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THE LINK POLYNOMIALS OF THE CHERN-SIMONS THEORY

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Abstract

The problem of computing the expectation values of the Wilson line operators associated with oriented links in the Chern-Simons theory is considered. It is shown that a set of simple rules permits the computation of these expectation values for a generic link whose components are associated with arbitrary representations of the gauge group. The resulting expressions obtained for some examples of knots and links are reported.

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A distinguished role between the topological field theories is played by the three-dimensional Chern-Simons (CS) model [1,2,3]. The CS action is

$$S = \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} \text{Tr} \left(A_\mu \partial_\nu A_\rho + i \frac{2}{3} A_\mu A_\nu A_\rho \right), \quad (1)$$

where $A_\mu = A_\mu^a T^a$, $\{T^a\}$ being the Hermitian generators of a simple compact group G in the fundamental representation*. Gauge invariance and general covariance are the fundamental properties of the action (1). The basic observables of the CS theory are the expectation values $\langle W(L) \rangle$ of Wilson line operators associated with oriented links in R^3 . Given an oriented knot C in R^3 and an irreducible representation ρ of G , the associated Wilson line operator is

$$W_\rho(C) = \text{Tr} \text{P exp} \left(i \oint_C A_\mu^a(x) T_\rho^a dx^\mu \right), \quad (2)$$

where $\{T_\rho^a\}$ are the generators of G in the ρ representation. In general, for any oriented link L in R^3 with m components $\{C_1, \dots, C_m\}$, one can associate to the i -th component C_i the irreducible representation ρ_i of G . The gauge-invariant and metric-independent observables in which we are interested are the vacuum expectation values

$$\langle W(L) \rangle \equiv \frac{\langle 0 | W_{\rho_1}(C_1) \dots W_{\rho_m}(C_m) | 0 \rangle}{\langle 0 | 0 \rangle} \quad (3)$$

We consider here the quantum CS field theory in R^3 and, as shown in eq.(3), the overall normalization of the vacuum expectation values of the Wilson line operators is the standard normalization used, for instance, in ordinary perturbation theory.

It has been suggested in [3] that, in the fundamental representations of $G = SU(N)$, $\langle W(L) \rangle$ is related to the Jones polynomial and its generalizations [4]. For arbitrary representations $\{\rho_i\}$, a second order computation of $\langle W(L) \rangle$ in the expansion parameter $\lambda = \left(\frac{2\pi}{k}\right)$ has been performed in [5] in the framework of the covariant BRS quantization.

* In our notations $\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$.

In the present paper, a simple recipe for deriving the full expression of $\langle W(L) \rangle$, associated to an arbitrary oriented link L in R^3 and for arbitrary representations $\{\rho_i\}$ of G , is presented. The proposed construction is based on a certain number of defining relations which uniquely determine $\langle W(L) \rangle$. These defining relations are derived from the results obtained in the CS field theory and can be used recursively for reconstructing in a simple way whatever $\langle W(L) \rangle$. A certain similarity exists between the standard method, known in knot theory for constructing the link polynomials and based on the recursive use of the skein relation, and the new set of rules described in this paper. The improvement, which permits us to extend the recursive relations for arbitrary representations, is represented by a simple method of taking into account the structure of the group G .

Apart from their meaning related to the physical context of the CS model, the resulting link invariants are of particular interest for knot theory. The link polynomial described by the defining relations can be called, using Drinfeld's terminology [6], the universal link polynomial. In fact, the link invariants described by $\langle W(L) \rangle$ coincide with those associated with the braid group representations described by the quasi-tensor category of the quasi-triangular quasi-Hopf algebras [6] associated to the quantum deformations of ordinary Lie algebras.

The paper is organized as follows. First, some general properties of $\langle W(L) \rangle$ are reviewed. Then, the monodromy matrices obtained in the canonical quantization approach are compared with Kohno's and Drinfeld's results. The defining relations of the universal link polynomial are introduced and the main reconstruction theorem is formulated. Finally, some applications and checks are illustrated by means of several examples.

