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Direct One-Loop Renormalization of $SU(2)_L \times U(1)_Y$ Four-Fermion Processes and Running Coupling Constants*

M. Kuroda⁺, G. Moulaka⁺⁺

Department of Theoretical Physics
University of Bielefeld

and

D. Schildknecht⁺⁺⁺

CERN - Theory Division

Abstract

Within the $SU(2)_L \times U(1)$ electroweak theory, by combining (bosonic) vacuum polarization with vertex and box contributions, we derive gauge-invariant (one-loop) dressed propagators for the $W^\pm$ boson and the γW^3 system in a straightforward manner. By introducing running couplings and masses, the corresponding four-fermion amplitudes are cast into a Born-term form. The subsequent "direct renormalization" expresses the running quantities in terms of their values at a chosen scale. Among other things, the value of Δr is rederived, and a comparison of our procedure and results with the results in the literature is carried out. Care is taken to give a transparent presentation of all physically relevant steps.

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⁺ Permanent address: Department of Physics, Meiji-Gakuin-University.

⁺⁺ Present and permanent address: Laboratoire de Physique Mathématique, Université des Sciences et Techniques du Languedoc, unité associée au CNRS, URA 040 768.

⁺⁺⁺ Permanent address: Department of Theoretical Physics, University of Bielefeld.

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1. Introduction

Renormalization in quantum field theory may be carried out via two technically different methods. The first one, mainly used in recent years, introduces counterterms in the basic Lagrangian and works with renormalized quantities throughout. The second method proceeds entirely via the bare Lagrangian. The second method has recently been used in the analysis on the significance of precision electroweak experiments by Gounaris and one of the present authors.^{1,2)} * It is of particular relevance when used in conjunction with "running", i.e., momentum dependent couplings and masses, thereby generalizing the notion of "running" parameters from the asymptotic domain where they are determined by renormalization-group methods to the non-asymptotic domain. The amplitude for a given process is expressed in terms of running couplings and masses, and to complete the procedure, the running parameters, originally related to the bare parameters in the Lagrangian, are expressed in terms of their values at a chosen (energy) scale, thereby eliminating the bare quantities. This procedure of carrying out renormalization will henceforth be called "direct renormalization", as it directly works with the bare Lagrangian without ever introducing counterterms. In our opinion this method is less formal than the method of counterterms, and thus, it is particularly transparent as regards its physical interpretation and also particularly close to the very heart of renormalization theory which simply amounts to predicting physical phenomena over a wide energy range in terms of a few parameters empirically determined at a particular energy scale.

The work^{1,2)} by Gounaris and one of the present authors has been aiming at an analysis of the role of the fermion vacuum polarization in precision electroweak experiments, in order to explore the question of how far future SLC-LEP experiments will be able to test the electroweak theory⁴⁾ beyond the fermion interactions which are empirically well known already, i.e., how far the precision experiments at the Z^0 peak will be able to restrict and/or directly measure the bosonic sector, i.e., the unknown vector-boson self-interactions and the interaction with a possibly existing Higgs scalar particle.

The present paper is of a somewhat different, more methodological nature. It aims at applying the method of direct renormalization and running coupling constants to the full one-loop four-fermion interaction (for massless external fermions) in the $SU(2)_L \times U(1)_Y$ electroweak theory. We will be particularly concerned with the question of how to incorporate the bosonic vacuum-polarization corrections to the $W^\pm, Z^0, \gamma$ propagators into the formalism. It is well known that the case of the bosonic vacuum polarization is less trivial than the case of the fermionic one, due to the gauge dependence of the bosonic loops. Fully gauge-invariant dressed vector-boson propagators may be defined, however, by combining the bosonic vacuum polarization with (obviously) universal (i.e., process independent) vertex and box contributions. The "genuine" propagators thus contain what one may call the "genuine" vacuum polarization in the sense of fully incorporating all process-independent corrections.

Our procedure of deriving the genuine propagators will be manifestly gauge invariant, and moreover, by clearly and systematically distinguishing between kinetic (mixing) terms and mass terms induced in the propagators via radiative corrections, the transition to the

* For use of the second method compare also Kennedy and Lynn³⁾. We refer to refs.1 and 2 for an extensive list of references on calculations of radiative corrections to electroweak processes.

introduction of the running parameters via factorization (charged-current interactions) and diagonalization (neutral-current interactions), and thus the renormalization procedure, will turn out to be extremely transparent in its structure.

Although being a straightforward extension of refs. 1,2, our present work is also closely related in its aim and in its spirit to the work of Kennedy and Lynn³⁾. We differ from their work in the very straightforward, particularly transparent and manifestly gauge-invariant method of derivation and also in the choice of the final running parameters. A detailed discussion on different choices for the running parameters and a comparison of our running parameters with the ones of Kennedy and Lynn will be given in Section 2.4.

In Section 2, which is fairly self-contained, we will present all physically essential steps in the procedure of deriving the four-fermion amplitudes at the one-loop level in as short and concise a form as possible. Section 2 will also include a discussion of different possible choices for the running quantities. The technical details of the derivation of the gauge-invariant dressed vector-boson propagators will be described in Section 3 and in Appendices A and B. In Section 4, we will give a short and transparent derivation of the radiative correction, Δr , to the vector-boson mass formulae within the presently applied formalism, including a comparison with the results in the literature. Some conclusions will be presented in Section 5.

2. The Gauge-Invariant Vector-Boson Propagators and Running Coupling Constants

As mentioned in the introduction, the vector-boson-loop corrections to the vector-boson ($W^\pm, Z^0, \gamma$) propagators (Fig. 1a) are not gauge invariant by themselves, but have to be combined with universal (i.e., process-independent) contributions from vertex and box diagrams (Figs. 1b,c) to obtain what may actually be called the "full" or "genuine" (gauge invariant) vacuum polarization.* The gauge-invariant vector-boson propagators thus defined by incorporating the genuine vacuum polarization contain the bulk of the radiative corrections to four-fermion processes (in the limit of light external fermions), the dominant fermion loops, as well as the full dependence on the mass of the Higgs-boson, which only contributes to the vacuum polarization of Fig. 1a. Accordingly, the scattering amplitude of Fig. 1 is to be split into two additive contributions, the first one of which, $M'_{CC,NC}$, contains the tree-level amplitude corrected by the contributions of the genuine vacuum polarization, while the second one, $\tilde{M}_{CC,NC}$, contains the non-universal (i.e., process-dependent) parts of the diagrams in Figs 1 b,c,

$$M_{CC,NC} = M'_{CC,NC} + \tilde{M}_{CC,NC}. \quad (2.1)$$

It is evident that the dominant part of the scattering amplitude is given by $M'_{CC,NC}$, the amplitude $\tilde{M}_{CC,NC}$ being frequently (but not always) negligibly small.

* The "genuine" vacuum polarization in this sense thus essentially contains all radiative corrections which are independent of the nature of the particles which are being scattered. The usual resummation procedure applied to obtain the one-loop dressed propagator thus involves a resummation of parts of the vertex and box diagrams (in such a manner that the corresponding running coupling constants fulfil the asymptotic requirements based on the renormalization group).

In Section 2.1, we will give the genuine gauge-invariant propagator for charged current processes and represent the corresponding amplitude, M'_{CC} , in terms of a running coupling constant and a running mass. Likewise, in Section 2.2, we will treat neutral-current processes. Although the treatment of the neutral-current interaction parallels that of the charged-current interaction, it is more involved, as it requires the analysis and diagonalization of a two-by-two propagator matrix.

2.1 Charged-Current Interactions

For some of the technical details concerning the derivation of the genuine amplitude for charged-current processes, we refer to Section 3. We will thus be able to discuss the physically essential points by conveniently starting from

$$M'_{CC} = \frac{g_0^2}{2} j_\pm^\mu \Delta_\pm(q^2) j_{\pm\mu}, \quad (2.2)$$

where the (bare) SU(2) gauge coupling and the charged weak currents have been denoted by g_0 and $j_\pm$, respectively. The gauge-invariant genuine dressed (inverse) propagator in (2.2) is given by

$$\Delta_\pm^{-1}(q^2) \equiv d_\pm(q^2) = q^2(1 + g_0^2 \delta_\pm(q^2)) - M_W^{(0)2}(1 + g_0^2 \mu_\pm), \quad (2.3)$$

where $M_W^{(0)}$ denotes the bare mass of the charged $W^\pm$. The radiative effects are contained in $\mu_\pm$ as well as in the function $\delta_\pm(q^2)$, the splitting of the radiative corrections being introduced in such a manner that

$$\lim_{q^2 \rightarrow 0} q^2 \delta_\pm(q^2) = 0, \quad (2.4)$$

while $\mu_\pm$ is independent of q^2 , i.e., a constant. The gauge-invariant explicit expressions for $\delta_\pm(q^2)$ and $\mu_\pm$ in terms of vacuum-polarization, vertex and box diagram contributions will be given in Section 3, as they are of no particular relevance within the present context. Moreover, the propagator (2.3) has a form to be expected anyway, even without explicit calculation.

It is now a simple matter to represent the amplitude (2.2) in terms of a "running" coupling constant, $g_W(q^2)$, and a running mass, $M_W(q^2)$, via factorization,

$$M'_{CC} = \frac{g_W^2(q^2)}{2} \cdot j_\pm^\mu \frac{1}{q^2 - M_W^2(q^2)} j_{\pm\mu}. \quad (2.5)$$

Here, according to (2.2), (2.3), we have

$$g_W^2(q^2) = \frac{g_0^2}{1 + g_0^2 \delta_\pm(q^2)}, \quad (2.6)$$

and

$$M_W^2(q^2) = \frac{M_W^{(0)2}(1 + g_0^2 \mu_\pm)}{1 + g_0^2 \delta_\pm(q^2)} = M_W^{(0)2}(1 + g_0^2 \mu_\pm) \frac{g_W^2(q^2)}{g_0^2}, \quad (2.7)$$

i.e., $g_W^2(q^2)$ and $M_W^2(q^2)$ have identical q^2 dependence, as a consequence of the representation for the dressed $W^\pm$ propagator given by (2.3).

So far, the running parameters (2.6), (2.7) have been expressed in terms of bare quantities, g_0 , $M_W^{(0)}$, and $\mu_\pm(q^2)$ and $\delta_\pm(q^2)$ contain (regularized) logarithmic infinities. By eliminating the bare quantities in favour of physical ones, i.e., representing the running quantities directly in terms of their values at a chosen energy scale, q_0^2 , we now obtain $g_W^2(q^2)$ and $M_W^2(q^2)$ in a manifestly finite form, the logarithmic infinities in $\delta_\pm(q^2)$ being cancelled,

$$g_W^2(q^2) = \frac{g_W^2(q_0^2)}{1 - g_W^2(q_0^2)(\Delta_g(q^2) - \Delta_g(q_0^2))}, \quad (2.8)$$

and

$$M_W^2(q^2) = M_W^2(q_0^2) \frac{g_W^2(q^2)}{g_W^2(q_0^2)}, \quad (2.9)$$

where the notation

$$\Delta_g(q^2) = -\delta_\pm(q^2), \quad (2.10)$$

has been introduced.

Using the mass of the $W^\pm$ -boson, M_W , determined and defined as the pole of the propagator in (2.5)

$$M_W^2 - \text{Re} M_W^2(M_W^2) = 0, \quad (2.11)$$

as a convenient and natural scale, $q_0^2 \equiv M_W^2$, the running mass in (2.9) is written as

$$M_W^2(q^2) = (M_W^2 + iM_W \Gamma_W) \frac{g_W^2(q^2)}{g_W^2(M_W^2)}, \quad (2.12)$$

where Γ_W denotes the total width of the $W^\pm$. From (2.12), we immediately conclude that the Fermi coupling, defined via the $W^\pm$ propagator in the $q^2 \rightarrow 0$ limit, directly determines the on-mass-shell ratio of the SU(2) gauge coupling and the $W^\pm$ mass, even when (one-loop-resummed) radiative corrections are taken into account¹⁾,

$$\frac{8G_F}{\sqrt{2}} \equiv \frac{g_W^2(0)}{M_W^2(0)} = \frac{g_W^2(M_W^2)}{M_W^2}. \quad (2.13)$$

It is worth noting that the product of $g_W^2(q^2)$ and the propagator in (2.5) may identically be rewritten in terms of a single running parameter, the coupling, $g_W(q^2)$, and a propagator expressed in terms of the empirical mass, $M_W^2 \equiv \text{Re} M_W^2(M_W^2)$, i.e.,

$$M'_{CC} = \frac{g_W^2(q^2)}{2} j_\pm^\mu \frac{1}{q^2 - M_W^2 - i\text{Im} M_W^2(q^2)} j_{\pm\mu}, \quad (2.14)$$

* Note that the "universal Fermi coupling" G_F defined by (2.13) differs from the conventional Fermi coupling deduced from measurements of $\mu\pi$ decay, G_μ , as G_μ contains process-dependent box contributions, ϵ_{box} , as well as an additional vertex contribution, ϵ_{vertex} , which is related to the infrared behaviour of the μ -decay amplitude, i.e., $G_\mu = G_F(1 + \epsilon_{\text{box}} + \epsilon_{\text{vertex}})$.

where

$$\hat{g}_W^2(q^2) = \frac{g_W^2(q^2)}{1 - \frac{\text{Re} M_W^2(q^2) - M_W^2}{q^2 - M_W^2}}. \quad (2.15)$$

For $q^2 \rightarrow M_W^2$, we obtain

$$\hat{g}_W^2(M_W^2) = \frac{g_W^2(M^2)}{1 - \frac{d\text{Re} M_W^2(q^2)}{dq^2} \Big|_{q^2=M_W^2}}, \quad (2.16)$$

the function $\hat{g}(q^2)$ thus being characterized by the property of coinciding with the residue of the physical pole of the $W^\pm$ propagator.

As to the asymptotic behaviour, the detailed discussion in Section 3 will show that $g_W(q^2)$ depends on q^2 logarithmically for $q^2 \rightarrow \infty$, as required by renormalization-group considerations. According to (2.15), the same is true for $\hat{g}_W(q^2)$.

2.2 Neutral-Current Interactions

The derivation of the genuine renormalized fermion-fermion-scattering amplitude in the neutral-current sector proceeds in analogy to the treatment of the charged-current sector. The formalism is more involved, however, due to the fact that the vacuum polarization induces off-diagonal $Z^0 - \gamma$ (besides $Z^0 - Z^0$ and $\gamma\gamma$) transitions, implying a description in terms of a two-by-two propagator matrix. Again referring to Section 3 for technical details, we immediately start with the expression for the genuine neutral-current amplitude, M'_{NC} , written in a form analogous to (2.2). Most conveniently, with respect to matters of electromagnetic gauge invariance, M'_{NC} is expressed in the j_3, j_Q base, where j_3 denotes the third component of the weak isospin current, and j_Q denotes the electromagnetic current. With the unrenormalized electromagnetic coupling being denoted by e_0 , the genuine neutral-current amplitude is given by

$$M'_{NC} = \begin{pmatrix} g_0 j_3 \\ e_0 j_Q \end{pmatrix}^t \Delta(q^2) \begin{pmatrix} g_0 j_3 \\ e_0 j_Q \end{pmatrix}, \quad (2.17)$$

where $\Delta(q^2)$ represents a two-by-two propagator matrix which describes the propagation of the third component of the $(W^\pm, W^3)$ triplet and the photon, as evidenced by the base used in (2.17). The inverse propagator, $\Delta^{-1}(q^2)$ reads

$$\Delta^{-1}(q^2) = q^2 \begin{pmatrix} 1 + g_0^2 \delta_{33}(q^2) & s_0(1 + g_0^2 \delta_{3Q}(q^2)) \\ s_0(1 + g_0^2 \delta_{Q3}(q^2)) & 1 + e_0^2 \delta_{QQ}(q^2) \end{pmatrix} - M_W^{(0)2} \begin{pmatrix} 1 + g_0^2 \mu_{33} & 0 \\ 0 & 0 \end{pmatrix}. \quad (2.18)$$

It has several remarkable features which are worth noting.

