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Model Independent Fitting to the Z Line Shape

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Abstract

An improved model independent formula to fit the Z line shape parameters is presented. The accuracy of the QED radiative corrections used in the formula is estimated, as is the absolute accuracy of the formula itself. The precision of the formula is compared with existing fitting formulae, in terms of cross sections produced and in terms of the Z parameters returned from a fit. The errors which the LEP-SLC experiments can expect to achieve on the Z parameters are examined as a function of total integrated luminosity. The formula presented here can fit the Z parameters with high accuracy: below 1 MeV for the Z mass, the order of 1 MeV for total width, and below 0.04% for the hadronic and leptonic peak cross sections. Such biases will remain much below the expected experimental uncertainties from LEP and SLC.

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1 Introduction

The use of model independent formulae to describe the Z resonance was suggested during the preparation for LEP-SLC physics [1]. When fitting to experimental data, a model independent formulation is a clean way of extracting the relevant physical information of the Z resonance. The reasons for adopting a model independent formulation are twofold:

- Tests of the Standard Model may be biased if one assumes its validity from the beginning.
- By assuming only very basic theoretical concepts, one is able to analyze the results in terms of new theoretical models.

The basic idea of the model independent formulae, as we will show, is the fact that for final states other than electrons, the Z line shape, before considering initial state radiation, is well described by a Breit-Wigner resonance, with s -dependent width. Near the peak (say ± 3 GeV around the Z pole), the Z resonance dominates the photon exchange and Z -photon interference contributions, though these smaller terms will be included in our analysis.

It is well known [2] that initial state radiation plays an important role and must also be taken into account. Our method, following Burgers [3], will be to numerically integrate the Breit-Wigner (with photon and interference terms) over the initial state radiation spectrum. As we shall see, accurate treatment of initial state radiation is necessary to obtain precise results on the fit parameters, in particular the Z mass.

Up to now, wide use of the different suggested calculations has been made by all the experiments and exciting results have been published [4]. At the level of statistics of the results published so far (which come from the data taken in 1989), the error on the Z total width from a single experiment is of the order of 50 MeV, and the error (aside from errors due to uncertainty in the total energy scale of LEP) on the Z mass is about 25 MeV. At this level of statistics existing formulae are sufficient for extracting the resonance parameters; the theoretical uncertainty using any of the formulae suggested in the literature is much smaller than the statistical one.

In the future, however, some $10\text{-}50\text{ pb}^{-1}$ of data are expected for scanning the Z at LEP. As we will show, at these high statistics the shortcomings of existing approximations start to become apparent. This motivated us to study possible upgrades of the existing programs and has led us to produce a FORTRAN program (which we call **MIZA**, Model Independent Z Approximation) which is a closer approximation to the resonance shape.

The paper is organized as follows. In Section 2 we examine an existing model independent formula for the Z line shape (which is the starting point for our formula), we discuss the improvements which we make to that formula, and we present our final expression. In Section 3 we estimate the accuracy of the QED calculation used in our fitting function. In the final two sections we evaluate the accuracy of our formula and two analytic functions for Z fitting, and we estimate the precision that the LEP-SLC experiments can expect to achieve on the Z parameters in future running near the resonance.

2 The New Model Independent Formula

2.1 Examination of Existing Formula

The new formula we will present here can be considered as an improved version of the expression suggested in the ‘ Z PHYSICS AT LEP I’ [2]. In that approach the non-radiative electroweak corrected total cross section for massless fermions at the centre-of-mass energy squared s is written as,

$$\sigma_1(s) = \frac{s}{(s - M_Z^2)^2 + s^2 \Gamma_Z^2 / M_Z^2} \left[\frac{12\pi \Gamma_{e\bar{e}} \Gamma_{f\bar{f}}}{M_Z^2} + \frac{I(1 + \delta_{QCD})(s - M_Z^2)}{s} \right] + \frac{4\pi}{3} Q_f^2 N_c \frac{\alpha^2(M_Z^2)}{s} (1 + \delta_{QCD}) \quad (1)$$

where N_c is the number of colours, and δ_{QCD} is the QCD correction factor¹.

In this expression, $\Gamma_{e\bar{e}}$, $\Gamma_{f\bar{f}}$, Γ_Z and M_Z should be understood as the actual physical parameters which are determined by fitting the data. Two different sorts of approximations are implicit in this expression:

- Due to the factorization implied by using the product of partial widths, some contributions cannot be naturally included. For this non-radiative expression, these contributions can originate only from heavy box diagrams, and their size has been computed to be less than a few $\times 10^{-3}\%$ [5].
- The only s dependence, kept in the corrected currents which have been replaced by width products, is linear. The rest is evaluated at $\sqrt{s} = M_Z$.

The pure photon exchange term is written using only QED. In principle the magnitude of this term could be determined from the data. In practice the photon term is evaluated using the known fermion charges and color factors. Since the QED sector is well known, the model independence of the formula is not jeopardized. In fact the photon term may be considered a known ‘background’ to the Z .

The factor I of the interference term can be treated in two different ways: it can be considered an additional physical parameter containing information independent from the four parameters mentioned above, in which case it should be fit from the data; or one can try to obtain I from a deeper insight into the theory.