Because of general covariance, one could expect at first sight that $\{\langle W(L) \rangle\}$ define invariants of ambient isotopy for oriented links. However, in order to maintain general covariance at the full quantum level, it is necessary to introduce a framing for each component of the link L [5]. As a result, $\langle W(L) \rangle$ actually describes an ambient isotopy invariant for links made with oriented twisted bands. As is well known, one can define a one-to-one correspondence between the equivalence classes of ambient isotopy of oriented links made of bands and the equivalence classes of regular isotopy of oriented link diagrams. In our convention, the twist variables $T(C_i)$ are given by

$$T(C_i) = w(C_i), \quad (4)$$

where $w(C_i)$ is the writhe of the component C_i of the link diagram L . For notational simplicity, L will denote both framed links in R^3 and the corresponding link diagram

classes of regular isotopy. Because of the above-mentioned correspondence, no ambiguities will ever arise. In this context and with the convention (4), the Wilson line expectation values will be denoted by

$$E(L) = \langle W(L) \rangle. \quad (5)$$

By definition, $E(L)$ is an invariant of regular isotopy for oriented link diagrams L and the problem that we want to solve is how to compute $E(L)$ in closed form.

The first two constraints on the dependence of $E(L)$ on the coupling constant* k are due to the effects of the symmetry which modifies the orientation of R^3 and to the reality of the action and functional measure. In particular, if $\tilde{L}$ denotes the mirror image of the link diagram L , then [5]

$$E(L; k) = E(\tilde{L}; -k); \quad (6)$$

whereas, if L^{-1} denotes the oriented link diagram obtained by reversing the orientations of all the components of L , then

$$E^*(L; k) = E(L^{-1}; -k). \quad (7)$$

Another property of $E(L)$, which will play a crucial role in what follows, comes from the exact scale invariance of the CS theory [8]. From the definition (5) and precisely with the normalization shown in eq.(3), it follows that, if L is the distant (disjoint) union of L_1 and L_2 , then

$$E(L) = E(L_1)E(L_2). \quad (8)$$

Eqs.(6-8) have been explicitly verified at second non-trivial order of perturbation theory [5].

In a canonical quantization approach, the states of the theory are associated with a punctured surface. The positions of the punctures are defined by the intersection points of the surface with the oriented link L on which the Wilson line operators are defined. Each puncture is characterized by the quantum numbers of a representation ρ_i of G inherited from $W(L)$. Because of general covariance, the CS theory does not

* In all the formulae, k represents the renormalized coupling constant of the theory. This means that k is the multiplicative factor in front of the renormalized action; equivalently, k enters the canonical commutation relations between the fields as shown in eq.(3.2) of [3] or eq.(2.6) of [7]. This also means that k appears in the Gauss law constraint as shown in eq.(3.4) of [3] or in eq.(2.26) of [7].

have a dynamical particle content. The non-trivial information carried by the states of the theory is represented by the monodromy matrices acting on the state wave functions when the punctures are exchanged according to the action of the braid group. Because of general covariance, for our purposes it is sufficient to consider the case of two punctures only. Let $\{T_1^a\}$ and $\{T_2^a\}$ be the generators of G in the representations ρ_1 and ρ_2 associated with the two punctures; V_1 and V_2 are the linear spaces on which ρ_1 and ρ_2 are defined. Under a braid exchange (corresponding to a crossing with writhe $w = 1$) of the two punctures, the states of the system transform as [7]

$$|i\rangle \mapsto |f\rangle = \Pi_{12} q^{\Omega_{12}} |i\rangle, \quad (9)$$

$$\text{where} \quad \Omega_{12} = T_1^a \otimes T_2^a, \quad (10)$$

$$q = \exp\left(-i\frac{2\pi}{k}\right), \quad (11)$$

and Π_{12} is the ordinary permutation operator which makes the connection between $V_1 \otimes V_2$ and $V_2 \otimes V_1$. Eqs.(9-11) have been obtained in [7] and are just a simple consequence of the canonical commutation relations between the fields; they are in agreement with both the general field theory properties (6,7) and the perturbative computations performed in the covariant BRS formalism. Eq.(11) means that in the CS theory there is no shift.

