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Gauge Independence of Plasma Parameters in Thermal Gauge Theories*

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ABSTRACT

The issue of potential gauge dependence of plasma parameters in thermal gauge theories is studied by deriving and analysing the appropriate Ward identities. Gauge invariance and gauge independence is proved for the leading temperature contributions to vertex functions in hot QCD and quantum gravity. In the case of QCD it is shown moreover that gauge independent dispersion relations can be extracted beyond leading order from the inherently gauge dependent gluon propagator and that the observed gauge dependence of the one-loop plasmon damping constant is due to an inconsistent approximation scheme. On the other hand, gauge independence of the results from appropriately resummed loop expansions can be verified quite generally, and is checked in an explicit calculation. In the case of quantum gravity at non-zero temperature, where it is more difficult to define a propagator, it is shown how dispersion relations can be extracted from the thermal energy-momentum tensor and the thermal graviton self-energy.

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When expanded in powers of φ , this equation gives all the tree-level Ward identities among the classical vertices.

Already at the classical level one needs to fix the gauge freedom in order to be able to define Green functions. This can be done by adding a quadratic gauge breaking term

$$S[\varphi] \rightarrow S_{gf}[\varphi] = S[\varphi] + \frac{1}{2} \eta_{\alpha\beta} F^\alpha[\varphi] F^\beta[\varphi], \quad (1.2)$$

with a functional F^α such that the Faddeev-Popov operator $F^\alpha_i[\varphi] D^i_j[\varphi]$ is invertible, i.e. possesses a Green function $\mathcal{G}$,

$$(F^\alpha_i D^i_j) [\varphi] \mathcal{G}^\beta_\gamma[\varphi] = -\delta^\alpha_\gamma, \quad (1.3)$$

where δ is to be understood as the appropriate product of Kronecker and Dirac deltas.

Although the gauge dependence of S_{gf} is unmistakable, it can be expressed by the following "Ward identity": An arbitrary infinitesimal variation $F^\alpha \rightarrow F^\alpha + \delta F^\alpha$ induces

$$\delta(S_{gf}) = -(S_{gf})_i D^i_\alpha \mathcal{G}^\alpha_\beta \delta F^\beta, \quad (1.4)$$

which shows that gauge independence will be achieved again only "on-shell", defined by $(S_{gf})_i = 0$. The proof of eq. (1.4) is trivial: Writing out $(S_{gf})_i = S_{,i} + \eta_{\gamma\delta} F^\gamma F^\delta_{,i}$, the first term is annihilated because of (1.1), and with eq. (1.3) the r.h.s. of eq. (1.4) then equals $-\eta_{\gamma\delta} F^\gamma F^\delta_{,i} D^i_\alpha \mathcal{G}^\alpha_\beta \delta F^\beta = \eta_{\gamma\delta} F^\gamma \delta F^\beta$ as it should.

In a quantum field theory, everything can be derived from an effective action $\Gamma = S + \text{rad. corr.}$. In perturbation theory this object is also known as the generating functional for one-particle-irreducible (1pi) vertex diagrams which thus are the quantum analogue of the classical vertices $S_{ijk} \dots$

In the compact DeWitt notation, its defining equation reads³

$$\exp i\Gamma[\varphi] = \int \mathcal{D}\varphi \det(F^\alpha_i[\varphi] D^i_j[\varphi]) \exp i\{S_{gf}[\varphi] - \Gamma_i[\varphi](\varphi - \bar{\varphi})^i\}. \quad (1.5)$$

[Introducing external sources J and identifying $\Gamma_{,i} = -J_i$, this equation is nothing more than the Legendre transformed path integral expression of the generating functional of connected Green functions $W[J] = \Gamma[\varphi] + J_i \bar{\varphi}^i$, $\bar{\varphi} = \delta W / \delta J_i$]

In terms of Γ , the quantum version of eq. (1.4) for potential gauge dependences reads⁴

$$\delta\Gamma[\varphi] = -\Gamma_{,i}[\varphi] (D^i_\alpha[\varphi] \mathcal{G}^\alpha_\beta[\varphi] \delta F^\beta[\varphi]) \equiv \Gamma_{,i}[\varphi] \delta X^i[\varphi], \quad (1.6)$$

where

$$(\mathcal{O}[\varphi]) = e^{-it[\varphi]} \int \mathcal{D}\varphi \mathcal{O}[\varphi] \det(F^\alpha_i D^i_j) e^{i(S_{gf}[\varphi] - \Gamma_i[\varphi](\varphi - \bar{\varphi})^i)}. \quad (1.7)$$

This relation holds generally for gauge theories with off-shell closed gauge algebras. If one would like to also consider variations of the matrix $\eta_{\alpha\beta}$ in the quadratic gauge breaking term, this is covered by $\delta F^\alpha[\varphi] \rightarrow \delta F^\alpha[\varphi] + \frac{1}{2} \eta^{\alpha\beta} \delta \eta_{\beta\gamma} F^\gamma[\varphi]$.

Eq. (1.6) is a generalization of the so-called Nielsen identities originally used to resolve the issue of gauge dependence of the minima of the effective potential of broken gauge theories⁵.

1. The Gauge Dependence of the Effective Action

In thermal gauge theories the issue of possible gauge dependence of hopefully physical plasma parameters has become acute through the unsuccessful attempts to determine the QCD plasmon decay constant¹ over the past years. As will be shown, this and similar questions can be decided by an analysis of the appropriate Ward identities.