In the second approach, there is no way of writing I in terms of the four physical parameters. Therefore one must actually break the model independence of the whole formula and specify a model for writing it. Within the Standard Model, the expression for I is, in leading terms,

$$I = \frac{-4\pi Q_f N_c \alpha^2(M_Z^2)}{6} (g_R^e + g_L^e)(g_R^f + g_L^f), \quad (2)$$

where the couplings g_R and g_L are

$$\begin{aligned} g_L^f &= \frac{I_f^3 - Q_f \sin^2 \theta_W}{\sin \theta_W \cos \theta_W}, \\ g_R^f &= \frac{-Q_f \sin^2 \theta_W}{\sin \theta_W \cos \theta_W} \end{aligned} \quad (3)$$

$\sin^2 \theta_W$ [6] [7] [8] being the effective weak mixing angle at M_Z^2 . In the Standard Model the effective mixing angle can be given by solving the following equation

$$\Gamma_{ll} = (1 + \kappa) \frac{\alpha(M_Z) \{1 + (1 - 4 \sin^2 \theta_W)^2\}}{48 \sin^2 \theta_W \cos^2 \theta_W} \cdot M_Z \quad (4)$$

where Γ_{ll} is the leptonic partial width and $\kappa = 0.2 \pm 0.3\%$ represents the additional electroweak effects [8].

¹We have changed the formula slightly by removing the QCD correction from the Z term. In this way the widths appearing are the fully corrected physical widths.

Process	$\sqrt{s}$ [GeV]	σ_Z [nb]	(%)	$\sigma_{interf.}$ [nb]	(%)	σ_γ [nb]	(%)
Hadrons	$M_Z - \Gamma_Z$	5.94	99.12	-0.031	-0.51	0.083	1.4
	M_Z	30.35	99.78	-0.011	-0.04	0.079	0.26
	$M_Z + \Gamma_Z$	10.03	99.02	0.024	0.24	0.075	0.74
Muons	$M_Z - \Gamma_Z$	0.283	92.93	-0.0007	-0.21	0.022	7.28
	M_Z	1.445	98.58	-0.0002	-0.016	0.021	1.44
	$M_Z + \Gamma_Z$	0.477	95.88	0.0005	0.10	0.020	4.02

Table 1: The Standard Model prediction for the hadron and muon production cross section taking $M_Z = 91.2$ and the effective $\sin^2 \theta_W = 0.23$. The values are from ZSHAPE [9].

For realistic $\sin^2 \theta_W$, the interference I is positive; and near the peak the whole term amounts to a very small fraction of the total cross section (see Table 1). Therefore, one way to treat the term is to compute I for any reasonable value of $\sin^2 \theta_W$ and consider the variation in the total cross section due to the difference with the actual value of $\sin^2 \theta_W$ as just an uncertainty of the formula. Since this variation is small, we can say that the model independence is only very softly broken. We will return to this point later.

Having discussed the existing formula, we now list the upgrades to that formula, which we will describe in detail in the following sections. As we will see, these modifications actually amount to all the basic improvements which can be made within a model independent framework:

1. Running the QED coupling constant
2. Avoiding double counting of QED corrections due to the use of physical widths
3. Applying the correct treatment of QED and QCD final state corrections

2.2 Running α

In all the formulae suggested so far, the QED coupling constant α which appears in the photon exchange term is taken as the effective value at the Z pole, i.e. $\alpha(M_Z^2) = 1/128.79$. While this is a very good approximation when the momentum-transfer-squared Q^2 of the non-radiative reaction $e^+e^- \rightarrow f\bar{f}$ is close to the Z pole, it does introduce a large difference when running at low Q^2 . To understand quantitatively how important it is to run α we want to compare the model independent kernel against as complete an electro-weak calculation as exists [9], [10]. The calculations from [9] and [10] agree to a fraction of a percent [11]. In Fig. 1 we show the relative difference between the kernel of the formula suggested in the [2] (equation (1) in this paper), σ_1 , and the equivalent non-radiative cross section from the Standard Model calculation σ_{SM}^2 . The dashed curve shows the ratio

$$\Delta K = \frac{\sigma_1 - \sigma_{SM}}{\sigma_{SM}} \quad (5)$$

for equation (1), i.e. the original kernel of [2]. The dotted curve shows ΔK after correcting σ_1 for running the QED constant. (We discuss the third line in Fig. 1 later.) The agreement is good for a wide range of the centre-of-mass energies.

The important point is that even if we want to study the e^+e^- interaction near the Z peak, initial state radiation lowers the effective center of mass energy and the reaction occurs as if we were scanning the non-radiative region from $Q^2 = (2m_f)^2$ to $Q^2 = s$. Of course, the weight for each Q^2 is very small for Q^2 far from s (hard radiation) since it is modulated by the radiator function. Nevertheless, it is clear that after the convolution the effect of running α is still rather

²We compare against ZSHAPE [9], which for top quark masses below about 150 GeV produces the same partial widths and full width as EXPOSTAR [10].

important. If one does not take it into account, the cross section is affected differently at both sides of the peak. This causes systematic shifts in the fit quantities especially in the total width.

Clearly then, the QED coupling constant should run and should be evaluated at s' which is the centre-of-mass energy squared after initial state radiation. The running of α does not actually break the model-independence of our formula since, as is well known, it is completely dominated by light fermion loops which are perfectly described by QED alone. Thus only QED (one of the ingredients of the model-independent approach) has to be assumed. The running QED coupling constant is computed as,

$$\alpha(s) = \frac{\alpha}{1 - \Delta\alpha}, \quad (6)$$

$$\Delta\alpha = \frac{\alpha}{3\pi} \sum_l Q_l^2 \left(\ln \frac{s}{m_l^2} - \frac{5}{3} \right) + \Delta\alpha_{hadrons}^{(5)},$$

where the index l runs over charged leptons and α is the QED coupling constant in the Thomson limit (i.e. $\alpha = 1/137.036$). The contribution from quarks is obtained by using dispersion relations from the low energy $e^+e^- \rightarrow \text{hadrons}$ data and can be parametrized for energies above 20 GeV as:

$$\Delta\alpha_{hadrons}^{(5)} = (0.0288 \pm 0.0009) + 0.002980 \left\{ \ln \frac{s}{s_0} + 0.006307 \left(\frac{s}{s_0} - 1 \right) \right\}, \quad (7)$$

where s_0 is taken to be 92^2 GeV^2 [12].