At this stage, it is perhaps useful to add a few comments on the meaning of eqs.(9-11). For equivalent representations ρ_1 and ρ_2 , the form of the "half-monodromy" (9) is well known in the context of the R -matrix approach. On one hand, Kohno has shown [9] that the monodromy representations χ of the braid group B_n , obtained by integrating the Knizhnik-Zamolodchikov connection, are equivalent to the R -matrix B_n representations ξ obtained in the algebraic approach. Each single generator of B_n is represented in χ by a matrix which is equivalent to the standard expression shown in eq.(9) acting on a certain appropriate subspace of the total space $V(n) = V_1 \otimes \dots \otimes V_n$. Of course, the form of the intertwining operator depends on the particular generator under consideration. At the same time, for fixed n , a unique intertwining operator connects χ with ξ . On the other hand, Drinfeld has shown [6] that eq.(9) represents the most general co-commutative isomorphism in the quasi-tensor category of the quasi-triangular quasi-Hopf algebras associated with the quantum deformations of ordinary Lie algebras. The complex variable q is the deformation parameter.

Now, general covariance implies that braid representations are realized on the physical state space of the CS theory. From eq.(9) it follows that these representations are

those associated to the quasi-tensor category considered by Drinfeld. This property is called the universality of the CS theory and the link and knot invariants described by $E(L)$ are what Drinfeld calls the universal link and knot invariants.

A remarkable consequence of Kohn's and Drinfeld's results concerns the role of Lie algebras in the R -matrix approach. The Knizhnik-Zamolodchikov connection is entirely expressed in terms of the (quadratic) Casimirs of the group G . In Khono's construction, the diagonal action of the group G belongs to the centralizer of the χ representations. This is a clear indication that the link invariants, associated with the braid representations obtained in the R -matrix approach and in CS theory, should admit also a derivation in terms of ordinary Lie algebras. Of course, in the CS theory everything is expressed in terms of the Lie algebra associated with the group G . In the perturbative computations of $\langle W(L) \rangle$, only the Casimirs of G enter. In the transition amplitudes of the CS theory, the charges associated with the generators of G define global symmetries for the system, and so on. In conclusion, what seems to emerge from Khono's and Drinfeld's works and from the CS theory also, is that the secret of the deep meaning of "quantum groups" lies, after all, in ordinary groups. In fact, as we shall see, the symmetry structure defined by the group G is the fundamental input underlying the construction of the universal link polynomial $E(L)$.

The last information needed to complete our data concerns the twisting. In the canonical approach, the variable measuring the effects of a change in the twist variable $T(C_i)$, associated to the link component C_i , has not been derived. Fortunately, partial information on this problem has been obtained in the covariant BRS formalism. A $\Delta T(C_i) = 1$ transformation results in the presence of a multiplicative factor α which, at second order in $\lambda = (\frac{2\pi}{k})$, has been computed in Ref[5] and the result is consistent with the expression

$$\alpha = q^{Q(\rho_i)} \quad , \quad (12)$$

where $Q(\rho_i)$ is the quadratic Casimir operator of the irreducible representation ρ_i associated with the component C_i :

$$T_i^a T_i^a = Q(\rho_i) \cdot 1 \quad . \quad (13)$$

I will assume that eq.(12) holds to all orders in λ . The meaning of the variable α is to provide the value of the relevant parameter which, in Turaev's construction [10], can be obtained by means of the enhancement matrix μ admitted by the R -matrix.