In order to cover all gauge theories (with a closed gauge algebra) at once, the Ward identities expressing the potential gauge dependence of the effective action will be derived in the compact notation of DeWitt², where φ^i denotes the complete set of fields in a gauge theory prior to gauge fixing, and the index i comprises all discrete and continuous labels of the fields. The summation convention then also implies integration over the space-time variables, and functional derivatives with respect to the fields φ is denoted by a comma and subsequent indices.

The classical action of the gauge theory under consideration be $S[\varphi]$ and the complete and independent set of generators of gauge transformations are denoted by $D^i_\alpha[\varphi]$. Invariance of the action is then expressed by

$$S_{,i}[\varphi] D^i_\alpha[\varphi] \equiv 0. \quad (1.1)$$

The modern way of deriving them is to use BRS techniques⁶, but actually the older methods of DeWitt² turn out to be more general. While in the BRS framework one only considers variations of constant parameters in a preassigned class of gauge choices, the derivation using the functional manipulations due to DeWitt is the same for all gauges and arbitrary variations thereof.

This derivation rests on the observation that variations in the gauge breaking term are equivalent to performing the (non-local) gauge transformation

$$\delta\varphi^i = D_\alpha^i \delta\varphi^\alpha(\varphi), \quad \delta\varphi^\alpha = -g^\alpha{}_\beta \delta F^\beta, \quad (1.8)$$

as is apparent from eq. (1.4). What is left to obtain eq. (1.6) from eq. (1.5), is then to verify that the Jacobian in the path integral associated with (1.8) just cancels the variation of the Faddeev-Popov determinant^{2,4}.

In order to investigate the consequences of the basic relation (1.6) within perturbation theory, in particular for perturbative thermal gauge theories to be considered in the following sections, it is useful to spell out its diagrammatic content.

$\Gamma[\bar{\varphi}]$ itself is given by a sum of 1pi primary (or vacuum) diagrams,

$$\Gamma[\bar{\varphi}] = S_{g,f}[\bar{\varphi}] - \frac{i}{2} \text{ (diagram: circle with dashed line)} + i \text{ (diagram: circle with solid line)} - \frac{1}{12} \text{ (diagram: circle with dashed line and cross)} + \frac{1}{2} \text{ (diagram: circle with solid line and cross)} - \frac{1}{8} \text{ (diagram: circle with dashed line and cross)} + \dots \quad (1.9)$$

where full lines stand for the propagator of φ -fields, $G^{ij}[\bar{\varphi}] = -[(S_{g,f})_{ji}]^{-1}$, and dashed lines correspond to the Faddeev-Popov ghost propagator introduced in eq. (1.3). These are connected by vertices representing $(S_{g,f})_{ijk}[\bar{\varphi}]$ for φ -lines, and $(F_\alpha^\alpha D_\beta^\beta)_{ijk}[\bar{\varphi}]$ for a ghost line with further φ -lines. The only exception to this rule are the one-loop contributions, where the single closed loop without vertices represents the trace of the logarithm of the respective propagator, rather than only the trace as the diagram itself would suggest.

All the quantities involved are to be considered as functionals in $\bar{\varphi}$, and the 1pi vertex diagrams are obtained by functional differentiations with respect to $\bar{\varphi}$, which either increase the number of prongs of an existing vertex or insert new vertices into an existing line.

A similar expansion into primary diagrams can also be obtained for the functional $\delta X^i[\bar{\varphi}]$ in eq. (1.6). For this an important simplification is achieved by subtracting the tree-level gauge breaking term from the effective action, $\Gamma[\bar{\varphi}] \rightarrow \Gamma[\bar{\varphi}] - \frac{1}{2} g_{\alpha\beta} F^\alpha F^\beta[\bar{\varphi}]$, which for linear gauge conditions is automatic when the gauge condition is imposed on $\varphi - \bar{\varphi}$ rather than φ alone: $F^\alpha = F_\alpha^\alpha(\varphi - \bar{\varphi})^i$.

This will still allow for background field gauges (or more precisely: mean-field dependent gauges)^{2,7}, for the correct procedure in implementing them is to introduce a field dependent gauge condition $F_\alpha^\alpha = F_\alpha^\alpha[\bar{\varphi}]$ and to make the identification $\bar{\varphi} = \varphi$ only after solving the functional integro-differential equation (1.5) for $\Gamma[\bar{\varphi}]$. Background field gauges can be used to achieve gauge invariance of the effective action, i.e. $\Gamma_i[\bar{\varphi}] D_\alpha^i[\bar{\varphi}] = 0$, which is to be

distinguished from the issue of gauge independence, $\delta\Gamma/\delta F = 0$, which does not follow from the former.

Assuming in addition that the gauge generators are at most linear in the fields, $D_{\alpha,j}^i \equiv 0$, we have to introduce three further vertices:

$$i \text{ (diagram: circle with dashed line)} - \alpha \text{ (diagram: circle with dashed line and cross)} = \delta F_\alpha^\alpha[\bar{\varphi}], \quad i \text{ (diagram: circle with dashed line and cross)} = D_{\alpha,j}^i$$