2.3 Avoiding Initial State QED Correction Double Counting

As we have seen, the pure Z-exchange non-radiative cross section near the peak can be expressed as,

$$\sigma_Z(s) = \frac{12\pi\Gamma_{e\bar{e}}\Gamma_{f\bar{f}}}{M_Z^2} \frac{s}{(s - M_Z^2)^2 + s^2\Gamma_Z^2/M_Z^2} \quad (8)$$

where the widths are the actual physical widths, that is, they include by definition all corrections to all orders. In order to obtain the experimentally observed cross section this expression is convoluted with the QED initial state radiator. Therefore, in practice, QED corrections in the $\Gamma_{e\bar{e}}$ term would be included twice. In order to avoid this problem, the physical width in the previous expression has to be explicitly 'uncorrected' for the QED corrections which it already contains. In practice, this can be done easily by the following substitution

$$\Gamma_{e\bar{e}} \rightarrow \frac{\Gamma_{e\bar{e}}}{(1 + \delta_{QED})} \quad (9)$$

where

$$\delta_{QED} = \frac{3\alpha}{4\pi} \simeq 0.17\%. \quad (10)$$

This change 'uncorrects' only to $O(\alpha)$ but since higher order QED corrections to the width decrease very rapidly, the effect of ignoring them is completely negligible. Not taking into account this double counting of the QED correction would introduce a systematic shift in the cross section of the order of 0.17% which propagates directly to the fit peak cross sections and therefore to the partial widths.

2.4 Final state corrections

As stated before, final state corrections are automatically included in the Z exchange term due to the use of physical widths, but they are not included in the two remaining terms. In order to include these corrections up to $O(\alpha)$, it is enough to multiply these terms by the factor

$$\Delta_{QED}^f = 1 + Q_f^2 \delta_{QED}. \quad (11)$$

For lepton pair production, the interference term can be written as³

$$I_{\bar{l}l} = \frac{-4\pi Q_f \alpha(s) \alpha(M_Z^2)}{6} (g_R^e + g_L^e) \cdot (g_R^f + g_L^f) \cdot (1 + \delta_{QED}). \quad (12)$$

In the case of hadrons, attention has to be paid to the way the sum runs over quark flavours. The flavour-dependent charge appears not just in the correction but also in the uncorrected terms. Therefore the pure photon exchange term for hadrons leads to

$$\sigma_\gamma(s) = \frac{4\pi}{3} \frac{\alpha^2(s)}{s} \left(\frac{11}{3} + \frac{35}{27} \delta_{QED} \right) \quad (13)$$

and in the interference term

$$I_{q\bar{q}} = \frac{-12\pi \alpha(s) \alpha(M_Z^2)}{6} (g_R^e + g_L^e) \sum_{f=u,d,s,c,b} (g_R^f + g_L^f) Q_f (1 + Q_f^2 \delta_{QED}). \quad (14)$$

In the case of hadrons, QCD corrections also have to be included in both the pure photon exchange and interference terms. This is properly done, by including the multiplicative factor Δ_{QCD} obtained [13] by extrapolating the lower energy $e^+e^- \rightarrow$ hadrons measured QCD correction. We have used:

$$\delta_{QCD}(M_Z^2) = 0.040 \pm 0.007, \quad (15)$$

which is equivalent to taking a value of $\alpha_s(M_Z^2) = 0.12 \pm 0.02$ in a second order QCD expansion:

$$\Delta_{QCD}(M_Z^2) = 1 + \delta_{QCD}(M_Z^2) = 1 + \frac{\alpha_s(M_Z^2)}{\pi} + 1.405 \left(\frac{\alpha_s(M_Z^2)}{\pi} \right)^2. \quad (16)$$

The inclusion of these final state corrections (and therefore their uncertainties) has, in practice, a very small effect for the cross section since it affects just the non-leading terms. This has been discussed here mainly for completeness.

2.5 Final expressions

After including all the modifications mentioned above, the explicit expression for the non-radiative cross section for lepton pair production becomes,

$$\sigma(s) = \frac{s}{(s - M_Z^2)^2 + s^2 \Gamma_Z^2 / M_Z^2} \left[\frac{12\pi \Gamma_{e\bar{e}} \Gamma_{\bar{l}l}}{M_Z^2} \cdot \frac{1}{1 + \delta_{QED}} + \frac{I_{\bar{l}l}(s - M_Z^2)}{s} \right] + \frac{4\pi}{3} \frac{\alpha^2(s)}{s} \cdot (1 + \delta_{QED}), \quad (17)$$

³In this expression one factor of α is evaluated at the Z mass and one is evaluated at $\sqrt{s}$. The factor $\alpha(s)$ comes from the photon amplitude while $\alpha(M_Z^2)$ is from the Z exchange and is just a bookkeeping parameter for the Z couplings. We thank W. Hollik for reminding us of this point.

Figure 1: Comparison of non-radiative cross sections using **ZSHAPE** as a reference: 1) the formulae (1) suggested in the Yellow Report [2] (dashed line), 2) after running the QED coupling constant (dotted line), 3) and after all corrections (solid line).

whereas for hadron production it is

$$\sigma(s) = \frac{s}{(s - M_Z^2)^2 + s^2 \Gamma_Z^2 / M_Z^2} \left[\frac{12\pi \Gamma_{e\bar{e}} \Gamma_{q\bar{q}}}{M_Z^2} \cdot \frac{1}{1 + \delta_{QED}} + \frac{I_{q\bar{q}}(s - M_Z^2)}{s} \cdot (1 + \delta_{QCD}) \right] + \frac{4\pi \alpha^2(s)}{3} \cdot \left(\frac{11}{3} + \frac{35}{27} \delta_{QED} \right) \cdot (1 + \delta_{QCD}). \quad (18)$$

The solid curve in Fig. 1 shows our kernel after all corrections, compared against the **ZSHAPE** kernel. The sizable disagreement for high $\sqrt{s}$ is due to the fact that our program does not treat the W^+W^- loop. Our aim is to produce a program for fitting the Z line shape, and as such **MIZA** is not intended for use far above the peak.