At this point, we can formulate the conditions defining the universal link polynomial $E(L)$ of regular isotopy. Each oriented link diagram L can be represented as the closure

of an element of B_n . A diagrammatic representation is used in which each oriented string is associated with a given irreducible representation of G . The five defining conditions are:

I. PROJECTION DECOMPOSITION

Given two irreducible representations ρ_1 and ρ_2 such that their tensor product admits the decomposition $\rho_1 \otimes \rho_2 = \sum_t \rho(t)$, then

This first condition is essentially a definition. The boxes introduced in the picture are projectors on the irreducible components $\rho(t)$. Two consecutive aligned projectors on $\rho(t)$ and $\rho(t')$ give rise to a single projector if $\rho(t) = \rho(t')$, otherwise they give zero.

II. CROSSING

This condition graphically represents eq.(9). In order to obtain the undercrossing, it is sufficient to replace each coefficient (appearing in the above relation) with its inverse.

It is assumed here that the braid group representation realized in the CS theory admits a projection decomposition in terms of the irreducible components as shown in the picture. This means that the combinations of the projectors, corresponding to the crossings, satisfy the defining relations of the braid group generators (i.e. the quantum Yang-Baxter equation). Note that, like in Kohn's construction, the particular subspaces identified by the projectors are in general related by non-trivial intertwining operators. The crucial observation here is that, in order to find $E(L)$, the explicit form of these intertwining operators is not necessary. In other words, the knowledge of the explicit expression of the associative isomorphism considered in [6] is not required for computing $E(L)$.

III. TWISTING

This condition illustrates the double role played by the variable α shown in eq.(12). On one side, α is related to the twist of the band and describes the covariance of $E(L)$ under Reidemeister moves of type I. On the other side, α gives information on the partial closure of the braid and, in this sense, it is related to the enhancement of Turaev's construction.

IV. PROJECTION COMPATIBILITY

In condition I, the new symbol of projection has been introduced; now it is necessary to specify in what sense the boxes really represent projectors more precisely. The simplest way of doing this is to require that the following satellite formula (for regular isotopy) holds

This condition states that the closure of the braid is compatible with the projection decomposition described in I.

V. FACTORIZATION

If L is the distant union of L_1 and L_2 , then

$$E(L) = E(L_1) E(L_2)$$

This condition is just a consequence of the definition (5) and has already been introduced in eq.(8).

I do not claim that the conditions (I-V) have been "rigorously" derived; they have been "very strongly" suggested from the explicit computations and the general properties of the CS theory. A discussion on this point can be found in [11]. By using the rules displayed by the conditions (I-V), any link diagram gets modified by the introduction of the projectors. These projectors can then be eliminated recursively and, as a result, the expression of $E(L)$ is obtained. At this stage, one could also forget about the CS theory and consider the conditions (I-V) as the defining relations of the $E(L)$ polynomial. My first claim is

STATEMENT 1. Conditions (I-V) uniquely determine a link polynomial $E(L)$ of regular isotopy; for $SU(N)$, $E(L; q) \in \mathbb{Z}^{\pm}[q^{\pm \frac{1}{N}}]$.

A "proof" of this statement is presented in [11]. Of course, one can easily construct [11], by means of $E(L)$, an invariant $X(L)$ of ambient isotopy. My second claim is

STATEMENT 2. The polynomial $E(L)$, considered in Statement 1, gives precisely the expression of the vacuum expectation values of Wilson line operators in the CS theory and it also describes the universal link polynomial considered by Drinfeld.

In what follows, some explicit examples are reported.