First of all, let us neglect all radiative-correction effects by simply putting $g_0^2 = 0$ in the expression for the inverse propagator, (2.18). In this limit we obtain the standard tree-level propagator matrix (before diagonalization), expressed in terms of off-diagonal γW^3 current mixing terms of strength

$$s_0 \equiv \sin\theta_W = \frac{e_0}{g_0}, \quad (2.19)$$

rather than in the conventional form in terms of mass mixing. * Turning to $g_0^2 \neq 0$, we note that in analogy to (2.4), the radiative-correction functions $\delta_{ab}(q^2)$ fulfil

$$\lim_{q^2 \rightarrow 0} q^2 \delta_{ab}(q^2) = 0, \quad (a, b = 3, Q) \quad (2.20)$$

while μ_{33} is independent of q^2 . The functions $\delta_{ab}(q^2)$ are gauge invariant and contain all universal (one-loop) radiative corrections, i.e., the vacuum polarization due to bosons and fermions in Fig. 1a, as well as the gauge-dependent universal vertex and box contributions contained in the diagrams shown in Figs. 1b,c. The explicit expressions for $\delta_{ab}(q^2)$ will be given in Section 3, where their gauge invariance will be established by explicitly keeping track of the gauge parameter during the calculation. The vanishing of three elements in the mass matrix in (2.18) is a consequence of electromagnetic gauge invariance. In fact, non-zero elements in this mass matrix, induced by radiative corrections, would imply, among other things, a parity violating coupling of the (real) photon, as well as a non-vanishing neutrino charge induced by radiative corrections (compare the discussion given in Appendix A). Electromagnetic gauge invariance of the spontaneously broken $SU(2)_L \times U(1)_Y$ theory forbids such effects to arise at any order of perturbation theory, thus necessitating the three matrix elements in the mass matrix (2.18) to be zero. We will come back to this point in connection with the derivation of the explicit expressions for $\delta_{ab}(q^2)$ and μ_{33} in Section 3.

In order to represent the genuine neutral-current amplitude (2.17) in terms of a photon-exchange and a Z^0 -exchange amplitude, the propagator matrix (2.18) is now cast into diagonal form by applying the appropriately chosen matrix D introduced in Appendix A,

$$\Delta_{diag}^{-1}(q^2) \equiv D \Delta^{-1}(q^2) D^t = q^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - M_Z^2(q^2) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (2.21)$$

Upon identifying the expression for the propagator (2.18) with the one in Appendix A, given by (A1), one finds from (A18) that the transformation D in the present case takes the form

$$D(q^2) = \begin{pmatrix} \frac{g_Z(q^2)}{g_0}, & -\frac{g_Z(q^2)}{e_0} s^2(q^2) \\ 0, & \frac{e(q^2)}{e_0} \end{pmatrix}. \quad (2.22)$$

* This is a consequence of the choice of the j^3, j_Q base. Parenthetically, we remark that (2.18) for $g_0^2 = 0$ precisely coincides with the form of the propagator one immediately obtains from the underlying Lagrangian, if the electroweak theory is based on γW^3 mixing⁵⁾ and W -dominance^{6,7)}, where the mixing parameter is given by $\lambda \equiv s_0$.

The functions $e(q^2)$, $g_Z(q^2)$ and $\bar{s}^2(q^2)$ are given by

$$\begin{aligned}
 e(q^2) &= e_0 \left[1 - \frac{e_0^2}{2} \delta_{QQ}(q^2) \right] \\
 &\equiv e_0 \left[1 + \frac{e_0^2}{2} \Delta_e(q^2) \right], \\
 g_Z(q^2) &= \frac{g_0}{c_0} \left[1 + \frac{g_0^2}{2} \{-\delta_{33}(q^2) + \frac{s_0^2}{c_0^2} (2\delta_{Q3}(q^2) - \delta_{33}(q^2) - s_0^2 \delta_{QQ}(q^2))\} \right] \\
 &\equiv \frac{g_0}{c_0} \left[1 + \frac{g_0^2}{2c_0^2} \Delta_Z(q^2) \right], \\
 \bar{s}^2(q^2) &= s_0^2 \left[1 + g_0^2 \delta_{3Q}(q^2) - e_0^2 \delta_{QQ}(q^2) \right] \\
 &\equiv s_0^2 \left[1 + g_0^2 \Delta_s(q^2) \right].
 \end{aligned} \tag{2.23}$$

As already indicated by the notation used in (2.22), (2.23), the functions introduced in (2.22), (2.23) will play the role of running coupling constants and a running mixing angle, respectively, in the neutral-current genuine amplitude. Indeed, in terms of the diagonalized propagator (2.21) the neutral-current amplitude, M'_{NC} , becomes

$$\begin{aligned}
 M'_{NC} &= \left(\frac{g_0 j_3}{e_0 j_Q} \right)_\mu^t D^t \Delta_{4\mu g} D \left(\frac{g_0 j_3}{e_0 j_Q} \right)_\mu \\
 &= e^2(q^2) j_Q^\mu j_Q^\mu + g_Z^2(q^2) (j_3 - \bar{s}^2(q^2) j_Q)^\mu \frac{1}{q^2 - M_Z^2(q^2)} (j_3 - \bar{s}^2(q^2) j_Q)_\mu.
 \end{aligned} \tag{2.24}$$

The expression for the “running” mass, $M_Z(q^2)$, is found by making use of (A.19). As in the charged-current sector, the q^2 -dependence of $M_Z^2(q^2)$ is determined by the corresponding coupling,

$$M_Z^2(q^2) = M_Z^{(0)2} (1 + g_0^2 \mu_{33}) \frac{c_0^2}{g_0^2} g_Z^2(q^2). \tag{2.25}$$

The expression (2.24) for the renormalized neutral-current amplitude, M'_{NC} , has the form of a Born-term amplitude, even though it incorporates all universal radiative corrections in terms of three q^2 -dependent couplings and a q^2 -dependent mass.

Writing the running coupling constants in (2.23) in a manifestly finite form by expressing them in terms of their values at a chosen scale, q_0^2 , thereby eliminating the logarithmic divergences in Δ_e , etc., we have

$$e^2(q^2) = \frac{e^2(q_0^2)}{1 - e^2(q_0^2) (\Delta_e(q^2) - \Delta_e(q_0^2))}, \tag{2.26}$$

$$g_Z^2(q^2) = \frac{g_Z^2(q_0^2)}{1 - g_Z^2(q_0^2) (\Delta_Z(q^2) - \Delta_Z(q_0^2))},$$

$$\bar{s}^2(q^2) = \frac{\bar{s}^2(q_0^2)}{1 - g_W^2(q_0^2) (\Delta_s(q^2) - \Delta_s(q_0^2))}. \tag{2.27}$$

as well as

Moreover, according to (2.25),

$$M_Z^2(q^2) = M_Z^2(q_0^2) \frac{g_Z^2(q^2)}{g_Z^2(q_0^2)} = (M_Z^2 + i M_Z \Gamma_Z) \frac{g_Z^2(q^2)}{g_Z^2(q_0^2)}, \tag{2.28}$$

where in the last step we have specialized to the scale $q_0^2 \equiv M_Z^2$, where M_Z is the measured Z^0 mass, which is identical to the pole of the (real part of the) Z^0 propagator in (2.24), and Γ_Z is the total width of the Z^0 . In analogy to (2.13), we conclude from (2.28) that a measurement of the neutral-current strength for $q^2 \rightarrow 0$ determines the ratio of the on-mass-shell coupling constant $g_Z(M_Z^2)$ and the mass of the Z^0 , M_Z , even when (universal) radiative corrections are taken into account.

Again, as in the charged-current case, the neutral-current genuine amplitude (2.24) may be represented in a form in which $M_Z^2(q^2)$ in the denominator is replaced by the physical mass, $M_Z^2 = \text{Re} M_Z^2(M_Z^2)$,

$$\begin{aligned}
 M'_{NC} &= e^2(q^2) j_Q^\mu j_Q^\mu \frac{1}{q^2 - M_Z^2(q^2)} \\
 &\quad + \bar{g}_Z^2(q^2) (j_3 - \bar{s}^2(q^2) j_Q)^\mu \frac{1}{q^2 - M_Z^2(q^2) - i \text{Im} M_Z^2(q^2)} (j_3 - \bar{s}^2(q^2) j_Q)_\mu,
 \end{aligned} \tag{2.29}$$

where

$$\bar{g}_Z^2(q^2) = \frac{g_Z^2(q^2)}{1 - \frac{\text{Re} M_Z^2(q^2) - M_Z^2}{q^2 - M_Z^2}}. \tag{2.30}$$

For $q^2 \rightarrow M_Z^2$, we obtain

$$\bar{g}_Z^2(M_Z^2) = \frac{g_Z^2(M_Z^2)}{1 - \frac{\text{Re} M_Z^2(q^2)}{q^2}}, \tag{2.31}$$

i.e., $\bar{g}_Z^2(q^2)$ coincides with the residue of the pole of the Z^0 propagator for $q^2 \rightarrow M_Z^2$.

As will be discussed in Section 3, the asymptotic behaviour of the running couplings coincides with the one obtained by renormalization-group considerations.

2.3 Relations among the Running Parameters

The charged-current and the neutral-current universal parts of the four-fermion scattering amplitudes given by (2.5), (2.14) and (2.24), (2.29), respectively, have been written in terms of four universal running couplings and two (running) masses. As the $SU(2)_L \times U(1)_Y$ theory contains two arbitrary gauge couplings and a single mass parameter in the gauge-boson sector, only three of the six parameters are independent from each other. Indeed, making use of the defining equations (2.6), (2.7) and (2.23), (2.25), one finds* that

$$\frac{\bar{s}^2(q^2) g_W^2(q^2)}{e^2(q^2)} = 1 + g_W^2(q_0^2) (\delta_{Q3}(q^2) - \delta_\pm(q^2)), \tag{2.32a}$$

* Bare quantities have been replaced by physical ones on the right-hand side of the relations in (2.32, etc.), in consistency with the order of approximation we are working in. As the left-hand sides in (2.32, etc.) are renormalized quantities, the right-hand sides are finite, as may be explicitly checked by using the defining equations given in Section 3.

$$\begin{aligned}
\rho &\equiv \frac{g_Z^2(q^2)}{M_Z^2(q^2)} \left(\frac{g_W^2(q^2)}{M_W^2(q^2)} \right)^{-1} \\
&= 1 + g_W^2(q_0^2) (\mu_{\pm} - \mu_{33}) \\
&\equiv 1 + g_W^2(q_0^2) \delta\rho, \\
\bar{\rho}(q^2) &\equiv \frac{M_W^2(q^2)}{M_Z^2(q^2) c_W^2(q^2)} \\
&= 1 + g_W^2(q_0^2) \{ (\mu_{\pm} - \mu_{33}) + \delta_{33}(q^2) - \delta_{\pm}(q^2) \\
&\quad - \frac{s_W^2}{c_W^2} (\delta_{Q3}(q^2) - \delta_{33}(q^2)) \} \\
&\equiv 1 + g_W^2(q_0^2) \delta\bar{\rho}.
\end{aligned} \tag{2.32b}$$

$$\equiv 1 + g_W^2(q_0^2) \delta\bar{\rho}. \tag{2.32c}$$

These relations correspond to tree-level relations corrected by terms proportional to $g_W^2(q_0^2)$. With the above choice of the running parameters, the ρ -parameter * becomes independent of q^2 . An additional corrected Born-term relation between the neutral-current and charged-current couplings, given by

$$\begin{aligned}
\bar{\rho}(q^2) &\equiv g_Z^2(q^2) \left(\frac{g_W^2(q^2)}{c_W^2(q^2)} \right)^{-1} \\
&= 1 + g_W^2(q_0^2) \left(\delta_{\pm}(q^2) - \delta_{33}(q^2) - \frac{s_W^2(q_0^2)}{c_W^2(q_0^2)} (\delta_{33}(q^2) - \delta_{Q3}(q^2)) \right),
\end{aligned} \tag{2.33}$$

is not an independent one, as $\bar{\rho}$ is related to ρ and $\bar{\rho}$ via

$$\bar{\rho}(q^2) = \frac{\rho}{\bar{\rho}(q^2)}.$$

If one chooses to use the form (2.14), (2.29) of the charged-current and neutral-current amplitudes in terms of four running parameters and the fixed (physical) masses, M_W and M_Z , the relations (2.32) are replaced by equivalent ones, which now contain $\hat{g}_W^2(q^2)$ and

* The difference $\mu_{\pm} - \mu_{33}$ contains ^{1,2)} the well-known m_t^2 contribution ⁸⁾ for $m_t \geq M_W$.

$\hat{g}_Z^2(q^2)$ from (2.15), (2.30). They are given by

$$\begin{aligned}
\hat{\rho}(q^2) &\equiv \frac{\hat{g}_Z^2(q^2)}{M_Z^2} \left(\frac{\hat{g}_W^2(q^2)}{M_W^2} \right)^{-1} \\
&= 1 + g_W^2(q^2) \{ \mu_{\pm} - \mu_{33} + \frac{q^2}{q^2 - M_Z^2} (\Delta_Z(q^2) - \Delta_Z(M_Z^2)) \\
&\quad + \frac{q^2}{q^2 - M_W^2} (\delta_{\pm}(q^2) - \delta_{\pm}(M_W^2)) \} \\
&\equiv 1 + g_W^2 \delta\hat{\rho}, \\
\hat{\bar{\rho}}(q^2) &\equiv \frac{M_W^2}{M_Z^2 c_W^2(q^2)} \\
&= 1 + g_W^2 \{ \mu_{\pm} - \mu_{33} + \delta_{33}(M_Z^2) - \delta_{\pm}(M_W^2) \\
&\quad - \frac{s_W^2}{c_W^2} (2\delta_{3Q}(M_Z^2) - \delta_{33}(M_Z^2) - s_W^2 \delta_{QQ}(M_Z^2)) \\
&\quad + \frac{s_W^2}{c_W^2} (\delta_{3Q}(q^2) - s_W^2 \delta_{QQ}(q^2)) \} \\
&\equiv 1 + g_W^2 \delta\hat{\bar{\rho}}.
\end{aligned} \tag{2.34}$$

2.4 Remarks on Different Choices of the Running Parameters

As already noted in Sections 2.2 and 2.3, the parameters $(g_W(q^2), M_W(q^2))$ and $(g_Z(q^2), M_Z(q^2))$ may equivalently be replaced by the set $(\hat{g}_W(q^2), \hat{M}_W)$ and $(\hat{g}_Z(q^2), \hat{M}_Z)$. There is no such different choice for $\epsilon(q^2)$ and $s^2(q^2)$. While the above choices of different running parameters are completely equivalent, one may obviously also introduce other different sets of running parameters which yield charged-current and neutral-current amplitudes which, apart from higher-order terms only, are equivalent to the above-mentioned ones. This is the case for the running ("starred") couplings used by Kennedy and Lynn ³⁾.

From the form of (2.24), as well as from the defining equations (compare (2.23), (3.22)), one immediately finds that

$$e^2(q^2) \equiv e^{*2}(q^2), \tag{2.35}$$

as well as

$$\bar{s}^2(q^2) \equiv s^{*2}(q^2), \tag{2.36}$$

where the stars refer to the quantities introduced by Kennedy and Lynn. Our functions $g_W(q^2)$ and $g_Z(q^2)$, however, and our running masses are different from the corresponding starred quantities, as evidenced by the fact that

$$\frac{s^{*2}(q^2) g_W^{*2}(q^2)}{e^{*2}(q^2)} = 1, \tag{2.37}$$

and

$$g_Z^2(q^2) \left(\frac{g_W^2(q^2)}{c^2(q^2)} \right)^{-1} = 1, \quad (2.38)$$

while the corresponding formulae of ours, (2.32), (2.33), contain correction terms on the right-hand sides. Comparing (3.37), (3.38) with (2.32), (2.33) one finds

$$g_W^2(q^2) = g_W^2(q^2) (1 + g_W^2(q^2) (\delta_\pm(q^2) - \delta_{Q3}(q^2))), \quad (2.39a)$$

$$g_Z^2(q^2) = \frac{g_W^2(q^2)}{c^2(q^2)} = g_Z^2(q^2) \left(1 + \frac{g_W^2(q^2)}{c^2(q^2)} (\delta_{33}(q^2) - \delta_{3Q}(q^2)) \right). \quad (2.39b)$$

Noting that

$$\frac{g_W^2(q^2)}{q^2 - M_W^2(q^2)} = \frac{g_W^2(q^2)}{q^2 - M_W^{*2}(q^2)} + 0(g_W), \quad (2.40)$$

where

$$M_W^{*2}(q^2) = M_W^2(q^2) + g_W^2(q^2) (q^2 - M_W^2(q^2)) (\delta_\pm(q^2) - \delta_{Q3}(q^2)),$$

one concludes that the charged-current amplitudes are identical in both formulations, apart from higher-order terms which are to be neglected consistently. The same conclusion is reached for the neutral-current amplitudes.