Referring to the above expressions, we see that the non-radiative peak cross section for fermion production from Z exchange at $s = M_Z^2$ is basically

$$\sigma_{peak}^{(0)} = \frac{12\pi \Gamma_{e\bar{e}} \Gamma_{f\bar{f}}}{M_Z^2 \Gamma_Z^2}. \quad (19)$$

We will make this change of variables later when we study fitting with the new expression and with existing formulae.

3 Accuracy of the QED corrections

As mentioned above, to obtain the observed cross section, the non-radiative kernel must be convoluted with the initial state radiator function. In order to obtain the most accurate approach, full second order QED radiative corrections with exponentiation of the leading infrared and virtual corrections has been performed [3]. This point is of paramount importance due to the

large effect of the initial state QED corrections. The accuracy of the QED corrections essentially determines the total accuracy of the approach.

In order to determine the absolute accuracy of our approach, it is not sufficient to compare the results with an existing calculation, the program **ZSHAPE** [9]. That comparison is not sufficient because both we and **ZSHAPE** use the most accurate QED initial state radiation calculation existing. So, the limit in the accuracy of this part is common to both the calculations.

Therefore, in order to give an absolute answer about the actual precision of our approach, the accuracy of the QED corrections has to be estimated. As usual, the corrections can be divided into three classes: initial-state corrections; final-state corrections; and initial/final state interference corrections.

3.1 Final state corrections

Final-state corrections are small (of the order of two per mil) if no strong acollinearity cuts are applied to the data. With our treatment, they are exactly included to all orders in the pure weak term, since they are included in the physical widths. In the pure QED part and in the QED-weak interference part, they are included up to order α which is clearly enough. The first term which is neglected is the $O(\alpha^2)$ contributions to the QED part. This implies an uncertainty of the order of 10^{-6} .

3.2 Initial state corrections

Initial state corrections are large and we include them using an exponentiation approach [14]. This includes a full second order calculation with the leading infrared and virtual corrections. This can be written in the following way :

$$\begin{aligned} \sigma_{obs}(s) &= \sigma_0(s)(1 + \delta_1 + \delta_2)x_0^\beta \\ &+ \int_{x_0}^1 dx \left[\beta x^{\beta-1}(1 + \delta_1 + \delta_2) - \frac{\beta}{2}(2-x)(1 + \delta_1 + \beta \ln x) + \delta_2^H \right] \sigma_0(s') \end{aligned} \quad (20)$$

where δ_1 and δ_2 are, respectively, the $O(\alpha)$ and $O(\alpha^2)$ explicit (i.e., not exponentiated) virtual + soft photon corrections and δ_2^H contains the $O(\alpha^2)$ hard photon corrections. The reduced center of mass energy is $s' = s(1-x)$, where x is the radiated energy fraction. Furthermore x_0 is the soft-hard photon cutoff, and

$$\beta = \frac{2\alpha}{\pi} \left(\ln \frac{s}{m_e^2} - 1 \right). \quad (21)$$

The first terms which are missing are already of order α^3 . However, due to the possible enhancement coming from large logs like $\ln(s/m^2) \simeq 24$, where m is the electron mass or even $\ln(M_Z/\Gamma_Z) \simeq 3.8$, these terms can be sizable. We will get a feeling on how large they are in the virtual plus soft part and in the hard part separately.

In the former, the first term missing is the explicit third order correction, which we will call δ_3 . Since no computation of δ_3 exists, we will estimate its value based on the known values of $\delta_1 \simeq 9\%$ and $\delta_2 \simeq -0.5\%$. A naive approach would give,

$$\begin{aligned} |\delta_2| &\simeq \frac{1}{2}\delta_1^2 \simeq 0.4\%, \\ |\delta_3| &\simeq \frac{1}{6}\delta_1^3 \simeq 0.01\% \end{aligned} \quad (22)$$

Since we get roughly the right number for the second order correction, we assume that our estimation for δ_3 is also sensible.

The first term that we are missing in the hard photon part of the calculation is the following:

$$I = \int_{x_0}^1 dx \sigma_0(s') \left[-\frac{\beta}{2}(2-x)(\delta_2 + \delta_1 \beta \ln x + \frac{1}{2}\beta^2 \ln^2 x) + \delta_3^H \right] \quad (23)$$

In the vicinity of the Z peak, this expression can be approximated by:

$$I \sim \sigma_0(s) \int_{x_0}^{\frac{\Gamma}{M}} dx \left[-\beta(\delta_2 + \delta_1 \beta \ln x + \frac{1}{2}\beta^2 \ln^2 x) + \delta_3^H \right] \quad (24)$$

where Γ and M are the Z width and mass, respectively. The last piece (δ_3^H) includes the pure third order corrections, without any large logs, and is, therefore, expected to be of the order $O(\alpha/\pi)^3$, that is, completely negligible. The remaining pieces can be evaluated as follows :

$$\begin{aligned} -\beta \delta_2 \int_{x_0}^{\frac{\Gamma}{M}} dx &\sim 10^{-5} \\ -\beta^2 \delta_1 \int_{x_0}^{\frac{\Gamma}{M}} dx \ln x &\sim 10^{-4} \\ -\frac{1}{2}\beta^3 \int_{x_0}^{\frac{\Gamma}{M}} dx \ln^2 x &\sim -4 \times 10^{-4} \end{aligned} \quad (25)$$

If we sum the terms, we see that the pieces not included in the exponentiation could amount to something around 0.3×10^{-3} . We will take this number as an estimate of the precision of our treatment for the initial state radiation corrections. Although we could include these corrections to the hard photon part that we have been discussing, it is not completely clear which of them are supposed to be included, since no third order calculation exists. We, then, choose not to include them and quote their value as our uncertainty.