(a fourth one would be necessary when $\delta F_\alpha^\alpha[\varphi]|_{\varphi=\bar{\varphi}} \neq 0$). With them δX^i is given by a sum of primary diagrams which are 1pi with respect to all other lines apart from the basic ghost line attached to δF_α^α . Up to 2-loop order:

$$\begin{aligned} \delta X^i = & i \text{ (diagram: circle with dashed line)} + i \text{ (diagram: circle with dashed line and cross)} \\ & + i \text{ (diagram: circle with dashed line and cross)} + i \text{ (diagram: circle with dashed line and cross)} + i \text{ (diagram: circle with dashed line and cross)} \\ & + \frac{1}{2} i \text{ (diagram: circle with dashed line and cross)} + \frac{1}{2} i \text{ (diagram: circle with dashed line and cross)} - i \text{ (diagram: circle with dashed line and cross)} \\ & + i \text{ (diagram: circle with dashed line and cross)} + i \text{ (diagram: circle with dashed line and cross)} + i \text{ (diagram: circle with dashed line and cross)} + i \text{ (diagram: circle with dashed line and cross)} \\ & + \frac{1}{2} i \text{ (diagram: circle with dashed line and cross)} + \frac{1}{2} i \text{ (diagram: circle with dashed line and cross)} - i \text{ (diagram: circle with dashed line and cross)} + \dots \end{aligned} \quad (1.10)$$

Because of $\delta F_\alpha^\alpha[\bar{\varphi}] = 0$, this diagrammatic expansion starts only at 1-loop order, for the gauge dependence at the tree-level is subtracted out. Thus the gauge dependence of an ℓ -loop n -point 1pi vertex function $\delta\Gamma_{i_1 \dots i_n}$ is determined through eq. (1.6) in terms of products of a vertex function with $\leq (n+1)$ prongs and loop order $< \ell$, and a function obtained by up to n vertex insertions into the primary diagrams of δX^i .

For example, the gauge dependence of a 1-loop 2-point vertex function $\Gamma_{ij}[0]$ is given by (assuming for simplicity $S_i[0] = 0$)

$$\begin{aligned} \delta\Gamma_{ij}^{1\text{-loop}}[0] = & i S_{im}[0] \left\{ m \text{ (diagram: circle with dashed line and cross)} + m \text{ (diagram: circle with dashed line and cross)} + m \text{ (diagram: circle with dashed line and cross)} \right. \\ & \left. + m \text{ (diagram: circle with dashed line and cross)} + m \text{ (diagram: circle with dashed line and cross)} + m \text{ (diagram: circle with dashed line and cross)} \right\} \\ & + i S_{ijm}[0] \left\{ m \text{ (diagram: circle with dashed line and cross)} + m \text{ (diagram: circle with dashed line and cross)} + m \text{ (diagram: circle with dashed line and cross)} \right\} \\ & + (i \leftrightarrow j) \end{aligned} \quad (1.11)$$

All which is left to do for a concrete field theory is to specify the various objects like gauge generators and vertices. In QED, for instance, the gauge generators $D_\alpha^{ij}[\varphi]$ of the compact notation have to be expanded as

$$D_\alpha^{ij} \rightarrow \frac{\partial}{\partial x_\mu^{(i)}} \delta(x^{(i)} - x^{(\alpha)}) \quad \text{when } i \leftrightarrow A_\mu \quad (1.12a)$$

$$\rightarrow -ie\psi(x^{(i)})\delta(x^{(i)} - x^{(\alpha)}) \quad \text{when } i \leftrightarrow \psi \quad (1.12b)$$

$$\rightarrow +ie\bar{\psi}(x^{(i)})\delta(x^{(i)} - x^{(\alpha)}) \quad \text{when } i \leftrightarrow \bar{\psi}. \quad (1.12c)$$

One immediate result for the case of QED with linear gauge conditions is that vertex functions with only external photon lines are gauge independent to all loop orders, for $D_{\alpha j}^{ij}$ is non-zero only for fermionic fields and apart from this no ghost vertices exist, hence one cannot build diagrams for δX_{jk}^{ij} which are $1\pi i$ (with exception of the basic ghost line) as required. Notice that this does not hold as soon as external fermion lines are involved due to the field dependence of D_α^{ij} when i corresponds to a fermionic field.

2. Gauge Invariance and Gauge Independence of Hard Thermal Loops

2.1. HIGH TEMPERATURE QCD

Most recently it has been observed by Braaten and Pisarski⁸ and Frenkel and Taylor⁹ that in high-temperature QCD the leading temperature contributions to 1-loop vertex functions, the "hard thermal loops"⁸, obey tree-level-type ghost-free Ward identities, and moreover appear to be gauge fixing independent. The latter (which does not already follow from the former!) has been checked for general covariant gauges, the Coulomb gauge and axial gauges. However, this gauge independence did not arise in an obvious way and had to be verified gauge by gauge so far.

As will be shown, the gauge dependence identities derived in the previous sections can be used to verify this gauge independence (and as a corollary also the gauge invariance) of these hard thermal loops in a very simple manner. The only further ingredients needed are the temperature-power-counting rules as given in Ref. 8, which, roughly, read as follows:^{*} in a Feynman diagram, loop momenta in the numerator give a factor T , each propagator $1/\square$ gives T^{-1} , and the integration over the loop momentum (including summation over the Matsubara frequencies) contributes T^3 unless there are two or more propagators and all of them of the same statistics, in which case one only receives T^2 .

By this, the leading temperature contribution to a 1-loop vertex function in QCD (or QED) goes like T^2 . Vertex functions with more than two external fermion lines, however, do not contribute to this order.

^{*}These rules pertain to the imaginary time formalism. For the (presumably unnecessary) complications in the real time formalism see the contributions of T. Evans, R. Kobes, and H. Nakkagawa et al. to this volume.