One may be disturbed by the fact that the running mass, $M^*(q^2)$, in (3.41) varies linearly with q^2 . Even though one might prefer to work with (asymptotically) logarithmically running quantities, it should be noted that both choices of parameters are equivalent, insofar as the corresponding amplitudes are concerned. Substituting the expressions for $g_W(q^2)$ and $M_W(q^2)$ into (2.40), one finds that the $q^2 \rightarrow \infty$ behaviour of the right-hand side in (2.40) coincides with that of the left-hand side which is given by $g_W^2(q^2)/q^2$ for $q^2 \rightarrow \infty$, where $g_W^2(q^2)$ varies logarithmically with q^2 .

Even though different choices of the running parameters thus largely appear as a matter of convention, it may be worthwhile to add a further comment on the relative merits of different choices. In our formulation in terms of $g_W(q^2)$, etc., the masses behave as the couplings as functions of q^2 , but the Born-term relations among the various parameters receive corrections, while the Born-term relations are enforced to hold for the starred quantities. In this connection we note that in the dominant-fermion-loop approximation 1.2), one finds a restoration of the Born-term relations for our running parameters in the (unrealistic) limit of $m_t \rightarrow 0$ (where m_t denotes the mass of the top quark), i.e., the violation of global SU(2) weak isospin symmetry by $m_t \geq M_W$ is directly displayed within the formalism as a violation of simple Born-term relations. This feature of our running parameters is intuitively quite satisfactory, as important symmetry properties are transparently displayed within the formalism.

3. Derivation of the Gauge-Invariant Dressed Vector-Boson Propagators

This Section will mainly be concerned with the derivation of the gauge-invariant dressed vector-boson propagators quoted in Section 2 (see (2.3) and (2.18)). We will start with the treatment of the neutral-current case in 3.1, turning to charged currents in 3.2. In 3.3

we will discuss the asymptotic behaviour of the running couplings, and a few remarks on the process-dependent residual amplitude $\bar{M}$ (compare the separation (2.1)) will follow in 3.4.

3.1. The γW^3 Propagator Matrix

Vacuum polarization (compare Fig. 1a) induces $Z^0 Z^0$ as well as $\gamma\gamma$ and off-diagonal $Z^0\gamma$ transitions in the propagators of the neutral vector bosons. Accordingly, we have a two-by-two propagator matrix which upon resummation of the lowest order vacuum-polarization diagram in Fig. 1a is given by

$$\Delta(q^2) = \begin{pmatrix} q^2 - \Pi_{ZZ} - M_Z^{(0)2} & -\Pi_{ZA} \\ -\Pi_{ZA} & q^2 - \Pi_{AA} \end{pmatrix}^{-1}. \quad (3.1)$$

It is to be used in conjunction with the doublet (j_Z, j_Q) containing the Z^0 as well as the electromagnetic current. For later reference we note the well-known transformation into the (j_3, j_Y) base, j_3 and j_Y being the third component of the weak-isospin and the hypercharge current, respectively. The transformation is given by

$$\hat{\Delta}(q^2) = R \Delta(q^2) R^{-1}, \quad (3.2)$$

where

$$\begin{pmatrix} j_3 & j_Y \\ e_0 & j_Q \end{pmatrix} = \begin{pmatrix} c_0 & -s_0 \\ s_0 & c_0 \end{pmatrix} \begin{pmatrix} g_0 j_3 \\ g_0 j_Y \end{pmatrix} \equiv R^t \begin{pmatrix} g_0 j_3 \\ g_0 j_Y \end{pmatrix}. \quad (3.3)$$

In standard notation, in (3.1), $M_Z^{(0)}$ denotes the unrenormalized Z^0 mass, while vacuum polarization due to fermions and bosons is described by the functions Π_{ZZ} , etc., defined via the vacuum-polarization tensor

$$\Pi_{ij}^{\mu\nu} = g^{\mu\nu} \Pi_{ij}(q^2, \xi) + (q^\mu q^\nu - \text{part}), \quad (3.4)$$

where the dependence on the gauge parameter, ξ , due to vector-boson loops has been noted explicitly. The $q^\mu q^\nu$ part in (3.4) will be unimportant for our subsequent discussion in which the masses of the external leptons and quarks coupled to the propagators will be neglected. The indices i, j in (3.4) refer to the photon, A , and the Z^0 boson, Z , and the corresponding functions are written as

$$\begin{aligned} \Pi_{AA} &= g_0^2 s_0^2 \Pi_{QQ} \equiv g_0^2 s_0^2 \tilde{\Pi}_{QQ}, \\ \Pi_{ZA} &= \frac{s_0}{c_0} g_0^2 (\Pi_{3Q} - s_0^2 \Pi_{QQ}), \\ \Pi_{ZZ} &= \frac{g_0^2}{c_0^2} (\Pi_{33} - 2s_0^2 \Pi_{3Q} + s_0^4 \Pi_{QQ}), \end{aligned} \quad (3.5)$$

where the indices $3, Q$ refer to the third component of the weak isospin current and the electromagnetic current, respectively, while g_0 denotes the unrenormalized SU(2) coupling, and,

$$s_0^2 \equiv \sin^2 \theta_W = 1 - c_0^2 = \frac{g_0^2}{g_0^2 + g_0'^2}, \quad (3.6)$$

with the $U(1)_Y$ coupling g'_0 . The (unrenormalized) electromagnetic coupling is then given by

$$e_0 = g_0 s_0. \quad (3.7)$$

As announced in the introduction and in Section 2.2, the vacuum-polarization in (3.1) will now be combined with gauge-dependent universal vertex and box corrections,* in order to form the gauge-invariant universal neutral-current amplitude, M'_{NC} , introduced in (2.1). We will show that this amplitude has the Born-term form (2.17) and contains the gauge-invariant genuine propagator (2.18).

The specification of the universal (and gauge-dependent) parts of the vertex and box diagrams is most conveniently carried out with respect to the (j_3, j_Y) base. As regards vertex corrections, with respect to this base, only the W_3 -fermion-fermion vertex receives a correction, $\Gamma'_3(q^2, \xi)$. Turning to box diagrams, one finds that there is an important universal gauge-dependent contribution to the W^+W^- box diagram. It may be written as a current-current amplitude multiplied by a function of q^2 to be called $B'_{NC}(q^2, \xi)$. The (universal and gauge-dependent) vertex correction, Γ'_3 , and the box correction, B'_{NC} , must be incorporated into M'_{NC} , while the residual amplitude, $\bar{M}_{NC}$, in (2.1) will contain all non-universal and finite vertex corrections to the W^2 - and B -boson couplings to fermions, as well as the sum of the $\gamma\gamma$, γZ^0 and $Z^0 Z^0$ exchange (box) diagrams and the process dependent parts of the W^+W^- exchange (box) diagram, which are finite and gauge invariant by themselves.

By construction, M'_{NC} is the dominant part of the full neutral-current amplitude, as it contains the tree-level amplitude corrected by the dominant fermion vacuum polarization as well as the bosonic vacuum polarization which increases logarithmically with the Higgs-boson-mass, m_H , for large values of m_H .

According to the above specification, M'_{NC} is given by

$$M'_{NC} = \left(\frac{g_0 j_3}{g_0 j_Y} \right)^t \begin{pmatrix} 1 + g_0^2 \Gamma_3 & 0 \\ 0 & 1 \end{pmatrix} \hat{\Delta}(q^2) \begin{pmatrix} 1 + g_0^2 \Gamma_3 & 0 \\ 0 & 1 \end{pmatrix} \left(\frac{g_0 j_3}{g_0 j_Y} \right)^\mu + g_0^2 j_{3\mu} B'_{NC} j_3^\mu + 0(g_0^5, g_0^6), \quad (3.8)$$

where $\hat{\Delta}(q^2)$ denotes the propagator (3.1) rotated to the (j_3, j_Y) base according to (3.2). The form (2.17), (2.18) for M'_{NC} is obtained from (3.8) in several steps. Upon combining the vertex and box contributions in (3.8), we will pass to the (j_3, j_Q) base, and, upon making use of electromagnetic gauge invariance, we will arrive at (2.17), (2.18), and, at the same time, we will give the explicit expressions for the genuine vacuum-polarization corrections $\delta_{33}(q^2)$, $\delta_{Q3}(q^2)$, etc., appearing in (2.18).

First of all, one easily verifies by direct computation that (3.8) may be rewritten as

$$M'_{NC} = \left(\frac{g_0 j_3}{g_0 j_Y} \right)_\mu \Gamma \hat{\Delta}(q^2) \Gamma^t \left(\frac{g_0 j_3}{g_0 j_Y} \right)_\mu + o(g_0^5, g_0^6), \quad (3.9)$$

* Our vertex corrections include half of the external fermion self-energy corrections, the rest of which are absorbed in the wave function renormalizations.

where* the matrix Γ is given by

$$\Gamma = \begin{pmatrix} 1 + g_0^2 \left(\Gamma_3 + \frac{1}{2} (q^2 - M_W^{(0)2}) B'_{NC} \right) & \frac{g_0 g_0}{2} M_W^{(0)2} B'_{NC} \\ 0 & 1 \end{pmatrix} \quad (3.10)$$

Secondly, upon passing to the (j_3, j_Q) base via the non-orthogonal transformation

$$\begin{pmatrix} g_0 j_3 \\ g_0 j_Y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{s_0}{c_0} & \frac{1}{c_0} \end{pmatrix} \begin{pmatrix} g_0 j_3 \\ g_0 j_Q \end{pmatrix} \equiv R_1^t \begin{pmatrix} g_0 j_3 \\ g_0 j_Q \end{pmatrix}, \quad (3.11)$$

we obtain

$$M'_{NC} = \begin{pmatrix} g_0 j_3 \\ g_0 j_Q \end{pmatrix}^t \Delta_{tot}(q^2) \begin{pmatrix} g_0 j_3 \\ g_0 j_Q \end{pmatrix}, \quad (3.12)$$

where the inverse of the genuine propagator $\Delta_{tot}(q^2)$ is given by

$$\Delta_{tot}^{-1}(q^2) = \left((R_1 \Gamma R)^t \right)^{-1} \Delta(q^2)^{-1} (R_1 \Gamma R)^{-1}, \quad (3.13)$$

$\Delta(q^2)$ denoting the original propagator matrix (3.1). Explicitly, upon substituting Γ from (3.10), as well as R and R_1 from (3.3) and (3.11), respectively, one finds**

$$\Delta_{tot}^{-1}(q^2) = \begin{pmatrix} d_{33}(q^2) & d_{3Q}(q^2) \\ d_{Q3}(q^2) & d_{QQ}(q^2) \end{pmatrix}, \quad (3.14)$$

where

$$\begin{aligned} d_{33}(q^2) &= q^2 - M_W^{(0)2} - g_0^2 \left[\Pi_{33} + (q^2 - M_W^{(0)2}) (2\Gamma_3 + (q^2 - M_W^{(0)2}) B'_{NC}) \right], \\ d_{3Q}(q^2) &= d_{Q3}(q^2) = q^2 \cdot s_0 - e_0 g_0 \left[\Pi_{3Q} - M_W^{(0)2} \Gamma_3 + q^2 (2\Gamma_3' + (q^2 - M_W^{(0)2}) B'_{NC}) \right], \\ d_{QQ}(q^2) &= q^2 (1 - e_0^2 (\hat{\Pi}_{QQ} + 2\Gamma_3 + q^2 B'_{NC})). \end{aligned} \quad (3.15)$$

By construction, the dressed propagator matrix (3.14), (3.15) is a gauge-invariant quantity. The gauge dependences of the various contributions in (3.15) simply cancel each other. The gauge invariance of (3.15) is explicitly verified in Appendix B, where the expressions for the various contributions to $\Delta_{tot}(q^2)$ are given in an arbitrary ξ gauge⁹. We note that (3.15) evidently involves a resummation of the "genuine vacuum polarization" which is formed by combining the vacuum-polarization in Fig. 1a with the universal parts of vertices and boxes from Figs. 1b and 1c. A remark on the uniqueness of this procedure of constructing the genuine $W^3\gamma$ propagator and M'_{NC} will be found below in connection with the $q^2 \rightarrow 0$ as well as the $q^2 \rightarrow \infty$ behaviour of the propagator in (3.14), (3.15).

* When verifying the transition from (3.8) to (3.9), it is sufficient to use the Born-term form for the propagator $\hat{\Delta}(q^2)$. The result (3.9) is then derived in a straightforward manner.

** The matrix inversion in (3.13) is a simple matter in the order g^4 we are working in.

In a final step, we will now make use of electromagnetic gauge invariance to cast (3.14), (3.15) into the intuitively transparent form (2.18) used in chapter 2. Noting that the elements of the propagator matrix (3.14) are regular at $q^2 = 0$, i.e.,

$$d_{ab}(q^2) - d_{ab}(0) = \frac{d_{ab}}{dq^2} \Big|_{q^2=0} q^2 + \dots, \quad (a, b = 3, Q) \quad (3.16)$$

we identically rewrite the elements of the inverse propagator Δ_{tot}^{-1} in the form

$$d_{ab}(q^2) = q^2 \left(\frac{d_{ab}(q^2) - d_{ab}(0)}{q^2} \right) + d_{ab}(0), \quad (a, b = 3, Q) \quad (3.17)$$

which clearly separates the term which vanishes for $q^2 \rightarrow 0$ from the remainder given by $d_{ab}(0)$. The properties of a propagator of this form have been examined in detail in Appendix A, and the result of Appendix A is simply stated: if the propagator (3.14) is to yield a massless photon coupled to the electromagnetic current, j_Q , upon diagonalization, no photon-mass and no mass-mixing terms in (3.14) are allowed, i.e.,

$$d_{3Q}(0) = d_{Q3}(0) = d_{QQ}(0) = 0. \quad (3.18)$$

As preservation of electromagnetic gauge invariance upon spontaneous symmetry breaking (in conjunction with gauge-invariant regularization) assures the existence of a massless photon coupled to the (conserved) electromagnetic current, j_Q , the relations (3.18) have necessarily to hold for the γW^3 mixing propagator. Noting in addition that the universal vertex and box contributions do not contain a photon pole, i.e.,

$$q^2 \Gamma_3'(q^2) \Big|_{q^2 \rightarrow 0} = q^2 B'_{NC}(q^2) \Big|_{q^2 \rightarrow 0} = 0, \quad (3.19)$$

the vanishing of $d_{3Q}(0)$ in (3.15) yields

$$\Pi_{3Q}(q^2 = 0) - M_W^{(0)2} \Gamma_3'(q^2 = 0) = 0, \quad (3.20)$$

while $d_{QQ}(q^2 = 0)$ vanishes without further restrictions.

Relation (3.20) is noteworthy, as it connects, as a consequence of electromagnetic gauge invariance, the $q^2 \rightarrow 0$ limit of the γW^3 mixing strength introduced by boson loops with the universal part of the vertex correction to the third component of the weak-isospin current, $\Gamma_3'(q^2 = 0)$, compare Fig. 2. Relation (3.20), among other things, guarantees that loop corrections do not lead to a non-vanishing electric charge of the neutrino or a parity violating interaction of real photons.

It is gratifying that the relations (3.19), (3.20) may also be derived explicitly by direct calculation. Upon identifying Γ_3' with the expression for the universal part of the vertex correction to the weak isospin current, Γ_3^{PI} (where PI stands for “process-independent”) and B'_{NC} with the universal part of the $W^+ W^-$ box diagrams, both being given in Appendix B (compare (B.27), (B.31)) in an arbitrary ξ gauge, one finds by direct computation that (3.19), (3.20) are fulfilled for any value of the gauge parameter ξ (in the finite and the ultraviolet parts).