3.3 Initial/final state interference corrections

Next, we will consider initial/final state interference corrections. These are completely neglected in our approach, since they are known to be small [15]. However, due to the level of precision we have reached in the other parts of the formula, it may well be that those are the largest corrections neglected.

To study this contribution, use has been made of the formulae appearing in ref. [15]. From there, we can extract expressions giving the numerical size of the corrections due to the different kind of initial/final state interference terms as a function of the maximum radiated energy allowed. The formulae include both virtual and real photon contributions (needed to cancel the infrared divergencies) and are integrated over the whole angular range.

Four different contributions can appear. They will be called

- $Z\gamma \otimes Z$, which includes the interference of γ - Z boxes with the lowest order Z diagram together with the interference of two initial/final state diagrams with Z propagator.
- $Z\gamma \otimes \gamma$, which includes the interference of γ - Z boxes with the lowest order γ diagram together with part of the interference of two initial/final state diagrams: one with a Z propagator and another with a photon propagator.
- $\gamma\gamma \otimes Z$, which includes the interference of γ - γ boxes with the lowest order Z diagram together with the rest of the interference of two initial/final state diagrams: one with a Z propagator and another with a photon propagator.

	Muons	Hadrons
Initial state	0.3	0.3
Final state	10^{-3}	10^{-3}
I/F Interference	2	0.2

Table 2: QED uncertainties (per mil)

- $\gamma\gamma \otimes \gamma$, which includes the interference of γ - γ boxes with the lowest order γ diagrams together with the interference of two initial/final state diagrams with γ propagator. This term gives a C -odd contribution which vanishes upon symmetrical integration in the polar angle.

For the first three contributions the formulae can be readily taken from ref. [15] for the production of a fermion pair $f\bar{f}$:

$$\begin{aligned}
\delta_{Z\gamma\otimes Z}(s) &= \frac{3\alpha}{\pi} Q_e Q_f \frac{(g_L^e)^2 - (g_R^e)^2}{(g_L^e)^2 + (g_R^e)^2} \cdot \frac{(g_L^f)^2 - (g_R^f)^2}{(g_L^f)^2 + (g_R^f)^2} \cdot \\
&\quad \text{Re} \left[z(z+1) \ln \frac{x_{\max} + z - 1}{z} + (z-1)(1-x_{\max}) \right] - \ln|z| - 2 \ln x_{\max} \\
\delta_{Z\gamma\otimes\gamma}(s) &= \frac{3\alpha}{\pi} Q_e^2 Q_f^2 \frac{g_L^e - g_R^e}{(g_L^e)^2 + (g_R^e)^2} \cdot \frac{g_L^f - g_R^f}{(g_L^f)^2 + (g_R^f)^2} \cdot \\
&\quad \text{Re} \left[(1-z^*) \left(z(z+1) \ln \frac{x_{\max} + z - 1}{z} + (z-1)(1-x_{\max}) - \ln(z) - 2 \ln x_{\max} \right) \right] \\
\delta_{\gamma\gamma\otimes Z}(s) &= \frac{3\alpha}{\pi} Q_e^2 Q_f^2 \frac{g_L^e - g_R^e}{(g_L^e)^2 + (g_R^e)^2} \cdot \frac{g_L^f - g_R^f}{(g_L^f)^2 + (g_R^f)^2} \cdot \text{Re}(1-z)(-2 \ln x_{\max} + x_{\max} + \frac{1}{2})
\end{aligned}$$

where the δ 's represent the relative contribution with respect to the tree level purely weak result, $x_{\max}$ is the maximum photon energy in units of the beam energy and we have defined

$$z = \frac{M_Z^2}{s} - i \frac{\Gamma_Z}{M_Z} \text{ and } z^* = \frac{M_Z^2}{s} + i \frac{\Gamma_Z}{M_Z}. \quad (26)$$

In figures 2–4 the sizes of the three corrections are given as a function of the centre-of-mass energy around the pole for two different values of $x_{\max}$, both for muon and hadron production.

As we see, quite unexpectedly, the largest contribution comes from the interference of the $\gamma\gamma$ boxes with the tree level Z diagram ($\delta_{\gamma\gamma\otimes Z}$) and not from the part with γ – Z boxes interfering with the Z ($\delta_{Z\gamma\otimes Z}$). The reason comes from the fact that the combination of initial state couplings appearing in the latter

$$\frac{(g_L^e)^2 - (g_R^e)^2}{(g_L^e)^2 + (g_R^e)^2} \quad (27)$$

is an order of magnitude smaller than the one appearing in the former

$$\frac{g_L^e - g_R^e}{(g_L^e)^2 + (g_R^e)^2} \quad (28)$$

In the case of muon pair production the same effect appears in the final state couplings as well.

The sum of the three contributions is plotted in Fig. 5. As we see, for muon production the correction is smaller than 2×10^{-3} in the region $\sqrt{s} = M_Z \pm 3$ GeV for totally inclusive measurements, i.e., when very loose cuts are applied to the data. If we now plot the total correction for hadronic final states, we find (Fig. 5) a correction which is an order of magnitude

Figure 2: Initial/final state interference corrections ($Z\gamma \otimes Z$ contribution) for muons and hadrons.