Restricting ourselves to vertex functions with only external gluons, all of the potential gauge dependence of hard thermal loops is now contained in the 1-loop contributions to δX_{jk}^{ij} , whose diagrammatic structure is

$$\sum_{\text{Perm.}} \left\{ \begin{array}{c} \text{diagram 1} \\ + \\ \text{diagram 2} \end{array} \right\} \quad (2.1)$$

Starting from Feynman gauge and considering variations of the gauge condition which are at most linear in derivatives (hence do not bring in inverse mass parameters which would complicate the power-counting), but arbitrary otherwise, the gauge independence of the hard thermal loops is seen immediately, without the need to investigate the tensorial structure of the loop integrals in great detail:

The diagrams (2.1) with m gluon vertex insertions contain $(m+2)$ propagators, m vertices which are at most linear in the loop momentum and one F_i^{α} -insertion which by hypothesis adds at most one further loop momentum. For $m > 0$, this adds up to $\leq 2 - (m+2) + m+1 = 1$ in powers of T . Without any gluon vertex inserted, $3 - 2 + 1 = 2$ powers of T appear to be possible, but such a diagram vanishes in virtue of the unbroken global color symmetry. Thus gauge dependences can occur only in the subleading order $\propto T$.

This argument is not changed when background gauges are considered, the only difference being that gluon vertices can now also be inserted into F_i^{α} itself. Gauge invariance (i.e. tree-level-type Ward identities) is thus a corollary of gauge independence, as the former can always be achieved through background field gauges.

Although these considerations do not yield additional physical insight into why the off-shell hard thermal loops have to be gauge independent, they at least simplify the derivation of this result.

2.2. PERTURBATIVE QUANTUM GRAVITY AT FINITE TEMPERATURE

In finite temperature quantum gravity, the actual high-temperature limit cannot be studied by perturbation theory, one instead has to require that $T \ll G^{-1/2} \sim m_{\text{Planck}}$, in order that higher loop orders fall off by powers of $GT^2 \ll 1$. However, since collective phenomena generally occur for momenta small compared to the temperature, the corresponding calculations in quantum gravity turn out to be technically a high-temperature expansion. Although the presence of a plasma necessarily induces curvature, a flat-space perturbation theory is again applicable in order to determine the leading temperature contributions for temperatures below the Planck temperature (which is $\sim 10^{32} K$), because then $R^{\mu\nu} \sim GT^4 \ll T^2$.

Recently I have performed the calculation of the hard thermal loops for 1- and 2-point graviton vertex functions in Ref. 10. There the graviton loop constitutes a potential source of gauge dependence, and to check gauge independence by explicit calculations would be quite tedious. However, the analogous reasoning as above turns out to work also in this case.

Starting from Feynman-de Donder gauge (background covariant or not), the main difference to the case of QCD is that vertices among gravitons always contribute up to 2 loop momenta and so do the graviton-ghost vertices. This entails that at 1-loop order the leading temperature behaviour according to the superficial power-counting laws of Ref. 8 is

$$\Gamma_{(1)} \sim T^4, \quad \Gamma_{(n)} \sim T^{4+(n-2)}, \quad n \geq 2. \quad (2.2)$$

As in QCD,⁸ the leading order diagrams for $n \geq 3$ are built from 3-vertices only, loops with higher vertices being suppressed by powers of T .

The argument for gauge independence of the leading temperature contributions of 1-loop vertex functions is almost as simple as before. For the gauge variation of an n -point 1-loop vertex function, one has to consider again the diagrams (2.1) with $m \leq (n-1)$ vertex insertions, replacing gluons by gravitons. The difference to QCD is just that vertices add twice as many loop momenta, and now also the structure functions in $D_{\alpha\beta}^i$ contribute one loop momentum. This gives $(2+\delta_{0m})-(m+2)+2m+1+1 = 3 \dots 3+(n-2)$ powers of T for the potential gauge dependences, which are thus down by one power of T in comparison with (2.2). In contrast to QCD, the case $m = 0$ fits into this power counting argument, and indeed tadpole-type diagrams do not necessarily vanish in quantum gravity.

Again, since tree-level-type Ward identities hold in background covariant gauges, they generally hold for the gauge independent leading temperature parts. The leading contributions to the 1- and 2-point graviton vertex functions (both $\sim T^4$) are thus related by

$$\Pi_{\alpha\beta\mu\nu}(Q)(\eta^{\mu\nu}Q^\nu + \eta^{\nu\epsilon}Q^\mu) = -\frac{1}{2}T_{\mu\nu}(\delta_{\alpha\beta}^{\nu\epsilon}Q^\mu + \delta_{\alpha\beta}^{\mu\epsilon}Q^\nu - \delta_{\alpha\beta}^{\mu\nu}Q^\epsilon), \quad (2.3)$$

which has been found by explicit calculation in Feynman-de Donder gauge in Ref. 10. Because the higher vertex functions have a different temperature behaviour, see (2.2), the divergence of their leading contributions vanishes rather than being related to a lower vertex function,

$$\Gamma_{\alpha_1\beta_1\dots\alpha_n\beta_n}(Q_1, \dots, Q_n)(\eta^{\alpha_1\epsilon}Q^{\beta_1} + \eta^{\beta_1\epsilon}Q^{\alpha_1}) = O(T^{3+(n-2)}). \quad (2.4)$$

(Because the leading part of $\Gamma_{(n)}$ turns out to be completely symmetric in its indices, its divergence vanishes with respect to each of the external momenta.)