Inserting the explicit expressions (3.15) for $d_{ab}(q^2)$ into (3.17) and making use of the constraints (3.19), (3.20), the gauge-invariant (inverse) γW^3 propagator matrix (3.14) may be written in the following transparent form,

$$\Delta^{-1}(q^2) = q^2 \begin{pmatrix} 1 + g_0^2 \delta_{33}(q^2) & s_0(1 + g_0^2 \delta_{3Q}(q^2)) \\ s_0(1 + g_0^2 \delta_{Q3}(q^2)) & 1 + g_0^2 \delta_{QQ}(q^2) \end{pmatrix} - M_W^{(0)2} \begin{pmatrix} 1 + g_0^2 \mu_{33} & 0 \\ 0 & 0 \end{pmatrix}, \quad (3.21)$$

where

$$\delta_{33}(q^2) = -\frac{1}{q^2} \Pi_{33}(q^2) + (q^2 - M_W^{(0)2}) (2\Gamma_3'(q^2) + (q^2 - M_W^{(0)2}) B'_{NC}(q^2)) - M_W^{(0)2} \mu_{33}, \quad (3.22)$$

$$\delta_{3Q}(q^2) = \delta_{Q3}(q^2) = -\left[\frac{1}{q^2} (\Pi_{3Q} - M_W^{(0)2} \Gamma_3') + 2\Gamma_3' + (q^2 - M_W^{(0)2}) B'_{NC} \right],$$

$$\delta_{QQ}(q^2) = -\left[\Pi_{QQ}(q^2) + 2\Gamma_3'(q^2) + q^2 B'_{NC}(q^2) \right],$$

with the q^2 -independent term

$$M_W^{(0)2} \mu_{33} = \Pi_{33}(0) - 2M_W^{(0)2} \Gamma_3'(0) + M_W^{(0)4} B'_{NC}(0). \quad (3.23)$$

The form of (3.21) coincides with (2.18). In addition, (3.22), (3.23) show how vertex and box contributions are combined with the original vacuum-polarization contributions and resummed to form a gauge-invariant γW^3 propagator matrix.

A final remark concerns the uniqueness of the procedure of defining $\Gamma_3'(q^2, \xi)$ as well as $B'_{NC}(q^2, \xi)$. Obviously, Γ_3' and B'_{NC} can always be modified by adding finite ξ -independent contributions, to be called $\Delta\Gamma_3^{PI}$ and ΔB_{WW} , respectively, *

$$\begin{aligned} \Gamma_3'(q^2, \xi) &= \Gamma_3^{PI}(q^2, \xi) + \Delta\Gamma_3^{PI}(q^2), \\ B'_{NC}(q^2, \xi) &= B_{WW}(q^2, \xi) + \Delta B_{WW}(q^2), \end{aligned} \quad (3.24)$$

provided the constraints (3.19), (3.20) are fulfilled, i.e.,

$$\begin{aligned} \Delta\Gamma_3^{PI}(q^2 \rightarrow 0) &= 0, \\ q^2 \Delta B_{WW}(q^2 \rightarrow 0) &= 0. \end{aligned} \quad (3.25)$$

For definiteness, we note that Γ_3^{PI} and B_{WW} have been given by (B.27) and (B.31). The freedom (3.24), (3.25) will be essential in connection with the ensuing discussion on the asymptotic behaviour of M'_{NC} , i.e. the asymptotic behaviour of the running parameters which will be required to coincide with the renormalization-group predictions.

* Obviously, the change induced via (3.24) in M'_{NC} has to be compensated by corresponding change in the residual amplitude, $\tilde{M}_{NC}$.

3.2 The $W^\pm$ Propagator

The procedure to be employed to derive the dressed $W^\pm$ propagator is analogous to the one used in the neutral-current case, without the complication, however, of having to deal with a propagator in matrix form. The universal part of the charged-current amplitude is given by

$$M_{CC}^{\prime} = \frac{g_0^2}{2} (1 + g_0^2 \Gamma_\pm^4) j_\pm^\mu \frac{1}{q^2 - M_W^{(0)2} - \Pi_{WW}(q^2)} j_{\pm\mu} + \frac{g_0^4}{2} B_{CC}^{\prime} j_\pm^\mu j_{\pm\mu}, \quad (3.26)$$

where

$$\Pi_{WW}(q^2) = g_0^2 \Pi_\pm(q^2), \quad (3.27)$$

denotes the $g^{\mu\nu}$ part of the $W^\pm$ -boson vacuum polarization, while $\Gamma_\pm^4(q^2, \xi)$ and $B_{CC}^{\prime}(q^2, \xi)$ denote the universal parts of the $W^\pm$ -fermion-vertex correction and the box diagram contributions, respectively. To the order of g_0^4 we are working in, (3.26) may be rewritten in the form already referred to in (2.2), (2.3),

$$M_{CC}^{\prime} = \frac{g_0^2}{2} j_\pm^\mu \Delta_\pm(q^2) j_{\pm\mu}, \quad (3.28)$$

where

$$\Delta_\pm^{-1}(q^2) \equiv d_\pm(q^2) = q^2 (1 + g_0^2 \delta_\pm(q^2)) - M_W^{(0)2} (1 + g_0^2 \mu_\pm), \quad (3.29)$$

with $\delta_\pm(q^2)$ now being specified as

$$\delta_\pm(q^2) = -\frac{1}{q^2} [\Pi_\pm(q^2) + (q^2 - M_W^{(0)2}) (2\Gamma_\pm^4(q^2) + (q^2 - M_W^{(0)2}) B_{CC}^{\prime}(q^2)) - M_W^{(0)2} \mu_\pm], \quad (3.30)$$

and the constant term

$$M_W^{(0)2} \mu_\pm = \Pi_\pm(0) - 2M_W^{(0)2} \Gamma_\pm^4(0) + M_W^{(0)4} B_{CC}^{\prime}(0), \quad (3.31)$$

has been split off in order to assume the $q^2 \rightarrow 0$ limit, $q^2 \delta_\pm(q^2) \rightarrow 0$, given by (2.4). By construction, it is clear that the dressed propagator (3.29) is a gauge-invariant quantity. The gauge-invariance of (3.29) is nevertheless explicitly verified in Appendix B, where $\Gamma_\pm^4(q^2, \xi)$, $\Pi_{WW}(q^2, \xi)$ and $B_{CC}^{\prime}(q^2, \xi)$ are given explicitly.

Note that similar to the neutral-current case, $\Gamma_\pm^4$ and $B_{CC}^{\prime}$ can be modified by adding finite ξ -independent functions, to be called $\Delta\Gamma_\pm^{PI}(q^2)$ and $\Delta B_{CC}^{PI}(q^2)$, respectively,

$$\begin{aligned} \Gamma_\pm^4(q^2, \xi) &= \Gamma_\pm^{PI}(q^2, \xi) + \Delta\Gamma_\pm^{PI}(q^2), \\ B_{CC}^{\prime}(q^2, \xi) &= B_{CC}^{PI}(q^2, \xi) + \Delta B_{CC}^{PI}(q^2), \end{aligned} \quad (3.32)$$

provided the same quantities are subtracted from the residual amplitude $\tilde{M}_{CC}$.

3.3 The Asymptotic Behaviour of the Running Couplings

The splitting of the four-fermion interaction in (2.1) into two additive and separately gauge-invariant terms has essentially been defined by introducing the genuine gauge-invariant dressed propagators given in (2.18), (3.21), (3.22) and (2.3), (3.29), (3.30). They contain what we have called the ‘‘genuine vacuum polarization’’, i.e., all process-independent radiative corrections. Nevertheless, as pointed out already in (3.24), (3.25) in connection with the discussion of the $q^2 \rightarrow 0$ behaviour of the genuine neutral-current propagator, the required gauge invariance and process independence of $M_{NC}^{\prime}$ does not uniquely determine the splitting (2.1). Indeed, one can always modify the genuine propagators by adding a process-independent, ξ -independent, ultraviolet-finite function of q^2 to the vertex and box contributions in the propagators fulfilling the $q^2 \rightarrow 0$ constraints (3.25), provided, of course, one compensates these modifications of $M_{NC,CC}^{\prime}$ by equivalent ones in the residual amplitudes, $\tilde{M}_{NC,CC}$. In what follows, we will briefly address the question of how far the asymptotic behaviour of the running couplings is affected by and can be used to restrict the ambiguity in the splitting (2.1).

We start by considering the running electromagnetic coupling $e(q^2)$, where the q^2 -dependence is determined according to (2.23), (3.22) by

$$\Delta_e(q^2) = -\delta_{QQ}(q^2) = \tilde{\Pi}_{QQ}(q^2) + 2\Gamma_3^4(q^2) + q^2 B_{NC}^{\prime}(q^2). \quad (3.33)$$

Imposing the constraint that (3.33) be asymptotically ($q^2 \rightarrow \infty$) determined by ultraviolet divergences, as obtained by applying the renormalization group to the running of $e(q^2)$, one finds that (3.33) is to fulfil

$$Re\Delta_e(q^2) \sim \frac{1}{48\pi^2} \left(21 - \frac{32}{3} N \right) \ln \left(\frac{\Lambda^2}{|q^2|} \right), \quad (3.34)$$

where N denotes the number of generations of leptons and quarks, and Λ denotes the ultraviolet cut-off. Identifying $\Gamma_3^4(q^2)$ and $B_{NC}^{\prime}(q^2)$ in (3.33) with the process-independent vertex and box contributions of Appendix B (compare (B.27), (B.30)), one indeed finds the expected ultraviolet-divergent term,

$$\begin{aligned} Re \left(\tilde{\Pi}_{QQ} + 2\Gamma_3^{PI} + q^2 B_{WW} \right) &\sim \frac{1}{48\pi^2} \left(21 - \frac{32}{3} N \right) \ln \frac{\Lambda^2}{|q^2|} \\ &+ \frac{1}{16\pi^2} \left(3 \ln \frac{|q^2|}{M_W^{(0)2}} - 2 \ln^2 \frac{|q^2|}{M_W^{(0)2}} \right). \end{aligned} \quad (3.35)$$

In addition, however, (3.35) contains infrared (so-called Sudakov-) logs. Renormalization-group asymptotic behaviour may, however, be easily imposed on $\Delta_e(q^2)$ by identifying Γ_3^4 in (3.33) with

$$\Gamma_3^4(q^2, \xi) \equiv \Gamma_3^{PI}(q^2, \xi) + \Delta\Gamma_3^{PI}(q^2), \quad (3.36)$$

where the (ultraviolet-finite, ξ -independent) function $\Delta\Gamma_3^{PI}(q^2)$ is to fulfil

$$2Re\Delta\Gamma_3^{PI} \sim \frac{-1}{16\pi^2} \left(3 \ln \frac{|q^2|}{M_W^{(0)2}} - 2 \ln^2 \frac{|q^2|}{M_W^{(0)2}} \right), \quad (3.37)$$

as well as the $q^2 \rightarrow 0$ constraint (3.25),

$$\Delta\Gamma_3^{PI}(q^2 \rightarrow 0) = 0. \quad (3.38)$$

For completeness, we explicitly give an example for the choice of $\Delta\Gamma_3^{PI}$,

$$16\pi^2 \Delta\Gamma_3^{PI} = \frac{1}{2} \phi_W(q^2) - 3 \int_0^1 dx \ln \left(1 - x(1-x) \frac{q^2}{M_W^{(0)2}} - i\epsilon \right), \quad (3.39)$$

where $\phi_W(q^2)$ is the finite, ξ -independent function defined in (B.21), which for $q^2 \rightarrow \infty$ fulfils

$$Re\phi_W(q^2) \sim 3 \ln \frac{|q^2|}{M_W^{(0)2}} + 2 \ln^2 \frac{|q^2|}{M_W^{(0)2}}, \quad (3.40)$$

while for $q^2 \rightarrow 0$

$$\phi_W(q^2 \rightarrow 0) = 0, \quad (3.41)$$

as required by (3.37), (3.38).

Similar arguments are to be applied to $g_W(q^2)$, i.e., $\Delta_g(q^2)$ given by (2.8) (2.10) and (3.30),

$$\begin{aligned} \Delta_g(q^2) &= -\delta_\pm(q^2) \\ &= \frac{1}{q^2} [\Pi_\pm(q^2) + (q^2 - M_W^{(0)2}) (2\Gamma_\pm'(q^2) + (q^2 - M_W^{(0)2}) B'_{CC}(q^2)) \\ &\quad - M_W^{(0)2} \mu_\pm], \end{aligned} \quad (3.42)$$

which according to the renormalization group for $q^2 \rightarrow \infty$ should behave as

$$Re\Delta_g(q^2) \sim \frac{1}{48\pi^2} \left(\frac{43}{2} - 4N \right) \ln \left(\frac{\Lambda^2}{|q^2|} \right). \quad (3.43)$$

Identifying $\Gamma_\pm(q^2)$ and $B'_{CC}(q^2)$ in (3.42) with the process-independent vertex and box contributions, $\Gamma_\pm^{PI}(q^2)$ and $B'_{CC}{}^{PI}(q^2)$ of Appendix B (compare (B.27), (B.28)), one finds for $q^2 \rightarrow \infty$,

$$\begin{aligned} Re \frac{1}{q^2} [\Pi_\pm(q^2) + (q^2 - M_W^{(0)2}) (2\Gamma_\pm^{PI}(q^2) + (q^2 - M_W^{(0)2}) B'_{CC}{}^{PI}(q^2)) - M_W^{(0)2} \mu_\pm] \\ \sim \frac{1}{48\pi^2} \left(\frac{43}{2} - 4N \right) \ln \left(\frac{\Lambda^2}{|q^2|} \right) + \frac{3}{16\pi^2} \ln \left(\frac{|q^2|}{M_W^2} \right), \end{aligned} \quad (3.44)$$

which indeed coincides with (3.43) as far as the ultraviolet-divergent part is concerned. The additional logarithm is removed by identifying $\Gamma_\pm(q^2)$ in (3.42) with

$$\Gamma_\pm'(q^2, \xi) = \Gamma_\pm^{PI}(q^2, \xi) + \Delta\Gamma_\pm^{PI}(q^2), \quad (3.45)$$

where $\Delta\Gamma_\pm^{PI}(q^2)$ is constrained via

$$Re\Delta\Gamma_\pm^{PI}(q^2) \sim -\frac{3}{16\pi^2} \ln \left(\frac{|q^2|}{M_W^2} \right). \quad (3.46)$$

An additional disease of $\delta_\pm(q^2)$ in (3.42), which is to be cured at the same time, is due to the fact that $\delta_\pm(q^2)$ (in contrast to $\delta_{QQ}(q^2)$) contains infrared divergences originating from photon-exchange contributions to $\Gamma_\pm$. With a photon mass denoted by M_γ , one finds that in the limit of $q^2 \rightarrow 0$,

$$\begin{aligned} Re \frac{1}{q^2} [\Pi_\pm(q^2) + (q^2 - M_W^{(0)2}) (2\Gamma_\pm^{PI}(q^2) + (q^2 - M_W^{(0)2}) B'_{CC}{}^{PI}(q^2)) - M_W^{(0)2} \mu_\pm] \\ \sim -\frac{s_W^2}{16\pi^2} \left[\frac{2M_W^{(0)2}}{q^2} \ln \left(1 - \frac{q^2}{M_W^{(0)2}} \right) + \frac{3}{2} \right] \ln M_\gamma^2, \end{aligned} \quad (3.47)$$

The condition that the choice of $\Delta\Gamma_\pm^{PI}(q^2)$ is not only to assure the proper asymptotic behaviour of $\delta_\pm(q^2)$, but is to remove its infrared divergence as well, imposes an additional constraint on $\Delta\Gamma_\pm^{PI}(q^2)$ to be deduced from (3.47). Both, the $q^2 \rightarrow \infty$ condition (3.46), as well as the $q^2 \rightarrow 0$ constraint (3.47), are fulfilled by

$$\begin{aligned} \Delta\Gamma_\pm^{PI}(q^2) &= \frac{s_W^2}{16\pi^2} [\Lambda_W(q^2, \xi = 1) - \Lambda_\gamma(q^2, \xi = 1)] \\ &\quad - \frac{3}{16\pi^2} \int_0^1 dx \ln \left(1 - x(1-x) \frac{q^2}{M_W^{(0)2}} - i\epsilon \right), \end{aligned} \quad (3.48)$$

where the function $\Lambda_V(q^2, \xi)$ with $V = \gamma, W$ is defined in Appendix B (see (B.23)). The second term in (3.48) removes the Sudakov log, while the first one removes the infrared divergence from the genuine amplitudes. It may be worth noting that the choice of the first term in the expression for $\Delta\Gamma_\pm^{PI}$ in (3.48) simply amounts to replacing the photon mass, M_γ , by the $W^\pm$ mass, M_W , within the ξ -independent part of $\Gamma_\pm^{PI}(q^2, \xi)$. This replacement may, alternatively to adding the first term in (3.48), be carried out by inserting¹⁰ the identity

$$\frac{1}{k^2} = \frac{1}{k^2 - M_W^2} - \frac{M_W^2}{k^2} \frac{1}{k^2 - M_W^2}, \quad (3.49)$$

for the photon propagator in the ξ -independent part of $\Gamma_\pm^{PI}(q^2, \xi)$, and by subsequently associating that part of $\Gamma_\pm^{PI}(q^2, \xi)$, which belongs to the second term on the right-hand side in (3.49), with the residual amplitude, $\bar{M}_{CC}$.