Figure 3: Initial/final state interference corrections ($Z\gamma \otimes \gamma$ contribution) for muons and hadrons.

Figure 4: Initial/final state interference corrections ($\gamma\gamma \otimes Z$ contribution) for muons and hadrons.

Figure 5: Initial/final state interference corrections (total) for muons and hadrons.

smaller (less than 0.2×10^{-3} in the above mentioned region for $x_{max} \simeq 1$). This is due to the different way quarks and leptons couple to the Z boson. Furthermore, the correction for u -type quark production and the correction for d -type quarks have opposite signs so that a partial cancellation occurs when computing the correction to inclusive hadron production without flavor tagging.

To summarize the results of this section, we have written in Table (2) the uncertainties in the QED corrections for muonic and hadronic final states and for $x_{min} = 1$, i.e., when no cut is made on the energy of the photon.

4 Precision of Cross Sections and Fit Resonance Parameters

In this section we compare the cross sections predicted by two analytic formulae in use in the literature and our model independent prescription, against the most complete electro-weak calculation. We also compare the results from these three programs for fitting the Z line shape.

4.1 Comparison of Cross Sections

Based on the model independent formula (17) and (18) we have constructed a FORTRAN routine we call **MIZA**. This program can be considered an updated version of **ZAPP** written by G. Burgers [3]. Burgers' original program takes into account only the Z exchange diagram. Our routine replaces the non-radiative kernel of **ZAPP** with equation (17) or (18); we use the same QED initial state calculation as **ZAPP**. In **ZAPP**, as in **MIZA**, QED initial-state radiation is performed through $O(\alpha^2)$ with exponentiation and is numerically convoluted with the non-radiative cross section using an efficient Gaussian integration algorithm.

We have studied two model independent cross section functions which exist in the literature: Cahn [16] (which is included in the code of **ZSHAPE** as approximation #1) and BCMS [17]. We have slightly modified the original BCMS formula by substituting s for M^2 where appropriate. In these programs, the initial-state radiative correction is performed with an analytic function, the QED coupling constant is fixed at M_Z , and the final-state radiative correction is neglected.

We have checked the accuracy of the programs by comparing them against **ZSHAPE** [9]. A set of muon and hadron cross sections were generated by using **ZSHAPE** with the following input parameters: $M_Z = 91.2$ GeV, $M_t = 100$ GeV, $M_H = 100$ GeV and $\alpha_s = 0.12$. These parameters lead to $M_W = 80.0$ GeV, $\Gamma_Z = 2.4876$ GeV, $\Gamma_{e\bar{e}} = 83.523$ MeV and $\Gamma_{q\bar{q}} = 1.7378$ GeV. Following ref. [6], this implies $\sin^2 \theta_W = 0.2316$ (Other definitions of the effective mixing angle exist; numerically they nearly equivalent [7] [8]). The relative difference in cross section between one of the three programs under study, σ_X , and **ZSHAPE** is defined as

$$\Delta R = \frac{\sigma_X - \sigma_{ZSHAPE}}{\sigma_{ZSHAPE}} \quad (29)$$

For the comparisons we have used, in each of the formulae under study, the physical widths calculated by **ZSHAPE**. Additionally we have evaluated the interference terms in those formulae using effectively the same value of the mixing angle as was used in **ZSHAPE**.

Comparison of observable cross sections is shown in Fig. 6 and Fig. 7 for muon and hadron production, respectively. The minimum invariant mass of the numerical integration in **MIZA** has been set to $4 \times m_\mu^2$ for muon pair production and $4 \times m_\pi^2$ for hadron production. The variation of the ratio ΔR for **MIZA** is below 1 per mil in the range of $M_Z \pm 2\Gamma_Z$ for muon production giving significant improvement compared with the analytic functions. For hadrons also **MIZA** shows improvement over the analytic approximations. The upward sloping tail of

Figure 6: Comparison of convoluted muon cross sections using **ZSHAPE** as a reference. The solid line shows our formula, the dot-dashed line is the formula (1) suggested in the Yellow Report [2], the dashed line is BCMS and the dotted line is Cahn's formula.

the **MIZA** curve for hadrons at low energy can be as much as 2 per mil higher at 86 GeV depending upon whether one uses the quark mass or the pion mass as the integration bound for the initial state radiation. From an experimental point of view, this bound should be set to some value in the Monte Carlo event generation when determining detector acceptances, and the same value should appear in the fitting function. In practice the bound in a Monte Carlo is somewhat ambiguous because of fragmentation effects. In any case, for **MIZA**, the curve is below 1 per mil for all the energies used in the scanning at LEP.

As we shall see the slope of the BCMS and Cahn curves will result in systematic shifts of the fit mass and width of a few MeV. The normalization shifts of these two functions will result in shifts in the fit peak hadronic cross section.

The dependence on the input parameter of the electro-weak theory ($\sin^2 \theta_W$ in our case) is shown in Fig. 8 for muon production (The curves for hadron production are essentially the same). The effective Weinberg angle input to **MIZA** was varied over a very large range and the comparison is to **ZSHAPE** for a single combination of top and higgs masses (equivalently a single value of the mixing angle). In **MIZA** $\sin^2 \theta_W$ appears only in the interference term. That the accuracy is still below 4 per mil on $M_Z \pm 2\Gamma_Z$ simply shows that the interference term is small. If we restrict $\sin^2 \theta_W$ to a realistic range the uncertainty introduced is at the level of 0.4 per mil.