The Ward identities in finite temperature quantum gravity thus have the opposite consequences as in QCD: in the latter the gluon self-energy is transverse and the divergence of higher vertex functions can be expressed through lower vertex functions, whereas in gravity the thermal graviton self-energy is nontransverse, its divergence being related to the thermal energy-momentum tensor, but the divergence of the leading parts of higher vertex functions is constrained to vanish.

In QCD the Ward identities entail that up to an overall factor gluons and quarks give rise to the same polarization tensor (and in fact also to the same n -point gluon vertex functions)⁸

$$\Pi^{\mu\nu}(Q) \propto I^{\mu\nu} - \frac{1}{2}\delta^{\mu\nu}\delta_{\alpha\beta}I^{\alpha\beta}, \quad I^{\mu\nu} = \int \frac{d^4K}{(2\pi)^4} \frac{K^\mu K^\nu}{K^2(K-Q)^2}. \quad (2.5)$$

A similar thing happens in gravity: all the different (massless) fields contribute the same gravitational polarization tensor, the differing overall factors corresponding to the different energy densities associated with these fields. This universal result can be given in terms of the integral

$$I^{\mu\nu\rho\sigma}(q_0, \vec{q}) = \int \frac{d^4K}{(2\pi)^4} \frac{K^\mu K^\nu K^\rho K^\sigma}{K^2(K-Q)^2} \quad (2.6)$$

and reads

$$\Pi^{\alpha\beta\mu\nu} = \frac{1}{2}I^{\alpha\beta\mu\nu} - \frac{1}{4}(\delta_\rho^\alpha\delta_\rho^\mu\eta^{\nu\beta} + \delta_\rho^\alpha\delta_\rho^\nu\eta^{\mu\beta} + \delta_\rho^\beta\delta_\rho^\mu\eta^{\nu\alpha} + \delta_\rho^\beta\delta_\rho^\nu\eta^{\mu\alpha})\eta_{\gamma\delta}I^{\gamma\delta\sigma\rho}, \quad (2.7)$$

omitting contractions of I which give zero. It can in fact be shown that the combination (2.7) is the only one satisfying the identity (2.3).

3. Gauge Independence of QCD Plasma Dispersion Relations

3.1. GAUGE DEPENDENCE IDENTITIES FOR THE GLUON PROPAGATOR

In linear response theory, the spectrum of collective phenomena of a finite temperature field theory is read from the poles of the propagators. In particular the dispersion relations of colour oscillations in a quark-gluon plasma, which is expected to exist at sufficiently high temperatures, are contained in the finite temperature gluon propagator.

Since Lorentz symmetry is broken by the presence of the plasma with rest-frame velocity n_μ , there are now four structure functions in the full gluon propagator, unless a non-covariant choice of gauge further reduces the 3-dimensional symmetry:

$$\Delta_{\mu\nu}(Q) = \frac{1}{a(Q)}A_{\mu\nu} + \frac{1}{b(Q)}B_{\mu\nu} + \frac{1}{c(Q)}C_{\mu\nu} + \frac{1}{d(Q)}D_{\mu\nu} \quad (3.1)$$

in momentum space ($Q_\mu = (q_0, \vec{q}_i)$) and omitting colour indices, where we employ the following tensorial basis,

$$A_{\mu\nu} \equiv P_{\mu\nu} - \frac{\tilde{n}_\mu \tilde{n}_\nu}{\tilde{n}^2}, \quad (3.2a)$$

$$B_{\mu\nu} \equiv \frac{\tilde{n}_\mu \tilde{n}_\nu}{\tilde{n}^2}, \quad (3.2b)$$

$$C_{\mu\nu} \equiv \tilde{n}_\mu Q_\nu + Q_\mu \tilde{n}_\nu, \quad (3.2c)$$

$$D_{\mu\nu} \equiv \frac{Q_\mu Q_\nu}{Q^2}, \quad (3.2d)$$

with $\tilde{n}_\mu \equiv P_{\mu\nu}n^\nu$, $P_{\mu\nu} \equiv g_{\mu\nu} - \frac{Q_\mu Q_\nu}{Q^2}$.

In (3.1) only $A_{\mu\nu}$ and $B_{\mu\nu}$ are transverse with respect to the 4-vector Q_μ , and they correspond to spatially transverse and longitudinal modes, respectively.

Decomposing the bare inverse propagator according to

$$[\Delta_{(0)}^{-1}]_{\mu\nu} = Q^2 P_{\mu\nu} + f_0 B_{\mu\nu} + f_c C_{\mu\nu} + f_d D_{\mu\nu}, \quad (3.3)$$

where the gauge fixing term is parametrized by the f 's, which are constrained by $f_b f_c = \tilde{n}^2 Q^2 f_c^2$, and writing the self-energy corrections as

$$\Pi_{\mu\nu}(Q) = \Pi_a(Q)A_{\mu\nu} + \Pi_b(Q)B_{\mu\nu} + \Pi_c(Q)C_{\mu\nu} + \Pi_d(Q)D_{\mu\nu}, \quad (3.4)$$

the structure functions of the full propagator are given by¹¹

$$a(Q) = Q^2 + \Pi_a, \quad (3.5a)$$

$$b(Q) = Q^2 + \Pi_b + \tilde{n}^2 Q^2 \left[\frac{f_c^2}{f_d} - \frac{(f_c + \Pi_c)^2}{f_d + \Pi_d} \right], \quad (3.5b)$$