We finally note that the proper renormalization-group required asymptotic behaviour of $e^2(q^2)$ and $g_W^2(q^2)$, obtained by appropriate choices of $\Delta\Gamma_3^{PI}$ and $\Delta\Gamma_\pm^{PI}$, also assures that as $q^2 \rightarrow \infty$,

$$\begin{aligned} \delta_{Q3}(q^2) - \delta_\pm(q^2) &\rightarrow 0, \\ \delta_{Q3}(q^2) - \delta_{33}(q^2) &\rightarrow 0. \end{aligned} \quad (3.50)$$

According to (2.39a), (2.39b) this means that

$$\frac{g_W^2(q^2)}{g^{*2}(q^2)} \rightarrow 1, \quad \frac{g_Z^2(q^2)}{g_Z^{*2}(q^2)} \rightarrow 1, \quad (3.51)$$

i.e., $g_W^2(q^2)$ and $g_Z^2(q^2)$ will have the same high-energy behaviour as $g^{*2}(q^2)$ and $g_Z^{*2}(q^2)$, respectively.

3.4 The Residual Amplitudes $\tilde{M}_{CC}$ and $\tilde{M}_{NC}$

For the sake of clarity it is useful to give the expression for the residual amplitudes appearing in (2.1), even though we will restrict ourselves to their (schematic) form, without working out details. The charged-current amplitude, evidently, contains the ultraviolet-finite process-dependent vertex correction, $\Gamma_{\pm}^{PD}(q^2, \xi)$ modified by the process-independent part, $\Delta\Gamma_{\pm}^{PI}(q^2)$, introduced in (3.32), as well as ξ -independent parts of $W\gamma$ and WZ box diagrams, which depend on additional kinematical variables apart from q^2 ,

$$\begin{aligned} \tilde{M}_{CC} = & g_W^4 (M_W^2) \left(\Gamma_{\pm}^{PD} - \Delta\Gamma_{\pm}^{PI} \right) j_{\pm}^{\mu} \frac{1}{q^2 - M_W^{(0)2}} j_{\pm\mu} \\ & + \{ \xi - \text{independent parts of } W\gamma, WZ \text{ boxes} \}. \end{aligned} \quad (3.52)$$

Here, in consistency with our treatment, we have replaced the bare coupling, g_0^4 , by the renormalized one, g_W^4 ($g_0^2 = M_W^2$), where for definiteness the scale $q_0^2 \equiv M_W^2$ is chosen.

Finally, we will give the schematic expression for the neutral current residual amplitude, $\tilde{M}_{NC}$,

$$\begin{aligned} \tilde{M}_{NC} = & \begin{pmatrix} g_0^0 & j_3 \\ g_0^0 & j_Y \end{pmatrix} \begin{pmatrix} g_0^2 (\Gamma_3^{PD} - \Delta\Gamma_3^{PI}) & 0 \\ 0 & g_0^2 \Gamma_B \end{pmatrix} R \Delta(q^2) R^{-1} \begin{pmatrix} g_0^0 & j_3 \\ g_0^0 & j_Y \end{pmatrix} \\ & + \begin{pmatrix} g_0^0 & j_3 \\ g_0^0 & j_Y \end{pmatrix} R \Delta(q^2) R^{-1} \begin{pmatrix} g_0^2 (\Gamma_3^{PD} - \Delta\Gamma_3^{PI}) & 0 \\ 0 & g_0^2 \Gamma_B \end{pmatrix} \begin{pmatrix} g_0^0 & j_3 \\ g_0^0 & j_Y \end{pmatrix} \\ & + \{ \xi - \text{independent part of } WW \text{ box} \} \\ & + \{ \gamma\gamma, \gamma Z, ZZ \text{ boxes} \}, \end{aligned} \quad (3.53)$$

where $\Delta(q^2)$ denotes the propagator matrix defined in (3.1) with the vacuum polarization contribution being omitted, and R denotes the transformation to the (j_3, j_Y) base given in (3.3). As in the charged current case, g_0 and g_0^0 are to be replaced by the renormalized expressions at, e.g., the scale $q^2 = M_Z^2$, and likewise $M_Z^{(0)}$ is to be replaced by M_Z .

Even though the bulk of the amplitude at the one-loop order is given by $\tilde{M}_{NC,CC}$, the relative magnitude of the residual amplitudes has to be carefully checked for each particular process and kinematic situation. It is, e.g., of importance in the analysis of the neutrino charge-radius contribution to neutrino scattering¹¹⁾. It is also important in order to obtain the proper expression for the radiative correction, Δr , to μ -decay, which will be discussed in the next section.

4. The Radiative Correction Δr

Suppose we use the electromagnetic coupling measured in the low-energy limit, $e(0)$, i.e., $\alpha(0) \equiv \frac{e(0)^2}{4\pi} \simeq \frac{1}{137}$, the Fermi constant measured in $\mu^\pm$ decay, G_F , and the Z^0 mass precisely measured in e^+e^- annihilation, M_Z , as input parameters. On the basis of the foregoing formulation of the one-loop corrected theory, it is now a simple matter to predict other observables, in particular the mixing angle at the Z^0 peak, $s^2(q^2 = M_Z^2)$, the Z^0 coupling to (light) fermions at the Z^0 peak, and the $W^\pm$ mass, M_W , essentially,

by combining the expression (2.13) for the Fermi coupling with the improved Born-term relations (2.32).

The Fermi coupling measured in $\mu^\pm$ decay, according to (2.13), is given by

$$G_F = G_F(1 + \epsilon_{\text{box}} + \epsilon_{\text{vertex}}). \quad (4.1)$$

Here, G_F denotes the "universal Fermion constant" at one-loop order, i.e., the $q^2 \rightarrow 0$ limit of the dressed $W^\pm$ propagator, or the ratio of the running $SU(2)$ coupling, $g_W^2(q^2 \rightarrow 0)$, and the running $W^\pm$ mass, $M_W^2(q^2 \rightarrow 0)$ in the $q^2 \rightarrow 0$ limit, while ϵ_{box} and ϵ_{vertex} are additional box and vertex corrections specific to μ decay. We first of all deal with G_F and subsequently discuss the expressions for ϵ_{box} and ϵ_{vertex} .

According to (2.9), G_F may directly be expressed as the ratio of the running $SU(2)$ coupling, $g_W^2(q^2)$, and the running mass, $M_W^2(q^2)$, both evaluated at the same, however arbitrary, scale q^2 . For the subsequent discussion we conveniently choose $q^2 = M_Z^2$ as the scale, thus expressing G_F as

$$4\sqrt{2}G_F = \frac{g_W^2(0)}{M_W^2(0)} = \frac{g_W^2(M_Z^2)}{M_W^2(M_Z^2)}. \quad (4.2)$$

Inserting for $g_W^2(M_Z^2)$ in (4.2) the corrected Born-term relation (2.32a) and replacing $e^2(M_Z^2)$ therein by $e^2(0)$ according to (2.26), and expressing $M_W^2(M_Z^2)$ in (4.2) in terms of $\bar{\rho}$ according to (2.32c), or alternatively, in terms of ρ according to (2.32b), we quickly obtain our final result for the one-loop-corrected universal Fermi constant, which is given by

$$4\sqrt{2}G_F = \frac{e^2(0)}{M_Z^2 e^2(M_Z^2) s^2(M_Z^2)} \cdot \frac{1}{1 - e^2(0) (\delta_{QQ}(0) - \delta_{QQ}(M_Z^2))} \cdot \begin{cases} \rho^{-1} \left[1 - \frac{e^2(0)}{s^2(M_Z^2) s^2(M_Z^2)} (\delta_{33}(M_Z^2) - \delta_{Q3}(M_Z^2)) \right], & (\text{using } \rho), \\ \bar{\rho} (M_Z^2)^{-1} \left[1 - \frac{e^2(0)}{s^2(M_Z^2)} (\delta_{\pm\pm}(M_Z^2) - \delta_{Q3}(M_Z^2)) \right], & (\text{using } \bar{\rho}). \end{cases} \quad (4.3)$$

The (q^2 -independent) ρ -parameter, i.e., the neutral-to-charged-current ratio measured for $q^2 \rightarrow 0$, according to (2.32b) is given by

$$\rho = 1 + \frac{e^2(0)}{s^2(M_Z^2)} \delta\rho = 1 + \frac{e^2(0)}{s^2(M_Z^2)} (\mu_{\pm} - \mu_{33}), \quad (4.4)$$

where the constants μ_{33} and $\mu_{\pm}$ are given in (3.23) and (3.31). As for $\bar{\rho}(q^2)$, we refer to (2.32c). Returning to (2.32b), we also note that the Z^0 coupling to fermions* is given by

$$\begin{aligned} g_Z^2(M_Z^2) &= 4\sqrt{2}G_F \rho M_Z^2 \\ &= \frac{e^2(M_Z^2)}{s^2(M_Z^2) s^2(M_Z^2)} \cdot \frac{1}{1 + \frac{e^2(M_Z^2)}{s^2(M_Z^2) s^2(M_Z^2)} (\delta_{33}(M_Z^2) - \delta_{Q3}(M_Z^2))}, \end{aligned} \quad (4.5)$$

* For the case of the b quark an additional (obviously process-dependent) vertex correction has to be taken into account.

where the first equality in (4.3) has only been used in the second step. Here, G_F has to be replaced by the measured μ -decay constant, G_μ , according to (4.1).

Alternatively to the mixing parameter at the Z^0 mass shell, $s^2(q^2 = M_Z^2)$, one may choose to represent G_F in terms of the neutral-to-charged-boson mass ratio¹⁰, i.e., in terms of

$$s_W^2 \equiv 1 - \frac{M_W^2}{M_Z^2}. \quad (4.6)$$

Combining the defining equation for the $\bar{\rho}$ parameter (2.32c) with the equation for the running of $M_W^2(q^2)$ given by (2.9), one finds for the connection between $s^2(M_Z^2)$ and s_W^2 , in terms of $\delta\bar{\rho}$ and $\delta\rho$,

$$\begin{aligned} s^2(M_Z^2) &= s_W^2 \left(1 + e^2(0) \frac{c_W^2}{s_W^4} \delta\bar{\rho}(M_Z^2) \right) \left(1 + e^2(0) \frac{c_W^2}{s_W^4} (\delta_\pm(M_Z^2) - \delta_\pm(M_W^2)) \right), \\ &= s_W^2 \left(1 + e^2(0) \frac{c_W^2}{s_W^4} \delta\rho \right) \\ &\quad \left(1 + \frac{e^2(0)}{s_W^4} (\delta_{33}(M_Z^2) - s_W^2 \delta_{Q3}(M_Z^2) - c_W^2 \delta_{\pm}(M_W^2)) \right), \end{aligned} \quad (4.7)$$

which upon substitution into (4.3) yields, again either using ρ or $\bar{\rho}$,

$$4\sqrt{2}G_F = \frac{e^2(0)}{M_Z^2 c_W^2 s_W^2} \frac{1}{1 - e^2(0) (\delta_{QQ}(0) - \delta_{QQ}(M_Z^2))} \left\{ \begin{aligned} &\left(1 + e^2(0) \frac{c_W^2}{s_W^4} \delta\rho \right)^{-1} \quad \text{(using } \rho), \\ &\left(1 + e^2(0) \frac{c_W^2}{s_W^4} \delta\bar{\rho}(M_Z^2) \right)^{-1} \quad \text{(using } \bar{\rho}), \end{aligned} \right\} \left(1 + \frac{e^2(0)}{s_W^4} (\delta_{Q3}(M_Z^2) - \delta_\pm(M_W^2) - \frac{c_W^2}{s_W^2} (\delta_\pm(M_Z^2) - \delta_\pm(M_W^2))) \right). \quad (4.8)$$

Evidently, (4.3) and (4.8) may now be used to numerically predict $\bar{s}_W^2(M_Z^2)$ and s_W^2 (i.e., the $W^\pm$ mass according to (4.6)), in terms of $e(0)$, G_μ and M_Z , once the additional corrections ϵ_{box} and ϵ_{vertex} have been taken into account.

Note that our representations (4.3) and (4.8) for G_F are physically particularly transparent, as the radiative corrections are directly represented in terms of the manifestly gauge-invariant quantities, the ρ -parameter (4.4) and $\delta_\pm(q^2)$, $\delta_{33}(q^2)$, $\delta_{QQ}(q^2)$ and $\delta_{Q3}(q^2)$, which have a clear physical significance in terms of the dressed vector-boson propagators (2.3), (3.30), (3.31) and (2.18), (3.22), (3.23).^{*} It is well known, of course, that the dominant corrections to the tree-level relation among G_μ , e , M_Z and s_W^2 are due to fermion loops, which enter the running of the electromagnetic coupling via $\delta_{QQ}(0) - \delta_{QQ}(M_Z^2)$ in (4.3), (4.8), and due to the $\rho(\bar{\rho})$ -parameter via the well known m_f^2 term fully contained in

^{*} While each one of these functions contains a (regularized) logarithmic infinity, their differences, appearing in (4.3) and (4.8), are of course finite

$\rho(\bar{\rho})$. An additional logarithmic m_f dependence is contained^{1,2} in $\bar{\rho}$ and in the additional last correction factor on the right-hand sides in (4.3) and (4.8). The dependence on the Higgs-boson mass, m_H , logarithmic for $m_H \gg M_W$, is contained in ρ and $\bar{\rho}$, and again in the last correction factor in (4.3) and (4.8). It is of interest to note that the Higgs-mass dependence in the last expression in (4.8), where $\bar{\rho}$ is used, is practically fully contained in $\delta\bar{\rho}(M_Z^2)$.^{*} In good approximation, upon neglecting part of the logarithmic m_f dependence, we thus have

$$4\sqrt{2}G_F \simeq \frac{e^2(0)}{M_Z^2 c_W^2 s_W^2} \frac{1}{1 - e^2(0) (\delta_{QQ}(0) - \delta_{QQ}(M_Z^2))} \cdot \frac{1}{1 + e^2(0) \frac{c_W^2}{s_W^4} \delta\bar{\rho}(M_Z^2)}, \quad (4.9)$$

i.e., the whole correction to the universal Fermi coupling may be visualized as being due to the running of $e(q^2)$ and the $\bar{\rho}$ parameter defined in (2.32c). In other words, if the mass ratio (4.6) is used, the whole radiative correction to the relation for G_F is (in very good approximation) determined by the running of $e^2(q^2)$ and the discrepancy between s_W^2 and $s^2(M_Z^2)$ which implies $\delta\bar{\rho}(M_Z^2) \neq 0$ and essentially determines its numerical value.