4.2 Comparison of Fit Parameters with Respect to **ZSHAPE**

To determine the way in which the observed differences in cross sections propagate to shifts in the fit parameters, we have generated cross sections using **ZSHAPE** and fit for the physical parameters using **MIZA**, Cahn, and BCMS. We simulate the LEP scan by generating muon and hadron cross section points at the centre-of-mass energies of M_Z , $M_Z \pm 1$, $M_Z \pm 2$ and $M_Z \pm 3$

Figure 7: Comparison of three line shape programs with **ZSHAPE** for the hadronic cross section. The meaning of lines is the same as in the previous figure.

Figure 8: Comparison of **MIZA** with **ZSHAPE** for muon pair production for the effective Weinberg angle [6] range between 0.225 and 0.235. The hadronic cross section shows the same behaviour.

Fitting Program	M_Z	Γ_Z	$\sigma_{peak}^{(0)}$
	[MeV]	[MeV]	[pb]
ZSHAPE Input	91200.0	2487.6	1989.6
Cahn	91204.0 +4.0	2494.8 +7.2	1988.5 -1.1
BCMS	91197.4 -2.6	2491.9 +4.3	1993.6 +4.0
MIZA	91199.6 -0.4	2487.7 +0.1	1990.3 +0.7

Table 3: Three parameter fit for muon pair production. The upper number shows the fitting results. The discrepancy from the **ZSHAPE** input parameter is given below the fit value.

Fitting Program	M_Z	Γ_Z	$\sigma_{peak}^{(0)}$
	[MeV]	[MeV]	[pb]
ZSHAPE Input	91200.0	2487.6	41396
Cahn	91202.8 +2.8	2493.3 +5.7	41305 -91
BCMS	91197.8 -2.2	2489.1 +1.5	41445 +50
MIZA	91199.8 -0.2	2487.0 -0.6	41406 +10

Table 4: Three parameter fit for hadron production. The discrepancy from the **ZSHAPE** input parameter is given below the fit value.

Fitting Program	M_Z	Γ_Z	$\Gamma_{\mu\bar{\mu}}$	$\Gamma_{q\bar{q}}$
	[MeV]	[MeV]	[MeV]	[MeV]
ZSHAPE Input	91200.0	2487.6	83.523	1737.8
Cahn	91202.9 +2.9	2493.4 +5.8	83.700 +0.177	1738.4 +0.6
BCMS	91197.8 -2.2	2489.3 +1.7	83.675 +0.152	1738.9 +1.1
MIZA	91199.8 -0.2	2487.0 -0.6	83.521 -0.002	1737.4 -0.4

Table 5: Fits to combined lepton and hadron cross sections. Hadrons and the three leptons are fit simultaneously, assuming lepton universality. The discrepancy from the **ZSHAPE** input parameter is given below the fit value.

GeV. The results for a 3 parameter (M_Z, Γ_Z and $\sigma_{peak}^{(0)}$) fit to muon and hadron cross sections separately are summarized in Table 3 and Table 4 for muons and hadrons, respectively. The results for a combined fit to muons and hadrons simultaneously are shown in Table 5, where we fit for $M_Z, \Gamma_Z, \Gamma_{\mu\bar{\mu}}$ and $\Gamma_{q\bar{q}}$. The discrepancies shown in the tables can change slightly in the last digit depending, again, on what bound is taken in the integration in the hadronic cross section. In general MIZA returns fit parameters much closer to the input than the analytic functions.

The effect of changing the input value of $\sin^2 \theta_W$ is not shown in the tables. This bias, shared by all the programs, introduces additional small discrepancies in the fit parameters. For the wide range of $\sin^2 \theta_W$ used in fig. 8, the changes in the fit mass and total width are about 2 MeV and 1 MeV respectively. We believe these are conservatively large estimates of the biases introduced by using a particular value of $\sin^2 \theta_W$ in the interference term of MIZA, and that the true biases are well below 1 MeV.

As a summary, the routine presented here can reproduce the input mass to better than 0.5 MeV and total width below the 1 MeV level, whereas the existing analytic functions reproduce these parameters with discrepancies of a few MeV. The analytic programs reproduce the hadronic width at about the 1 MeV level; the discrepancy in the hadronic width introduced by using MIZA is about 0.5 MeV. For the muon width, MIZA reproduces the input value at the level of a few keV while the existing programs return widths differing from the ZSHAPE input by about a tenth of an MeV. MIZA reproduces the hadronic peak cross section with a bias of about .01 nb, while the analytic functions have biases 5 to 10 times as large, at the .1 nb level.

It is important to stress here, since we are comparing with ZSHAPE, that the fact that the agreement in the peak cross sections is better than 0.4 per mil doesn't necessarily mean that the absolute accuracy of the formula is that good. On the one hand, the fact that we use physical parameters guarantees that the accuracy in the non-QED corrections of our formula is, by construction, better than that in ZSHAPE (of course we don't know the values of those parameters until they are determined by fitting to data). But since the QED initial state radiation used is exactly the same, the absolute accuracy of our approach is determined by the accuracy of the QED part, which in the previous section has been estimated to be better than 2 per mil for muons and four times better for hadrons.

4.3 Uncertainty on Z Resonance Parameters Obtained by MIZA Due to Uncertainty of the QED Corrections

In the previous section we studied the ability of MIZA, Cahn, and BCMS to return the input parameters of the Standard Model calculation from ZSHAPE. In this section we estimate the uncertainty of Z resonance parameters fit by MIZA in light of the fact that the QED part of the calculation is incomplete.

For this purpose we have modified MIZA by including the uncertainties given in section 3, regarding the initial-final state QED corrections (the equations for $Z\gamma \otimes Z$, $Z\gamma \otimes \gamma$, and $\gamma\gamma \otimes Z$). This permits us to estimate the size of the errors which will propagate to the fit parameters due to the fact that those corrections are not complete. We compare the fit parameters returned by MIZA before and after including the expressions of section 3. These changes are an estimate of this uncertainty. In tables 6, 7, and 8 we show the change in the fit parameters for muons, hadrons, and the combined fit to hadrons and muons simultaneously. In all cases the accuracy is better than .03%.