$$c(Q) = -\frac{f_d + \Pi_d}{f_c + \Pi_c} b(Q), \quad (3.5c)$$

$$d(Q) = \frac{f_d + \Pi_d}{Q^2 + \Pi_b + f_b} b(Q). \quad (3.5d)$$

At the scale $Q \sim gT$, the dispersion relations are completely determined by the leading temperature contributions to the 1-loop self-energy, the "hard thermal loops", which according to the discussion of the previous section are gauge covariant and gauge fixing independent. Gauge covariance of the self-energy implies $\Pi_c = \Pi_d = 0$, and gauge independence of the remaining Π_a and Π_b is thus sufficient for gauge independence of the leading order dispersion relations $a(Q) = 0$ and $b(Q) = 0$ which have been studied extensively in the literature^{12,13,14}. In fact, at leading order the entire functions $a(Q)$ and $b(Q)$ are gauge independent, hence also for example the residues of the dispersion relations¹⁵.

Beyond leading order, however, one cannot expect gauge covariance and gauge independence of $\Pi_{\mu\nu}$, and explicit calculations exhibit that gauge dependence starts at order T , which persists at one-loop order even at the location of the plasma dispersion relations and which has given rise to the puzzle of a seemingly gauge dependent plasmon damping constant¹. Much of the ensuing work has concentrated on the quest for a manifestly gauge independent substitute for the inherently gauge dependent gluon propagator, with the result that even different gauge independent schemes have yielded different answers. However, as Pisarski and others^{8,16,17} have recently demonstrated, the real problem is that the ordinary loop expansion is not complete and needs resummation of the leading temperature corrections.

This conclusion can also be drawn by applying the gauge dependence identity, eq. (1.6), to the issue of potential gauge dependences of the plasma dispersion relations⁴.

Returning for the moment to the condensed notation of the previous sections, the gauge dependence of the propagator Δ^{ij} , $\Delta^{ijT}_{jk} = -\delta^i_k$, is related to the gauge dependence of the self-energy through

$$\delta\Delta^{ij}|_{J=0} = \Delta^{im}\delta\Gamma_{mn}\Delta^{nj}|_{J=0}. \quad (3.6)$$

Differentiating the basic relation (1.6)* twice at fixed $J_i = 0$,

$$\begin{aligned} \delta\Gamma_{mn}|_{J=0} &= (\Gamma + \delta\Gamma)_{mn}|_{(\Gamma+\delta\Gamma)_{j=0} - \Gamma_{mn}|_{\Gamma_j=0}} \\ &= (\Gamma_{in}\delta X^i_{,m} + \Gamma_{im}\delta X^i_{,n})|_{\Gamma_j=0} \end{aligned} \quad (3.7)$$

gives

$$\delta\Delta^{ij}|_{J=0} = -(\Delta^{im}\delta X^j_{,m} + \delta X^i_{,m}\Delta^{mj})|_{J=0}. \quad (3.8)$$

Specifying to QCD again and extracting the transverse structure functions from (3.8), one obtains

$$\delta a(Q) = -a(Q)A^b_\nu(Q)\delta X^\nu_{,\mu}(Q) \equiv a(Q)\delta Y(Q) \quad (3.9)$$

for the gauge dependence of the spatially transverse structure function, and

$$\delta b(Q) = -2b(Q)\left[\frac{\tilde{n}^\mu}{\tilde{n}^2} - \frac{f_c + \Pi_c}{f_d + \Pi_d}Q^\mu\right]\tilde{n}_\nu\delta X^\nu_{,\mu} \equiv b(Q)\delta Z(Q) \quad (3.10)$$

for that of the spatially longitudinal one.

Now if $\delta Y(Q)$ and $\delta Z(Q)$ are well-behaved at the location of the plasma dispersion relations, these particular gauge dependence identities imply that the solutions to $a(Q) = 0$ and $b(Q) = 0$, which determine the dispersion relations, are also solutions of $(a + \delta a)(Q) = 0$ and $(b + \delta b)(Q) = 0$, hence gauge independent. No such relations, by the way, hold for the nontransverse structure functions c and d .

The pole structure of δY and δZ is determined by $\delta X^\nu_{,\mu}$ and in the case of δZ in addition by a simple pole $1/Q^2$ contained in the $\tilde{n}$'s as well as by possible poles of $1/(f_d + \Pi_d)$. Thus one has to exclude these two kinematical poles from the gauge independence proof for the spatially longitudinal mode b . Indeed, $b(Q)$ turns out to be proportional to Q^2 , and this factor cancels in the Coulomb gauge propagator, but not in the covariant one, and also the potential pole $1/(f_d + \Pi_d)$ is evidently a gauge artefact, as it is gauge dependent already at the classical level.

The more important question is whether the potential pole structure of $\delta X^\nu_{,\mu}(Q)$ could possibly coincide with the expectantly physical dispersion relations given by $a(Q) = 0$ and $b(Q)/Q^2 = 0$. The diagrammatic representation of $\delta X^\nu_{,\mu}$ is obtained by inserting one vertex in all possible ways into the primary diagrams of eq. (1.10), where all resulting tadpole-like diagrams vanish due to unbroken global colour symmetry. Since the primary diagrams were 1-particle reducible only with respect to the basic ghost line attached to δF^a_μ , the diagrams for $\delta X^\nu_{,\mu}$ are built from ghost propagators and their radiative corrections carrying the external momentum Q , and, besides that, from 1pi subdiagrams whose internal propagators are off the (plasmon) mass shell even when the external momentum is on it. Hence potential gauge