We turn to the additional corrections ϵ_{box} and ϵ_{vertex} introduced in (4.1). The contribution of the ultraviolet-finite, process-dependent and gauge-invariant ZW-exchange box diagrams to $\mu^\pm$ decay, called ϵ_{box} in (4.1), is very much simplified by the fact that the $W^\pm$ and Z^0 masses are much larger than any energy variable or mass scale involved in the corresponding amplitude. The analytic expression can be found, e.g., in ref.12, and thus ϵ_{box} is given by the simple formula,

$$\epsilon_{\text{box}} = -\frac{e^2(0)}{16\pi^2 s_W^2} \left[\frac{5}{2s_W^2} - 4 - c_W^2 \right] \ln c_W^2. \quad (4.10)$$

The correction ϵ_{vertex} is calculated as follows. Provided we use the expression (3.48) for $\Delta\Gamma_\pm^P(q^2)$, all the photonic vertex corrections to the μ -decay amplitude are put into the residual part, $\bar{\Gamma}_\pm(q^2)$, which in the case of μ decay (compare (B.26)) reads as

$$\bar{\Gamma}_\pm(q^2) = \frac{1}{16\pi^2} \left[\frac{1 - 2c_W^2}{4c_W^2} \phi_Z(q^2) + s_W^2 \Lambda_\gamma(q^2, \xi = 1) - s_W^2 \Lambda_W(q^2, \xi = 1) \right]. \quad (4.11)$$

When defining G_μ in terms of the μ -life time¹⁰, the infrared divergent part is removed from the definition of G_μ using the identity (3.49), and ϵ_{vertex} is obtained from that part of $\bar{\Gamma}_\pm(q^2)$, which is infrared finite at $q^2 = 0$ (the relevant q^2 -region for μ -decay), as well as from the infrared divergent diagram in $\bar{\Gamma}_\pm(q^2)$ with the replacement of photon mass, M_γ , by the $W^\pm$ mass, M_W . Noting that among the diagrams contributing to the function $\Lambda_\gamma(q^2, \xi)$, only the external-fermion-line-regularization diagram, (Fig.3a), i.e.,

^{*} The Higgs-boson contribution to the factor containing $\delta\bar{\rho}$, for $m_H \gg M_W$, in (4.8) is enhanced by a factor of about 10 relative to the Higgs-boson contribution in the last factor on the right-hand side.

the function S_γ in the notation of Appendix B (see (B.20)), is infrared divergent at $q^2 = 0$, one finds

$$\begin{aligned} \epsilon_{\text{vertex}} &= \frac{e^2(0)}{16\pi^2 s_W^2} 2 \left[\frac{1-2c_W^2}{4c_W^2} \phi_Z(0) + s_W^2 \Lambda_\gamma(0, \xi=1) - s_W^2 \Lambda_W(0, \xi=1) \right. \\ &\quad \left. - \frac{s_W^2}{2} S_\gamma(0, \xi=1) + \frac{s_W^2}{2} S_W(0, \xi=1) \right] \\ &= \frac{e^2(0)}{16\pi^2} 2 \left[\Lambda_\gamma(q^2=0, \xi=1) - \Lambda_W(q^2=0, \xi=1) + \frac{1}{2} \ln \frac{M_W^2}{M_Z^2} \right], \end{aligned} \quad (4.12)$$

where the factor 2 is to take into account the corrections at the μ -vertex and the e -vertex in μ -decay, and the identity $\phi_Z(0) = 0$ (compare (B.32c)) is used in passing to the second line. Using (B.23) and (B.32b), one obtains

$$\epsilon_{\text{vertex}} = \frac{6e^2(0)}{16\pi^2}, \quad (4.13)$$

thus yielding the correction

$$\epsilon_{\text{box}} + \epsilon_{\text{vertex}} = \frac{e^2(0)}{16\pi^2} \left(6 - \frac{1}{s_W^2} \left(\frac{5}{2s_W^2} - 4 - c_W^2 \right) \ln c_W^2 \right). \quad (4.14)$$

Introducing the standard notation Δr for the radiative correction in (4.8),

$$\begin{aligned} 4\sqrt{2}G_\mu &= 4\sqrt{2}G_F(1 + \epsilon_{\text{box}} + \epsilon_{\text{vertex}}) \\ &= \frac{e^2(0)}{M_Z^2 c_W^2 s_W^2} \frac{1}{1 - \Delta r}, \end{aligned} \quad (4.15)$$

we now compare our expression for Δr , deduced from (4.8) and (4.14), with the one given in the literature. For this purpose, we insert the expression for δ_{QQ} etc., as well as $\delta\rho$, in terms of vacuum-polarization, vertex and box contributions (compare (3.22), (3.27) and (3.5) as well as (4.4) and (3.23), (3.31)) into our formula for Δr and obtain

$$\begin{aligned} \Delta r &= \frac{c_W^2}{s_W^2} \left[\frac{\Pi_{ZZ}(M_Z^2, \xi)}{M_Z^2} - \frac{\Pi_{WW}(M_W^2, \xi)}{M_W^2} \right] + \frac{\Pi_{WW}(M_W^2, \xi) - \Pi_{WW}(0, \xi)}{M_W^2} \\ &\quad - e(0)^2 \tilde{\Pi}_{QQ}(0, \xi) + 2e^2(0) \frac{c_W^2}{s_W^2} \Gamma_3^V(0, \xi) \\ &\quad + \frac{e^2(0)}{s_W^2} \left[2(\Gamma_\pm^V(0, \xi) - \Gamma_3^V(0, \xi)) - M_W^2 B'_{CC}(0, \xi) \right] \\ &\quad + \frac{e^2(0)}{16\pi^2} \left[6 - \frac{1}{s_W^2} \left(\frac{5}{2s_W^2} - 4 - c_W^2 \right) \ln c_W^2 \right]. \end{aligned} \quad (4.16)$$

By reinserting the gauge parameter ξ in (4.16), we indicate the manifestly gauge-invariant form of our expression for Δr , i.e., the evaluation of the right-hand side of (4.16) for any value of ξ yields the same numerical value of Δr . The first two lines in (4.16) agree with the

expression usually found in the literature^{10,13}, when ξ is specified to the t' Hooft-Feynman gauge, $\xi = 1$. It remains to explicitly evaluate the third line of (4.16) for $\xi = 1$. From the defining equations (3.45) and (B.27), (B.32b), it is straightforward to show that

$$\begin{aligned} \Gamma_\pm^V(0, \xi=1) - \Gamma_3^V(0, \xi=1) &= \frac{1}{16\pi^2} \left[s_W^2 \Lambda_\gamma(0, 1) + \Delta \Gamma_\pm^{PI}(0) + c_W^2 \Lambda_Z(0, 1) \right] \\ &= \frac{c_W^2}{16\pi^2} \left[3 + \frac{1}{c_W^2} \left(\frac{3}{2} - 3 - \frac{c_W^2}{2} \right) \ln c_W^2 \right], \end{aligned} \quad (4.17)$$

while

$$B'_{CC}(0, \xi=1) = 0.$$

Combining (4.17) with the last term in (4.16) immediately yields

$$\begin{aligned} \frac{e^2(0)}{s_W^2} \left[2(\Gamma_\pm^V(0) - \Gamma_3^V(0)) - M_W^2 B'_{CC}(0) \right] + \frac{e^2(0)}{16\pi^2} \left[6 - \frac{1}{s_W^2} \left(\frac{5}{2s_W^2} - 4 - c_W^2 \right) \ln c_W^2 \right] \\ = \frac{e^2(0)}{16\pi^2 s_W^2} \left[6 + \left(\frac{7}{2s_W^2} - 2 \right) \ln c_W^2 \right], \end{aligned} \quad (4.18)$$

which agrees with the result given in the literature^{10,13}.

In summary, the use of running couplings and masses yields an illuminatingly simple derivation of the one-loop corrected relation among the electromagnetic coupling, $e^2(0)$, the universal Fermi constant, G_F , the Z^0 mass, M_Z , and the effective mixing parameter $s_W^2(M_Z^2)$ or, alternatively, $s_W^2 \equiv 1 - M_W^2/M_Z^2$. Introducing, in addition, the process-dependent corrections to $\mu^\pm$ decay, yields a manifestly gauge-invariant expression for Δr , which upon evaluation for $\xi = 1$ fully agrees with the expression given in the literature and derived within the usual counterterm approach.

5. Conclusions

The subject matter of the one-loop radiative corrections to four-fermion processes has received much attention within the published literature. The emphasis on the underlying physics within the present treatment of the renormalization with a minimum amount of formalism may be refreshing, however.

In particular, we have derived manifestly gauge-invariant expressions for the one-loop dressed vector-boson propagators which contain all process-independent corrections, dubbed the effective or “genuine” vacuum-polarization corrections. The dressed propagators allow one to express the dominant part of the one-loop corrected four-fermion processes in Born-term form by using the intuitively appealing concept of running couplings and running masses. Different possible choices for some of the running quantities have been commented upon. The running parameters are in general related to each other by relations which have the form of corrected Born-term relations. The splitting between the dominant genuine amplitude and the residual amplitude, containing additional, not always negligible process-dependent corrections, is fairly unique, apart from the ambiguity inherently involved in removing the infrared divergences from the genuine amplitude.

The present formalism is particularly well-suited for a simple and transparent derivation of a (manifestly gauge-invariant) expression for the radiative correction to the mutual interdependence of $M_W, s^2(M_Z^2)$ and the input parameters $G_\mu, e(0)$ and M_Z . A comparison with the results in the literature shows (the expected) complete agreement with the published results for Δr , usually based upon a calculation within the counterterm approach.

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Appendix A The Diagonalization of the γW^3 Propagator Matrix

Upon introducing radiative corrections, the universal part of the neutral-current fermion-fermion interaction contains the two-by-two (inverse) propagator matrix (3.17) of the form

$$\Delta_{tot}^{-1} = q^2 \begin{pmatrix} c_1 & \lambda \\ \lambda & c_2 \end{pmatrix} - \begin{pmatrix} m_1^2 & \mu^2 \\ \mu^2 & m_2^2 \end{pmatrix}, \quad (A.1)$$

where m_1^2 , m_2^2 and μ^2 are constants, and the matrix proportional to q^2 vanishes for $q^2 \rightarrow 0$. We stress that the propagator (3.17),(A.1) is to be understood with respect to the base

$$\begin{pmatrix} j_3 \\ j_Q \end{pmatrix}, \quad (A.2)$$

spanned by the third component of the weak isospin current and the electromagnetic current, i.e., it describes γW^3 propagation. In the general form (A.1), the (inverse) propagator includes off-diagonal, so-called current-mixing terms ($\lambda \neq 0$), as well as mass-mixing terms ($\mu^2 \neq 0$). The choice of the base (A.2), i.e., describing electroweak interactions in terms of a γW^3 mixing propagator, is particularly useful, in order to display the implications of electromagnetic gauge invariance and of the coupling of the photon to the conserved electromagnetic current, j_Q . In passing we note that in Born approximation the (inverse) propagator (A.1) reduces to

$$\Delta^{-1}(q^2) = q^2 \begin{pmatrix} 1 & s_0 \\ s_0 & 1 \end{pmatrix} - \begin{pmatrix} M_W^{(0)2} & 0 \\ 0 & 0 \end{pmatrix}, \quad (A.3)$$

and thus, it coincides with the form of the propagator directly obtained from the basic Lagrangian, if electroweak interactions are described in terms of global $SU(2)_{WI}$ weak isospin symmetry broken by γW^3 mixing⁵⁾ supplemented by the W-dominance condition⁶⁾ for the mixing parameter, which implies $\lambda = s_0 = e_0/g_0$. The mixing effect in this case is due to the gauge-invariant $\lambda F^{\mu\nu} W_{\mu\nu}$ -term in the basic Lagrangian.

In this Appendix, we will show that only current mixing ($\lambda \neq 0$) is allowed in (A.1), while mass mixing is forbidden, i.e.,

$$m_2 = \mu \equiv 0, \quad (A.4)$$

if (A.1) is to yield a massless photon coupled to the electromagnetic current, j_Q , upon diagonalization. Our proof of (A.4), will be based on the explicit diagonalization of the (inverse) propagator in (A.1).

The diagonalization of (A1) is carried out by generalization of the procedure used in refs. 1, 2. In a first step we diagonalize the kinetic term via

$$D_1 \Delta_{tot}^{-1} D_1^T = q^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - M^2, \quad (A.5)$$

where the non-orthogonal matrix D_1 is given by

$$D_1 \equiv \begin{pmatrix} \frac{1}{\sqrt{c_1(1-\lambda^2)}}, & \frac{-\lambda}{\sqrt{c_2(1-\lambda^2)}} \\ 0, & \frac{1}{\sqrt{c_2}} \end{pmatrix}, \quad (A.6)$$

with

$$\lambda' = \frac{\lambda}{\sqrt{c_1 c_2}}, \quad (A.7)$$

and M^2 turns out to have the form

$$M^2 = D_1 \begin{pmatrix} m_1^2 & \mu^2 \\ \mu^2 & m_2^2 \end{pmatrix} D_1^T = \begin{bmatrix} \frac{m_1^2}{c_1(1-\lambda'^2)} - \frac{2\lambda'\mu^2}{\sqrt{c_1 c_2}(1-\lambda'^2)} + \frac{\lambda'^2 m_2^2}{c_2(1-\lambda'^2)}, & \frac{\mu^2}{\sqrt{c_1 c_2}(1-\lambda'^2)} - \frac{\lambda' m_2^2}{\sqrt{c_2^3}(1-\lambda'^2)} \\ \frac{\mu^2}{\sqrt{c_1 c_2}(1-\lambda'^2)} - \frac{\lambda' m_2^2}{\sqrt{c_2^3}(1-\lambda'^2)}, & \frac{m_2^2}{c_2} \end{bmatrix}. \quad (A.8)$$

Imposing the condition

$$\mu^2 = m_1 m_2, \quad (A.9)$$

which is absolutely necessary to obtain a vanishing photon mass upon diagonalizing the mass matrix, M^2 in (A.8) is simplified to become

$$M^2 = \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}, \quad (A.10)$$

where

$$a = \frac{m_1}{\sqrt{c_1(1-\lambda'^2)}} - \frac{\lambda' m_2}{\sqrt{c_2(1-\lambda'^2)}}, \quad (A.11)$$

$$b = \frac{m_2}{\sqrt{c_2}}.$$

Noting that M^2 in (A.10) is diagonalized by the orthogonal matrix

$$R_1 = \frac{1}{\sqrt{a^2 + b^2}} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad (A.12)$$

which yields

$$R_1 M^2 R_1^T = \begin{pmatrix} a^2 + b^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad (A.13)$$

full diagonalization of the (inverse) propagator matrix (A.1) is obviously achieved via

$$D = R_1 D_1 = \frac{1}{\sqrt{a^2 + b^2}} \begin{bmatrix} \frac{a}{\sqrt{c_1(1-\lambda'^2)}} - \frac{\lambda' m_2}{\sqrt{c_2(1-\lambda'^2)}}, & -\frac{\lambda' m_2}{\sqrt{c_2^3}(1-\lambda'^2)} \\ -\frac{\lambda' m_2}{\sqrt{c_2^3}(1-\lambda'^2)}, & \frac{m_2}{\sqrt{c_1 c_2}(1-\lambda'^2)} \end{bmatrix}, \quad (A.14)$$

i.e.,

$$\Delta_{diag}^{-1} \equiv D \Delta_{tot}^{-1} D^T = \begin{pmatrix} q^2 - (a^2 + b^2) & 0 \\ 0 & q^2 \end{pmatrix}, \quad (A.15)$$

where a and b have been given in (A.11). The conclusion $m_2 = 0$ now follows immediately from the expression for D in (A.14) by noting that the neutral-current amplitude formed with the propagator (A.1) is given by

$$M'_{NC} = \begin{pmatrix} g_0 j_3 \\ e_0 j_Q \end{pmatrix}_\mu^T D^T \Delta_{diag} D \begin{pmatrix} g_0 j_3 \\ e_0 j_Q \end{pmatrix}_\mu. \quad (A.16)$$

Indeed, $m_2^2 = \mu^2 = 0$ (where (A.9) has been used in addition), must necessarily be required, as asserted in (A.4), in order to prevent parity violation for reactions with real photons and an electric charge of the neutrino, among other things, to be induced by an interaction term of the form $j_3 j_Q / q^2$.

With respect to a reference needed in the main text, we note the form of D , if $\mu = m_2 = 0$ is adopted in the expression for the (inverse) propagator in (A.1). For $\mu = m_2 = 0$, the transformation D is given by

$$D \equiv \begin{bmatrix} \frac{1}{\sqrt{c_1(1-\lambda'^2)}} & -\frac{\lambda'}{\sqrt{c_2(1-\lambda'^2)}} \\ 0 & \frac{1}{\sqrt{c_2}} \end{bmatrix}, \quad (A.17)$$

and upon diagonalization, the propagator (A.1) becomes

$$\Delta_{diag} = \begin{bmatrix} \frac{1}{q^2 - \frac{m_1^2}{c_1(1-\lambda'^2)}}, & 0 \\ 0, & \frac{1}{q^2} \end{bmatrix}. \quad (A.18)$$

A final remark in this Appendix is related to electromagnetic gauge invariance, which has not been used explicitly within the demonstration given above. It is true that electromagnetic gauge invariance within the framework of an (effective) Lagrangian containing $\lambda, m_{1,2}, \mu$, etc., as free parameters, allows one to immediately deduce (A.4), i.e., to rule out mass-mixing terms in the basic Lagrangian, and the diagonalization then explicitly establishes gauge invariance for the diagonalized (physical) photon field. In the main text, however, relation (A.4) is applied to derive a constraint on mixing terms in the γW^3 propagator, which are induced via radiative corrections. As electromagnetic gauge invariance of the $SU(2)_L \times U(1)_Y$ electroweak theory remains after symmetry breaking, the (physical) photon is to couple only to the (conserved) electromagnetic current, j_Q , as assumed within the above demonstration, which thus yields the most direct way of convincing oneself of the validity of (A.4) and (3.18) in the main text without direct reference to an effective Lagrangian.