The initial state QED corrections are negligibly small for muons, but for hadrons may be as large as the initial-final corrections.

Overall then, we estimate uncertainty on the fit parameters due to incomplete QED corrections to be of the order of 0.03% for both muons and hadrons.

	ΔM_z [MeV]	$\Delta \Gamma_z$ [MeV]	$\Delta \sigma_{peak}^0$ [pb]
Uncertainty	1.6	0.7	0.2

Table 6: Uncertainty of fit parameters from MIZA for the muon line shape due to incomplete initial-final state QED corrections.

	ΔM_z [MeV]	$\Delta \Gamma_z$ [MeV]	$\Delta \sigma_{peak}^0$ [pb]
Uncertainty	0.1	0.06	0.1

Table 7: Uncertainty of fit parameters from MIZA for the hadronic line shape due to incomplete initial-final state QED corrections.

5 Projected Precision of Z Resonance Parameters from Experiments

In order to compare the theoretical biases introduced by using a particular fitting function against future experimental errors, in this section we study the errors on the Z parameters that the LEP-SLC experiments can expect to achieve. We have again used the standard LEP 7-point scan, giving one half of the the luminosity to the peak point, with the other half divided equally among the points at $M_Z \pm 1$, $M_Z \pm 2$ and $M_Z \pm 3$ GeV. We assume a low-angle Bhabha cross section of 25 nb for calculating the statistical error on the luminosity. We have taken two sets of systematic errors, one more optimistic than the other.

In case (1) we take

- $\Delta L/L = 1\%$, overall luminosity scale error
- $\Delta_{p-p} = 0$, no point-to-point error
- Systematic on hadron selection = 0.5%
- Systematic on each lepton selection = 1%

whereas for case (2) we assume

- $\Delta L/L = 3\%$, overall luminosity scale error
- $\Delta_{p-p} = 1\%$, point-to-point error, added in quadrature with statistical errors
- Systematic on hadron selection = 1.5%
- Systematic on each lepton selection = 3%

We then fit with MIZA to the exact ZSHAPE data, using the above systematic errors and the statistical errors calculated for a given luminosity. We assume lepton universality and fit hadrons, muons, taus and s-channel electrons simultaneously. We exclude the electrons at $M_Z \pm 3$ GeV because these points may be troublesome in the experiments due to uncertainty in the t-channel subtraction. Of course there is no problem in the simulated data for t-channel

	ΔM_z [MeV]	$\Delta \Gamma_z$ [MeV]	$\Delta \Gamma_{ll}$ [MeV]	$\Delta \Gamma_{q\bar{q}}$ [MeV]
Uncertainty	0.1	0.04	0.005	0.2

Table 8: Uncertainty of fit parameters from MIZA for the combined fit to hadrons and leptons due to incomplete initial-final state QED corrections.

subtraction; we want to exclude the statistics of these points to simulate the expected experimental errors as closely as possible. In any case, excluding the $M_Z \pm 3$ GeV points for electrons makes only a small difference. The expected errors on M_Z , Γ_Z , $\sigma_{q\bar{q}}^0$, $\Gamma_{q\bar{q}}/\Gamma_{ll}$, $\Gamma_{q\bar{q}}$, and Γ_{ll} are shown in Fig. 9; the solid line shows the result for the systematic errors of case (1) above, the dotted line is for case (2).

The figure shows that the experiments have far to go before the errors on the mass and width become limited by the systematics of individual the experiments; the curves for ΔM_Z and $\Delta \Gamma_Z$ do not flatten out until after 100 pb^{-1} . After $\approx 20 \text{ pb}^{-1}$ the errors on those quantities can be just a few MeV, and an accurate fitting program, such as MIZA, will be necessary in order to avoid relatively large systematic biases from the fitting program itself. The error on the mass shown in figure 9 does not include the error due to uncertainty in beam energy. The beam energy error gives the largest contribution to the mass error at LEP, being about 30 MeV [18].

The peak hadronic cross section on the other hand will soon be limited by the systematic error on luminosity measurement⁴. The hadronic and leptonic widths will begin to be systematically limited after about 10 pb^{-1} , essentially limited by the luminosity measurement and selection systematics.

6 Conclusion

A new fitting formula to extract the physical parameters of the Z resonance has been presented. Its actual performance and limitations have been discussed in detail. The relative accuracy of our formula, comparing with the Standard Model calculation ZSHAPE [9], has been computed to be better than 1 per mil in the region of LEP scanning. Also the absolute accuracy of our formula has been estimated to be better than 2 per mil for muon production and better than 1 per mil for hadron production in this region.

The new formula has been compared to two existing analytic functions and has been shown to be more accurate both in terms of cross section calculation and fit Z parameters. The errors on Z parameters which the LEP-SLC experiments can expect to achieve have been estimated. With enough luminosity these errors are small enough to become comparable with the biases introduced by present fitting functions. The function presented here introduces biases which will remain well below the experimental errors for the near future.

The computer program MIZA is available as a FORTRAN function from any of the authors.

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⁴At the time of this writing the LEP experiments each had about 5 pb^{-1} of integrated luminosity and publications were forthcoming.

Figure 9: Projected errors on Z resonance parameters that each LEP-SLC experiment can expect to attain, as a function of integrated luminosity. Errors are calculated with for two assumptions on systematic errors: the solid line shows the result for case(1), the dotted line is for case(2) (see text). Hadrons and each of the three leptons are fit simultaneously, assuming lepton universality. The error on the mass of about 30 MeV due to the LEP beam energy scale uncertainty is not included in ΔM_Z .

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