*Strictly speaking one should not subtract the tree-level gauge fixing term from $\Gamma[\bar{\varphi}]$ here, because Γ_{ij} would not remain perturbatively invertible. Consequently in what follows δX^i also contains diagrams with vertices $\delta F^a_\mu[\bar{\varphi}] \neq 0$. See Ref. 4 for more details.

dependences in the solutions of $a(Q) = 0$ and $b(Q)/Q^2 = 0$ are those given by the poles of the (full) ghost propagator. According to the discussion of the previous sections, at finite temperature the ghost propagator as well as the 1pi-subdiagrams of $\delta X''_\mu$ receive no leading temperature corrections $\sim g^2 T^2$, so their dispersion relations will be independent of the physical ones given by transverse structure functions of the gluon propagator, whose leading order is $Q^2 \sim g^2 T^2$. This renders plausible the assumption that δY and δZ are well-behaved at the location of the physical plasma dispersion relations.

The identities (3.9) and (3.10) also reveal that the gauge dependence found in the one-loop calculations at sub-leading order is in fact to be expected. Truncating at one-loop order, $\delta a(Q)$, e.g., is determined by

$$\delta a^{(1)}(Q) = a^{(0)}(Q) \delta Y^{(1)}(Q) \quad (3.11)$$

($\delta Y^{(0)}$ vanishes identically). However, the leading order of the plasma dispersion relations is given by $a^{(0)} + a^{(1)} = 0$, and the temperature corrections in $a^{(1)}$ cannot be considered small compared to $a^{(0)}$, but rather set the scale for the dispersion relations. Since $\delta Y \sim g^2 T$, this implies that the r.h.s. of eq. (3.11) does not vanish at the order $g^2 T$, where one is looking for plasmon damping.

As Pisarski^{18,8} has pointed out, the conventional loop expansion has to be substituted by an expansion which resums all the hard thermal loops $\sim g^2 T^2$, so that it is an expansion not in g alone, but in the excess of g 's over T 's. This program has been successfully completed in Ref. 16 within Coulomb gauge, and the above argument shows that this result can be regarded as a final, gauge independent answer, and indeed gauge independence has been checked explicitly in a certain class of gauges in Ref. 8.

3.2. PARTIALLY RESUMMED HIGH- T QCD

Gauge independence of the resummed loop expansion is a natural consequence of the fact that there corrections to $a^{(0)}$ and $b^{(0)}$ are indeed mere corrections, making the r.h.s. of (3.11) vanish at the resummed 1-loop order.

In order to demonstrate this in an explicit calculation, it already suffices* to consider a partial resummation which shifts the lowest order mass shell to the location of the dispersion relation where one is evaluating the damping part.

*Only with respect to gauge independence, but not as concerns the correctness of the result – one important lesson to be drawn from the whole controversy around the ‘correct’ definition of the plasmon damping constant¹ is that gauge independence does not yet imply physicality, the former being only necessary but not sufficient. Hence working in a manifestly gauge independent framework like e.g. the Vilkovisky-DeWitt effective action^{3,19} or Cornwall’s ‘gauge invariant propagator’²⁰, can have the drawback of having deprived oneself of the potentially useful check of cancelling gauge parameters. Evidently also all the manifestly gauge independent frameworks still require complete resummation of the leading temperature corrections.

Calculating for instance the plasmon damping constant at $\vec{q} = 0$, such a partial resummation is achieved by¹⁷

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{4} F_{\mu\nu} \frac{m_{pl}^2}{D(A)^2} F^{\mu\nu} (1 - \ell) \quad (3.12)$$

and identifying ℓ with a loop counting parameter which is set to one only after having performed the loop expansion to a given loop order. Nonperturbatively, nothing has been changed, but the Feynman rules, which are defined at $\ell = 0$, now contain temperature contributions which are compensated only one loop higher and thus reshuffle the old loop expansion. Because (3.12) is gauge invariant, gauge fixing proceeds as usual. In covariant gauge the propagator now reads

$$\Delta_{\mu\nu}^{(0)}(Q) = \frac{-1}{Q^2 - m_{pl}^2} \left(g_{\mu\nu} - \frac{Q_\mu Q_\nu}{Q^2} \right) - \alpha \frac{Q_\mu Q_\nu}{Q^4}, \quad (3.13)$$

and also the vertices are modified and contain non-local pieces proportional to m_{pl}^2 .

Performing the redefined one-loop calculation and extracting only the gauge parameter dependent pieces of the structure function b , one finds¹⁷

$$\begin{aligned} & i C_A g^2 \int_T d^4 K \left\{ \alpha^2 (Q^2 - m_{pl}^2)^2 \frac{[k_0 Q^2 - q_0 K \cdot Q]^2}{q^2 Q^2 K^4 (Q - K)^4} \right. \\ & + \alpha \frac{Q^2 - m_{pl}^2}{q^2 Q^2 (Q - K)^4} \left[q_0^2 \frac{k^2 - (q \cdot k)^2}{k^2} \left(1 - \frac{Q^2 - m_{pl}^2}{K^2 - m_{pl}^2} \right) \right. \\ & \left. \left. + \left(\frac{q_0^2 (q \cdot k)^2}{k^2} - q^4 \right) \frac{K^2 - Q^2}{K^2 - m_{pl}^2} + \frac{[k_0 Q^2 - q_0 K \cdot Q]^2 2K^2 - Q^2 - m_{pl}^2}{K^2 - m_{pl}^2} \right] \right\} \\ & = 0 \quad \text{at } q_0 = m_{pl}, \vec{q} = 0, \end{aligned} \quad (3.14)$$

so gauge independence arises at the location of the plasmon pole with $\vec{q} = 0$ already before performing the integration and the thermal sum implicit in $\int_T d^4 K$. Gauge dependences remain, however, for $\vec{q} \neq 0$, where due to the oversimplification of (3.12) one is still off the plasmon dispersion relations, which are not just given by a mass hyperboloid.