Appendix B Radiative Correction Functions and Gauge Invariance

In this Appendix, we will present the explicit expression for vacuum polarization, vertex and box contributions and show that the dressed inverse propagator $d_{ab}(q^2)$ given in (3.15), as well as $d_{\pm}(q^2)$ given in (3.29), are gauge-invariant quantities. Even though the existence of gauge-invariant dressed propagators is guaranteed by the gauge invariance of the formalism, it is gratifying to see gauge invariance manifestly satisfied for the propagators explicitly constructed in terms of vacuum polarization, vertex and box contributions.

In order to show the gauge invariance of $d_{\pm}(q^2)$ and $d_{ab}(q^2)$, it is sufficient to prove the gauge invariance of the following four quantities

$$\begin{aligned} S_0(q^2) &\equiv \Pi_{\pm}(q^2) + 2(q^2 - M_W^{(0)2})\Gamma'_{\pm}(q^2) + (q^2 - M_W^{(0)2})^2 B'_{CC}(q^2), \\ S_1(q^2) &\equiv \tilde{\Pi}_{QQ}(q^2) + 2\Gamma'_3(q^2) + q^2 B'_{NC}(q^2), \\ S_2(q^2) &\equiv \Pi_{3Q}(q^2) - \Pi_{QQ}(q^2) - M_W^{(0)2}(\Gamma'_3(q^2) + q^2 B'_{NC}(q^2)), \\ S_3(q^2) &\equiv \Pi_{33}(q^2) + \Pi_{QQ}(q^2) - 2\Pi_{3Q}(q^2) + M_W^{(0)4} B'_{NC}(q^2), \end{aligned} \quad (B.1)$$

where

$$\tilde{\Pi}_{QQ}(q^2) = \Pi_{QQ}(q^2)/q^2, \quad (B.2)$$

since

$$\begin{aligned} d_{\pm}(q^2) &= q^2 - M_W^{(0)2} - g_0^2 S_0(q^2), \\ d_{33}(q^2) &= q^2 - M_W^{(0)2} - g_0^2(q^2 S_1(q^2) + 2S_2(q^2) + S_3(q^2)), \\ d_{3Q}(q^2) &= d_{Q3}(q^2) = s_0(q^2 - g_0^2(q^2 S_1(q^2) + S_2(q^2))), \\ d_{QQ}(q^2) &= q^2(1 - e_0^2 S_1(q^2)). \end{aligned} \quad (B.3)$$

Note that $d_{3Q}(q^2) = d_{Q3}(q^2)$ and $d_{QQ}(q^2)$ depend on $S_1(q^2)$ and $S_2(q^2)$ only. Provided M'_{NC} is correctly constructed, the gauge invariance of $d_{33}(q^2)$ follows immediately from the gauge invariance of $d_{3Q}(q^2)$ and $d_{QQ}(q^2)$. We will nevertheless show the gauge invariance of $d_{33}(q^2)$ directly, by also verifying the gauge invariance of $S_3(q^2)$.

In the following the expressions for $\Pi_{\pm}(q^2)$, $\tilde{\Pi}_{QQ}(q^2)$, $\Pi_{3Q}(q^2)$, $\Pi_{QQ}(q^2)$, $\Gamma'_{\pm}(q^2)$, $\Gamma'_3(q^2)$, $\Gamma_B(q^2)$, $B'_{CC}(q^2)$ and $B'_{NC}(q^2)$ are given in the R_{ξ} -gauge.⁹ The fermion masses in the external legs are neglected. We will use the following notation

$$\ln \Lambda^2 = \frac{-2}{N-4} - \gamma_E + \ln 4\pi, \quad (B.4)$$

$$u \equiv \frac{q^2}{M_W^{(0)2}}, \quad (B.5)$$

$$D(x, \xi) = 1 - x + x\xi - x(1-x)u, \quad (B.6)$$

$$D_V(x, \xi, \xi') = x\xi + (1-x)\xi' \frac{M_V^{(0)2}}{M_W^{(0)2}} - x(1-x)u, \quad V = Z, \gamma, \quad (B.7)$$

where N denotes the dimension in dimensional regularization and γ_E the Euler constant. Q_f , T_{3f} and m_f stand for the charge, the third component of the weak isospin and the

mass of the fermion running in the loops of the neutral gauge boson propagators. We shall also introduce the following gauge-dependent finite function, $F_V(q^2, \xi)$, with $V = W, Z, \gamma$,

$$\begin{aligned} F_V(q^2, \xi) &\equiv \int_0^1 dx \ln \frac{x\xi M_W^{(0)2} + (1-x)M_V^{(0)2} - x(1-x)q^2 - i\epsilon}{xM_W^{(0)2} + (1-x)M_V^{(0)2} - x(1-x)q^2 - i\epsilon} \\ &+ \frac{M_W^{(0)2}}{M_V^{(0)2}} \int_0^1 dx \ln \frac{x\xi M_V^{(0)2} + (1-x)M_W^{(0)2} - x(1-x)q^2 - i\epsilon}{xM_V^{(0)2} + (1-x)M_W^{(0)2} - x(1-x)q^2 - i\epsilon} \\ &+ H_V(\xi, \xi) - H_V(\xi, 1) - H_V(1, \xi) + H_V(1, 1), \end{aligned} \quad (B.8)$$

where

$$\begin{aligned} H_V(\xi, \xi') &= \frac{M_W^{(0)2}}{2M_V^{(0)2}} \int_0^1 dx \left\{ \frac{\xi' x M_V^{(0)2}}{M_W^{(0)2}} + \xi(1-x) - x(1-x)u \right\} \\ &\quad \ln \left(\frac{\xi' x M_V^{(0)2}}{M_W^{(0)2}} + \xi(1-x) - x(1-x)u - i\epsilon \right). \end{aligned} \quad (B.9)$$

Vacuum polarization

$$\begin{aligned}
16\pi^2 \Pi_{\pm}^{\xi}(q^2) = & \left[\left(\frac{25}{6} - \xi \right) q^2 + \sum_{(I,i)} \left(\frac{m_i^2 + m_I^2}{2} - \frac{q^2}{3} \right) + \frac{\xi + 3}{4} \left(2M_W^{(0)2} - M_Z^{(0)2} \right) \right] \\
& \ln \frac{\Lambda^2}{M_W^{(0)2}} \\
& - \frac{q^2}{2} - \left(3M_W^{(0)2} + \frac{M_Z^{(0)2}}{4} \right) \ln c_0^2 + \frac{M_H^{(0)2}}{4} \ln \frac{M_H^{(0)2}}{M_W^{(0)2}} \\
& - \sum_{(I,i)} \int_0^1 dx [(1-x)m_i^2 + xm_I^2 - 2x(1-x)q^2] \\
& \ln \{ (1-x) \frac{m_i^2}{M_W^{(0)2}} + x \frac{m_I^2}{M_W^{(0)2}} - x(1-x)u - i\epsilon \} \\
& + \frac{1}{2} \int_0^1 dx \{ 2M_W^2 - (1-x)M_W^2 - xM_H^2 + x(1-x)q^2 \} \\
& \ln \{ 1-x + x \frac{M_H^{(0)2}}{M_W^{(0)2}} - x(1-x)u - i\epsilon \} \\
& - s_0^2 \int_0^1 dx \{ [5 - 10x(1-x)]q^2 + (7-8x)M_W^{(0)2} \} \\
& \ln(1-x-x(1-x)u-i\epsilon) \\
& + \int_0^1 dx \left\{ \left[\frac{2-5c_0^2-14c_0^4}{2c_0^2} + \frac{16c_0^4-15c_0^2-1}{2c_0^2} x \right] M_W^{(0)2} \right. \\
& \quad \left. + \left[\left(10c_0^2 + \frac{1}{2} \right) x(1-x) - 5c_0^2 \right] q^2 \right\} \\
& \ln \left(1-x + \frac{x}{c_0^2} - x(1-x)u - i\epsilon \right) \\
& + 16\pi^2 \left(\Pi_{\pm}^{\xi}(q^2) + \Pi^{\text{rad}} \right),
\end{aligned} \tag{B.10}$$

where the gauge-dependent finite part $\Pi_{\pm}^{\xi}(q^2)$ is given by

$$\begin{aligned}
16\pi^2 \Pi_{\pm}^{\xi}(q^2) = & \frac{3}{8}(\xi-1)q^2 + \frac{M_Z^{(0)2}}{4} [\xi \ln \xi - (\xi-1)(1+\ln c_0^2)] \\
& + M_W^{(0)2} \left[\frac{3\xi+1}{2} \xi \ln \xi - \frac{(\xi-1)(2\xi+3)}{4} - \frac{3}{4}(\xi^2-1) \ln c_0^2 \right] \\
& + M_W^{(0)2} (c_0^2 F_Z(q^2, \xi) + s_0^2 F_{\gamma}(q^2, \xi)) \\
& + c_0^2 M_W^{(0)2} \left[\int_0^1 dx I_1(x, \xi) \ln \left(x\xi + (1-x) \frac{1}{c_0^2} - x(1-x)u - i\epsilon \right) \right. \\
& \quad \left. + \int_0^1 dx I_2(x, \xi) \ln \left(x + (1-x) \frac{\xi}{c_0^2} - x(1-x)u - i\epsilon \right) \right. \\
& \quad \left. + \int_0^1 dx I_3(x) \ln \left(x + (1-x) \frac{1}{c_0^2} - x(1-x)u - i\epsilon \right) \right] \\
& + s_0^2 M_W^{(0)2} \left[\int_0^1 dx I_4(x, \xi) \ln (x\xi - x(1-x)u - i\epsilon) \right. \\
& \quad \left. + \int_0^1 dx I_5(x, \xi) \ln \left(x + (1-x) \xi \frac{M_{\gamma}^{(0)2}}{M_W^{(0)2}} - x(1-x)u - i\epsilon \right) \right. \\
& \quad \left. + \int_0^1 dx I_6(x) \ln \left(x + (1-x) \frac{M_{\gamma}^{(0)2}}{M_W^{(0)2}} - x(1-x)u - i\epsilon \right) \right],
\end{aligned} \tag{B.11}$$

with the functions I_i defined by

$$\begin{aligned}
I_1(x, \xi) = & -\frac{9}{4} D_Z^2(x, \xi, 1) - \left(\frac{11}{2} x^2 - 3 \right) D_Z(x, \xi, 1) u + D_Z(x, \xi, 1) \\
& - 1 - x^2 (x^2 - 2) u^2 - \frac{1}{2c_0^2} D_Z(x, \xi, 1) + \frac{s_0^4}{c_0^4}, \\
I_2(x, \xi) = & -c_0^2 \left[\frac{9}{4} D_Z^2(x, 1, \xi) + \left(\frac{11}{2} - 11x + \frac{5}{2} \right) D_Z(x, 1, \xi) u \right. \\
& \left. + 1 - \frac{1}{2} D_Z(x, 1, \xi) - (1-x)^2 (1+2x-x^2) u^2 \right] \\
I_3(x) = & -I_1(x, 1) - I_2(x, 1) \\
I_4(x, \xi) = & -\frac{9}{4} D_{\gamma}^2(x, \xi, 0) - \left(\frac{11}{2} x^2 - 3 \right) D_{\gamma}(x, \xi, 0) u - x^2 (x^2 - 2) \\
& u^2 + D_{\gamma}(x, \xi, 0) \\
I_5(x, \xi) = & \frac{M_W^{(0)2}}{M_{\gamma}^{(0)2}} \left[-\frac{9}{4} D_{\gamma}^2(x, 1, \xi) - \left(\frac{11}{2} x^2 - 11x + \frac{5}{2} \right) D_{\gamma}(x, 1, \xi) u \right. \\
& \left. + \frac{1}{2} D_{\gamma}(x, 1, \xi) - 1 + (1-x)^2 (1+2x-x^2) u^2 \right] \\
I_6(x) = & -I_4(x, 1) - I_5(x, 1).
\end{aligned} \tag{B.12}$$

The tadpole contribution Π^{tad} is given by

$$16\pi^2 \Pi^{\text{tad}} = M_W^{(0)2} \left(3 \ln \frac{\Lambda^2}{M_W^{(0)2}} + 1 \right) + \frac{M_Z^{(0)2}}{2c_\theta^2} \left(3 \ln \frac{\Lambda^2}{M_Z^{(0)2}} + 1 \right) + \frac{3}{4} M_H^{(0)2} \left(\ln \frac{\Lambda^2}{M_H^{(0)2}} + 1 \right) \\ + \frac{\xi}{4} M_Z^{(0)2} \left(\ln \frac{\Lambda^2}{\xi M_Z^{(0)2}} + 1 \right) + \frac{\xi}{2} M_W^{(0)2} \left(\ln \frac{\Lambda^2}{\xi M_W^{(0)2}} + 1 \right). \quad (B.13)$$

Note that $\Pi_{\pm}^{\xi}(q^2)$ contains an infrared divergence as $M_\gamma \rightarrow 0$ which is, however, cancelled by the same infrared divergence arising from $\Lambda_\gamma(q^2, \xi)$ defined later in (B.23).

$$16\pi^2 \tilde{\Pi}_{QQ}(q^2) = \{4 - \xi - \frac{4}{3} \sum_f Q_f^2\} \ln \frac{\Lambda^2}{M_W^{(0)2}} \\ + \frac{2}{3} + 8 \sum_f Q_f^2 \int_0^1 dx \cdot x(1-x) \ln \left(\frac{m_f^2}{M_W^{(0)2}} - x(1-x)u - i\epsilon \right) \\ + \int_0^1 dx \left[\left(-\frac{u}{2} - 6 + 8x^2 \right) (1-x(1-x)u) \right. \\ \left. + 10x^2 - 12x - 1 + 2u(1-x)^2 \right] \ln(1-x(1-x)u - i\epsilon) \\ - \frac{u}{2} \int_0^1 dx (\xi - x(1-x)u) \ln(\xi - x(1-x)u - i\epsilon) \\ + \int_0^1 dx \left[(u + 6 - 8x^2) (1-x + x\xi - x(1-x)u) + 2x^2 - 2u(1-x^2)^2 \right] \\ \ln(1-x + x\xi - x(1-x)u - i\epsilon) \\ + \frac{1}{2}(\xi - 1). \quad (B.14)$$

$$16\pi^2 \Pi_{3Q}(q^2) = \left[q^2 \left\{ \frac{25}{6} - \xi - \frac{2}{3} \sum_f T_{3f} Q_f \right\} + M_W^{(0)2} \frac{\Lambda^2}{2} \right] \ln \frac{\Lambda^2}{M_W^{(0)2}} \\ + q^2 \left[-\frac{1}{2} + 4 \sum_f Q_f T_{3f} \int_0^1 dx x(1-x) \right] \ln \left(\frac{m_f^2}{M_W^{(0)2}} - x(1-x)u - i\epsilon \right) \\ + \int_0^1 dx \{ 11x(1-x) - 5 - \frac{u}{2} + \frac{u^2}{2} x(1-x) \} \\ \ln(1-x(1-x)u - i\epsilon) \\ + M_W^{(0)2} \int_0^1 dx \left\{ \frac{9}{2} D^2(x, 1) + (11x^2 - 6) D(x, 1) u + 2(1-x^2)^2 u^2 - 9 \right\} \\ \ln(1-x(1-x)u - i\epsilon) \\ - \frac{q^2}{2} u \int_0^1 dx (\xi - x(1-x)u) \ln(\xi - x(1-x)u - i\epsilon) \\ + \int_0^1 dx [q^2 u \{ 1 - x + x\xi - x(1-x)u \} \\ - M_W^{(0)2} \{ \frac{9}{2} D^2(x, \xi) + (11x^2 - 6) D(x, \xi) u + 2(1-x^2)^2 u^2 \}] \\ \ln(1-x + x\xi - x(1-x)u - i\epsilon) \\ + \frac{3}{8}(\xi - 1) q^2 + M_W^{(0)2} \left\{ -\frac{1}{4}(\xi - 1)(2\xi + 3) + \frac{3}{2}\xi^2 \ln \xi + \xi \ln \xi \right\}. \quad (B.15)$$

