gavai, F. Karsch, M. Lüscher, G. Mack, A. Patel and F. Wegner for their helpful comments and discussions. The numerical simulations in this paper were performed at the CERN computer centre, HLRZ Jülich and RUS Stuttgart.

References

- [1] H. B. Nielsen and A. Patkos, Nucl. Phys. B195 (1982) 137.
- [2] G. Mack, Nucl. Phys. B235 (1984) 197.
- [3] K. G. Wilson in *Recent Developments in Gauge Theories*, ed. G. 'tHooft (Plenum, New York, 1980); R. H. Swendsen, Phys. Rev. Lett. 47 (1981) 1775.
- [4] M. Falcioni et al., Nucl. Phys. B265 (1986) 187.
- [5] A. Gonzalez-Arroyo and M. Okawa, Phys. Rev. D35 (1987) 672.
- [6] H.-J. Pirner, J. Wroldsen and M. Ilgenfritz, Nucl. Phys. B294 (1987) 905.
- [7] Y. M. Makeenko and A. A. Migdal, Phys. Lett. 88B (1979) 135;
- [8] C. B. Lang and M. Salmhofer, Phys. Lett. B205 (1988) 329.
- [9] G. J. Daniell, A. G. J. Hey and J. E. Mandula, Phys. Rev. D30 (1984) 2230.
- [10] R. Balian, J. M. Drouffe and C. Itzykson, Phys. Rev. D11 (1975) 2104; Phys. Rev. D19 (1979) 2514.

Figure Captions

Figure 1. The three terms in the block spin transformation used to define link variables ϕ on a lattice with spacing $2a$.

Figure 2. A pictorial representation of the colour dielectric version of the Migdal-Makeenko equation (eqn. (6)).