Although a complete resummation has been achieved by now¹⁶, this example of a partial resummation gives a good illustration of how gauge independence works out in accordance to the gauge dependence identities (3.9) and (3.10).

4. Derivation of Dispersion Relations in Quantum Gravity at Finite Temperature

In the case of gravity at finite temperature there has been some controversy on the correct definition of dispersion relations even at the level of the leading temperature corrections^{21,22,23,24}, in particular on the issue of the correct definition of the Jeans mass within thermal perturbative quantum gravity, which is expected as the analogue of the electric Debye mass, but as giving rise to instabilities of spatial inhomogeneities instead of spatial screening.

To this end only the static limit of the thermal graviton self-energy has been considered hitherto. In Ref. 21 the static limit of Π_{0000} was interpreted as giving the Jeans mass, but there it was assumed that Π was transverse so that the only nonvanishing static component of Π would have to be just this 0000-component. However, this is not the case as has been shown above, eq. (2.3), because the 1-point function, the energy-momentum tensor associated with the plasma, does not vanish. Correspondingly, it has been shown later by the authors of Ref. 22, that Π_{0000} by itself is dependent on the parametrization of the gravitational field (although its leading temperature contributions are gauge independent, as has been verified above). They instead proposed to focus on the 0000-component of the graviton propagator, but again due to the nonvanishing 1-point function, difficulties appeared which forced them to just neglect contributions from the 1-point function to the propagator, which of course cannot be correct.

In Ref. 23, 24 it was later pointed out that the equations for small disturbances obtained from the Einstein equations are well defined also at finite temperature, in contrast to the perturbative graviton propagator. These are given by

$$\left\{ \begin{aligned} & \frac{1}{2} (g^{\alpha\beta} g^{\mu\nu} - g^{\alpha(\mu} g^{\nu)\beta}) \nabla^2 - \frac{1}{2} (g^{\alpha\beta} \nabla^{\mu} \nabla^{\nu}) + g^{\mu\nu} \nabla^{\alpha} \nabla^{\beta} \\ & + g^{\alpha(\mu} \nabla^{\nu)} \nabla^{\beta} - C^{\alpha(\mu\nu)} \beta + \frac{1}{12} g^{\alpha\beta} g^{\mu\nu} R - \frac{1}{3} g^{\alpha(\mu} g^{\nu)\beta} R \\ & - 16\pi G \Pi^{\alpha\beta\mu\nu} (\nabla^0, \vec{\nabla}) \} \delta g_{\alpha\beta} = 8\pi G \delta T^{\mu\nu}, \end{aligned} \right. \quad (4.1)$$

where $C^{\alpha\mu\nu\beta}$ is the conformal curvature (or Weyl) tensor.

In order to make these equations invertible, one needs to fix the diffeomorphism invariance, which because of the Ward identity (2.3) is not broken by the leading temperature contributions to the thermal graviton self-energy $\Pi^{\alpha\beta\mu\nu}$. Hence one can, exactly as in the classical theory, require that the perturbation $\delta g_{\alpha\beta}$ be transverse and traceless,

$$\nabla^{\alpha} \delta g_{\alpha\beta} = 0, \quad g^{\alpha\beta} \delta g_{\alpha\beta} = 0. \quad (4.2)$$

By this and also using the Einstein equations and assuming a vanishing conformal curvature tensor, which is undetermined by the Einstein equations, eq. (4.1) reduces to

$$\left\{ \frac{1}{2} g^{\alpha(\mu} g^{\nu)\beta} \nabla^2 + 16\pi G \Pi^{\alpha\beta\mu\nu} \right\} \delta g_{\alpha\beta} = -8\pi G \delta T^{\mu\nu}. \quad (4.3)$$

The thermal graviton self-energy thus indeed acts as a (non-constant) mass matrix for small perturbations $\delta g_{\alpha\beta}$, but because of the constraints (4.2), it still has to be decomposed

into transverse-traceless parts and parts dropping out from eq. (4.3).

This fact has not been taken into account at all in Ref. 23, whereas in Ref. 24, because of their restriction to a zero momentum limit, the first constraint in (4.2) could not be covered.

At finite temperature there are now three transverse-traceless modes (instead of the two transverse ones in the QCD case). In Ref. 10 I have found that in the static limit only the structure functions corresponding to the spatially longitudinal mode develops a Jeans mass as expected, but the full dynamical case, while being to some extent similar to that of a QCD plasma, turned out to be quite intricate. As in the QCD case, there is a minimal plasma frequency above which the plasma can support propagating normal modes, but below the plasma frequency the two extra collective modes of the graviton plasma, which have no analogue in the $T=0$ case, are not screened in the static limit, whereas the spatially transverse-traceless mode, which is similar to ordinary gravitational waves, is screened – just opposite to the situation in hot QCD. The static Jeans mass moreover just marks the onset of instabilities in the two extra collective modes; also the extra mode which does not exhibit a static Jeans mass develops instabilities beyond large enough wavelengths of spatial inhomogeneities.

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