$$\begin{aligned}
16\pi^2 \Pi_{33}(q^2) = & \left[q^2 \left(\frac{25}{6} - \xi \right) + \frac{\xi + 3}{4} (2M_W^{(0)2} - M_Z^{(0)2}) + \sum_f \left(\frac{m_f^2}{2} - \frac{q^2}{6} \right) \right] \\
& \ln \frac{\Lambda^2}{M_W^{(0)2}} \\
& - \left(\frac{4}{3}s_0^4 + \frac{1}{2} \right) q^2 - \frac{M_Z^{(0)2}}{4} \ln \frac{M_W^{(0)2}}{M_Z^{(0)2}} - \frac{M_H^{(0)2}}{4} \ln \frac{M_W^{(0)2}}{M_H^{(0)2}} \\
& + \sum_f \int_0^1 dx \{ q^2 x(1-x) - \frac{m_f^2}{2} \} \ln \left(\frac{m_f^2}{M_W^{(0)2}} - x(1-x)u - i\epsilon \right) \\
& + \frac{1}{2} \int_0^1 dx \{ 2M_Z^{(0)2} - (1-x)M_H^{(0)2} - xM_Z^{(0)2} + x(1-x)q^2 \} \\
& \ln \{ (1-x) \frac{M_H^{(0)2}}{M_W^{(0)2}} + x \frac{M_Z^{(0)2}}{M_W^{(0)2}} - x(1-x)u - i\epsilon \} \\
& + (\xi - 1) \left\{ \frac{2 + 2c_0^2 - c_0^4}{8} q^2 \right. \\
& \quad \left. - \left(\frac{1 + s_0^4}{2} \xi + \frac{3 + 5s_0^4}{4} + \frac{1}{4c_0^2} (1 + \ln c_0^2) \right) M_W^{(0)2} \right\} \\
& + M_W^{(0)2} \int_0^1 dx A(x) \ln(1-x(1-x)u - i\epsilon) \\
& + M_W^{(0)2} \int_0^1 dx B(x, \xi) \ln(\xi - x(1-x)u - i\epsilon) \\
& + M_W^{(0)2} \int_0^1 dx C(x, \xi) \ln(1-x + x\xi - x(1-x)u - i\epsilon) \\
& + 16\pi^2 \Pi^{\text{tad}},
\end{aligned} \tag{B.16}$$

where

$$\begin{aligned}
A(x) = & \frac{9}{2} (c_0^4 + 2s_0^2) D(x, 1)^2 \\
& + \{ (c_0^4 + 2s_0^2) (11x^2 - 6)u - \frac{u^2}{2} - s_0^4 (8x^2 - 6) - 23s_0^4 - 11 \} D(x, 1) \\
& + 2u^2 (1-x^2)^2 + \left(2s_0^4 x^2 + 20s_0^4 + \frac{11}{2} \right) u - 10s_0^4 - \frac{17}{2},
\end{aligned} \tag{B.17}$$

$$B(x, \xi) = \frac{1-u^2}{2} (\xi - x(1-x)u), \tag{B.18}$$

$$\begin{aligned}
C(x, \xi) = & -\frac{9}{2} (c_0^4 + 2s_0^2) D(x, \xi)^2 \\
& - \{ (c_0^4 + 2s_0^2) (11x^2 - 6)u - u^2 - s_0^4 (8x^2 - 6) + s_0^4 \} D(x, \xi) \\
& - 2u^2 (1-x^2)^2 - 2us_0^4 x^2 + 2s_0^4,
\end{aligned} \tag{B.19}$$

and Π^{tad} is given in (B.13). We note that the tadpole contributions which are independent of q^2 and usually neglected, are included in $\Pi_{\pm}(q^2)$ and $\Pi_{33}(q^2)$. This is necessary in order to explicitly show the gauge invariance of $S_0(q^2)$, $S_1(q^2)$, $S_2(q^2)$ and $S_3(q^2)$.

Vertex corrections

The vector-boson-fermion-fermion vertex correction, $\Gamma_V(q^2)$, receives, in general, three contributions, the external-fermion-line self-energy correction, $S_V(q^2)$ (Fig.3a) and the vertex corrections $G_V(q^2)$ and $G_{VV'}(q^2)$, corresponding to the diagrams without and with trilinear vector boson vertex (Fig.3b,3c), respectively. Here, V or VV' stand for the kind of vector boson running in the loop. In the expression for $\Gamma_V(q^2)$, the functions S_V, G_V and $G_{VV'}$ appear in specific combinations. It is thus convenient to define the following ones,

$$\begin{aligned}
\phi_V(q^2) &= S_V(q^2, \xi) + G_V(q^2, \xi), \\
\Lambda_V(q^2, \xi) &= G_{VV'}(q^2, \xi) + \frac{1}{2} [S_V(q^2, \xi) + S_W(q^2, \xi)] - \frac{\xi + 3}{2} \ln \frac{\Lambda^2}{M_W^2},
\end{aligned} \tag{B.20}$$

with $V = W, Z$ or γ . Here, $\phi_V(q^2)$ is a gauge-invariant finite function of q^2 , while $\Lambda_V(q^2, \xi)$ is a finite, but gauge-dependent function of q^2 with the divergent part explicitly separated as shown in (B.20). The function $\phi_V(q^2)$ is given by

$$\begin{aligned}
\phi_V(q^2) = & -\frac{7}{2} - \frac{2}{w} + \left(3 + \frac{2}{w} \right) \ln(-w + i\epsilon) \\
& + 2 \left(1 + \frac{1}{w} \right)^2 \left[Sp(1 + w + i\epsilon) - \frac{\pi^2}{6} \right],
\end{aligned} \tag{B.21}$$

with $w = \frac{q^2}{M_V^2}$ and the Spence function

$$Sp(z) = - \int_0^z \frac{\log(1-t)}{t} dt, \tag{B.22}$$

while $\Lambda_V(q^2, \xi)$ is given by

$$\begin{aligned} \Lambda_V(q^2, \xi) = & 1 - 2 \left(1 + \frac{1-u}{u} \ln(1-u-i\epsilon) \right) \\ & - 2 \int_0^1 dx \left\{ 1 + \frac{1-x + x \frac{M_V^{(0)2}}{M_W^{(0)2}} - x(1-x)u}{1-xu} - \frac{x^2 u^2 + x \left(\frac{M_V^{(0)2}}{M_W^{(0)2}} - 1 \right) xu}{(1-xu)^2} \right\} \\ & \ln \left(1 - x + x \frac{M_V^{(0)2}}{M_W^{(0)2}} - x(1-x)u - i\epsilon \right) \\ & + 2 \int_0^1 dx \left\{ \frac{M_V^{(0)2}}{M_W^{(0)2}} x - \frac{x^2 u^2 + \left(\frac{M_V^{(0)2}}{M_W^{(0)2}} - 1 \right) xu}{(1-xu)^2} \right\} \ln \left(x \frac{M_V^{(0)2}}{M_W^{(0)2}} \right) \\ & + \xi \ln \xi - \frac{\xi}{2} \left(1 + \ln \frac{M_W^{(0)2}}{M_V^{(0)2}} \right) \\ & + F_1(q^2, \xi) - F_1(q^2, 1) + F_2(q^2, \xi) - F_2(q^2, 1) + u F_V(q^2, \xi), \end{aligned} \quad (B.23)$$

where $F_V(q^2, \xi)$ is given in (B.8), and $F_1(q^2, \xi)$ and $F_2(q^2, \xi)$ are given as

$$\begin{aligned} F_1(q^2, \xi) = & \int_0^1 dx \left\{ \left(\frac{3}{2}x - \frac{5}{2}x^2 \right) u - \frac{3}{2}x\xi - \frac{3M_V^{(0)2}}{2M_W^{(0)2}}(1-x) \right\} \\ & \ln \left((1-x) \frac{M_V^{(0)2}}{M_W^{(0)2}} + x\xi - x(1-x)u - i\epsilon \right), \\ F_2(q^2, \xi) = & \frac{M_W^{(0)2}}{M_V^{(0)2}} \int_0^1 dx \left\{ \left(\frac{3}{2}x - \frac{5}{2}x^2 \right) u - \frac{3M_V^{(0)2}}{2M_W^{(0)2}}x\xi - \frac{3}{2}(1-x) \right\} \\ & \ln \left(1 - x + x\xi \frac{M_V^{(0)2}}{M_W^{(0)2}} - x(1-x)u - i\epsilon \right). \end{aligned} \quad (B.24)$$

For completeness, we also show the explicit form of $S_V(q^2, \xi)$,

$$S_V(q^2, \xi) = \frac{3}{2} - \xi \left(1 + \ln \frac{\Lambda^2}{M_V^{(0)2}} - \ln \xi \right) \quad V = W, Z, \gamma. \quad (B.25)$$

Using (B.21), (B.22) and (B.25), one can easily find the expressions for $G_V(q^2, \xi)$ and $G_{VV}(q^2, \xi)$.

The vertex corrections, $\Gamma_3(q^2)$, and $\Gamma_{\pm}(q^2, \xi)$, are divided into two parts, the part contributing to the genuine propagators, Γ_3^V , $\Gamma_{\pm}^V$, and the residual part $\tilde{\Gamma}_3$, $\tilde{\Gamma}_{\pm}$, while $\Gamma_B(q^2)$ is purely process dependent and therefore gauge invariant. They are given as

follows,

$$\begin{aligned} \Gamma_{\pm}^V(q^2, \xi) = & \Gamma_{\pm}^{PI}(q^2, \xi) + \Delta \Gamma_{\pm}^{PI}(q^2), \\ \tilde{\Gamma}_{\pm}(q^2) = & \frac{1}{16\pi^2} \left[(s_0^2 Q_f Q_f) \phi_{\gamma}(q^2) + \frac{1}{c_0^2} (T_{3f} - s_0^2 Q_f) (T_{3f'} - s_0^2 Q_{f'}) \phi_Z(q^2) \right] \\ & - \Delta \Gamma_{\pm}^{PI}(q^2), \\ \Gamma_3^V(q^2, \xi) = & \Gamma_3^{PI}(q^2, \xi) + \Delta \Gamma_3^{PI}(q^2), \\ \tilde{\Gamma}_3(q^2) = & \frac{1}{16\pi^2} \left[(s_0 Q_f)^2 \phi_{\gamma}(q^2) + \left(\frac{T_{3f} - s_0^2 Q_f}{c_0} \right)^2 \phi_Z(q^2) \right] - \Delta \Gamma_3^{PI}(q^2), \\ \Gamma_B(q^2) \equiv & \tilde{\Gamma}_B(q^2) \\ = & \frac{1}{16\pi^2} \left[(s_0 Q_f)^2 \phi_{\gamma}(q^2) + \frac{1}{c_0^2} (T_{3f} - s_0^2 Q_f)^2 \phi_Z(q^2) + \frac{1}{2} \phi_W(q^2) \right], \\ & \text{left-handed,} \\ = & \frac{1}{16\pi^2} \left[(s_0 Q_f)^2 \phi_{\gamma}(q^2) + \left(\frac{s_0^2 Q_f}{c_0} \right)^2 \phi_Z(q^2) \right], \quad \text{right-handed,} \end{aligned} \quad (B.26)$$

where the process-independent parts Γ_3^{PI} and $\Gamma_{\pm}^{PI}$ are given as

$$\begin{aligned} 16\pi^2 \Gamma_3^{PI}(q^2, \xi) = & \frac{\xi + 3}{2} \ln \frac{\Lambda^2}{M_W^{(0)2}} + \Lambda_W(q^2, \xi) - \frac{1}{2} \phi_W(q^2), \\ 16\pi^2 \Gamma_{\pm}^{PI}(q^2, \xi) = & \frac{\xi + 3}{2} \ln \frac{\Lambda^2}{M_W^{(0)2}} + s_0^2 \Lambda_{\gamma}(q^2, \xi) + c_0^2 \Lambda_Z(q^2, \xi). \end{aligned} \quad (B.27)$$

$\Delta \Gamma_3^{PI}$ and $\Delta P_{\pm}^{PI}$ are gauge-invariant non-divergent functions defined in (3.39) and (3.48), respectively.

Box corrections

$$B'_{CC}(q^2) = B_{CC}^{PI}. \quad (B.28)$$

$$16\pi^2 M_W^{(0)2} B_{CC}^{PI}(q^2) = -c_0^2 F_Z(q^2, \xi) - s_0^2 F_{\gamma}(q^2, \xi), \quad (B.29)$$

where $F_V(q^2, \xi)$ is defined in (B.8).

$$B'_{NC}(q^2) = B_{WW}(q^2). \quad (B.30)$$

$$\begin{aligned} 16\pi^2 M_W^{(0)2} B_{WW}(q^2) = & \int_0^1 dx \left\{ 2 - \frac{1-x(1-x)u}{2} \right\} \ln(1-x(1-x)u - i\epsilon) \\ & + \int_0^1 dx \{-1-x+x\xi-x(1-x)u\} \\ & \ln(1-x+x\xi-x(1-x)u - i\epsilon) \\ & - \frac{1}{2} \int_0^1 dx (\xi - x(1-x)u) \ln(\xi - x(1-x)u - i\epsilon). \end{aligned} \quad (B.31)$$

At $q^2 = 0$,

$$16\pi^2 \Pi_{3Q}(0) = M_W^{(0)2} \left\{ \frac{\xi+3}{2} \ln \frac{\Lambda^2}{M_W^{(0)2}} + \frac{\xi+5}{4} - \frac{\xi(\xi+2)}{2(\xi-1)} \ln \xi \right\}, \quad (B.32a)$$

$$\Lambda_W(0, \xi) = \frac{\xi+5}{4} - \frac{\xi(\xi+2)}{2(\xi-1)} \ln \xi, \quad (B.32b)$$

$$\phi_V(0) = 0, \quad V = W, Z, \quad (B.32c)$$

$$16\pi^2 M_W^{(0)2} B_{WW}(0) = \frac{7-\xi}{4} + \frac{\xi(\xi-4)}{2(\xi-1)} \ln \xi. \quad (B.32d)$$

From (B.27) and (B.32), one immediately confirms that the relation (3.20) is indeed satisfied for $\Gamma_3 = \Gamma_3^{PI}$.

When $\xi = 1$ ('t Hooft-Feynman gauge) as $q^2 \rightarrow \infty$

$$16\pi^2 \Pi_{\pm} \rightarrow q^2 \left(\frac{19}{6} - \frac{1}{3} \sum_{(I,i)} \ln \frac{\Lambda^2}{|q^2|} \right), \quad (B.33a)$$

$$16\pi^2 \hat{\Pi}_{QQ} \rightarrow \left(3 - \frac{4}{3} \sum_f Q_f^2 \right) \ln \frac{\Lambda^2}{|q^2|}, \quad (B.33b)$$

$$16\pi^2 \Pi_{3Q} \rightarrow q^2 \left(\frac{19}{6} - \frac{2}{3} \sum_f T_{3f} Q_f \right) \ln \frac{\Lambda^2}{|q^2|}, \quad (B.33c)$$

$$16\pi^2 \Pi_{33} \rightarrow q^2 \left(\frac{19}{6} - \frac{1}{6} \sum_f \ln \frac{\Lambda^2}{|q^2|} \right), \quad (B.33d)$$

$$\Lambda_W(q^2, \xi=1) \rightarrow \ln \left(\frac{|q^2|}{M_W^{(0)2}} \right), \quad (B.33e)$$

$$\phi_W(q^2) \rightarrow 3 \ln \left(\frac{|q^2|}{M_W^{(0)2}} \right) + 2 \ln^2 \left(\frac{|q^2|}{M_W^{(0)2}} \right), \quad (B.33f)$$

$$16\pi^2 q^2 B_{WW} = 0, \quad (B.33g)$$

$$16\pi^2 q^2 B_{CC}^{PI} = 0. \quad (B.33h)$$

Note that the expressions for the Π 's, as well as for Γ_V^{PI} ($V = W, 3, B$) and for B_{WW} and B_{CC}^{PI} are not unique, when the Feynman parameter integrals are not performed. They can be expressed in many seemingly different but equivalent forms.

Once the explicit ξ dependence of each function is given, it is straightforward to check the gauge invariance of $S_0(q^2)$, $S_1(q^2)$, $S_2(q^2)$ and $S_3(q^2)$ by explicitly performing the integrals, or more simply by showing that the derivatives of $S_0(q^2)$, $S_1(q^2)$, $S_2(q^2)$ and $S_3(q^2)$ with respect to ξ are zero.

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Figure captions

Fig.1 Schematic representation of (a) the tree-level amplitude and the vacuum polarization which contains fermionic as well as bosonic loops, (b) the vertex correction, (c) the box correction to four-fermion interactions.

Fig.2 Electromagnetic gauge invariance requires cancellation of the two diagrams in the limit of $q^2 \rightarrow 0$.

Fig.3 The vertex corrections receive contributions from the external-fermion selfenergy (a), and the vertex diagrams without, (b), and with, (c), trilinear vector-boson couplings.

Fig. 1

Fig. 2

(a)

(b)

(c)

Fig. 3