Let us consider for simplicity the group $G = SU(N)$ and let F denote the fundamental representation of $SU(N)$; A and S represent the antisymmetric and the symmetric representations appearing in $F \otimes F = A \oplus S$. Let $E_0[\rho] = E(U; \rho)$ be the universal polynomial of the unknotted knot U with zero writhe associated with the ρ -representation of $SU(N)$. Conditions (I-V) imply that [11]

$$E_0[F] = \frac{q^{\frac{N}{2}} - q^{-\frac{N}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \quad (14)$$

$$E_0[A] = \frac{q^{\frac{N-1}{2}} - q^{-\frac{N-1}{2}}}{q - q^{-1}} E_0[F] \quad (15)$$

$$E_0[S] = \frac{q^{\frac{N+1}{2}} - q^{-\frac{N+1}{2}}}{q - q^{-1}} E_0[F] \quad (16)$$

For $N = 2$, eqs. (14-16) are easily generalized; for the generic representation J of $SU(2)$ ($2J = 0, 1, 2, \dots$ and $Q(J) = J(J+1)$) one has

$$E_0[J] = \frac{q^{J+\frac{1}{2}} - q^{-(J+\frac{1}{2})}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \quad (17)$$

Consider the trefoil knot 3_1 with writhe $w = 3$ associated to the representation J of $SU(2)$. The universal polynomial $E(3_1; J)$ is given by [11]

$$E(3_1; J) = \frac{q^{3J^2}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \sum_{n=0}^{2J} (-1)^n q^{\frac{1}{2}n(n-4J-1)} \left(q^{2J-n+\frac{1}{2}} - q^{-(2J-n+\frac{1}{2})} \right) \quad (18)$$

For the 4_1 knot with zero writhe and in the adjoint ($J = 1$) representation of $SU(2)$, one gets

$$E(4_1; J = 1) = q^7 - q^5 + q + 1 + q^{-1} - q^{-5} + q^{-7} \quad (19)$$

Consider now the two-component link 2_1^2 (Hopf) with writhe $w = 2$ and let the representations of $SU(2)$, associated with the two components of this link, be characterized by J_1 and J_2 with, for instance, $J_1 \geq J_2$. The universal link polynomial

$E(2_1^2; J_1, J_2)$ takes the form

$$E(2_1^2; J_1, J_2) = \frac{q^{2J_1J_2}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \sum_{m=0}^{2J_2} q^{m^2 - m(2J_1 + 2J_2 + 1)} \cdot \left[q^{J_1 + J_2 - m + \frac{1}{2}} - q^{-(J_1 + J_2 - m + \frac{1}{2})} \right] \quad (20)$$

A more extended list of examples will be reported in [11]. Some final checks and comments are in order.

Eqs. (14-20) are in agreement with eqs. (6,7) and the explicit computations performed in [5]. Eq. (14) is in agreement with the expression proposed in [3] provided the correct value of q , given in eq. (11), is inserted in the formulae of Ref. [3]. For $N = 3$, the $A = 3^*$ representation is the complex conjugate of the $F = 3$ representation. Therefore, $E_0[3]$ should be equal to $E_0[3^*]$. In fact, for $N = 3$, the expressions (14) and (15) coincide. For $N = 2$, the A representation is the trivial ($J = 0$) representation of $SU(2)$. In this case, eq. (15) gives $E_0[J = 0] = 1$, in agreement with the exact field theory result.

In general, for $N = 2$, the universal link polynomial can be called the extended Jones polynomial because, when all the representations associated with the components of the links coincide with the fundamental ($J = \frac{1}{2}$) representation of $SU(2)$, one gets the Jones polynomial (in our convention, the normalization of the unknot is different from that used in [4]). A first observation on the extended Jones polynomial concerns the value of the unknot. Since $|q| = 1$, eq. (17) coincides precisely with the character formula for $SU(2)$. For integer values of the coupling constant k , the $SU(2)$ class entering eq. (17) is determined by the generator of the finite subgroup C_k of $SU(2)$.

A more detailed discussion on the properties of the universal link polynomial will be performed in [11].

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REFERENCES

- [1] A.S. Schwarz, Lett. Math. Phys. 2 (1978) 247; Comm. Math. Phys. 67 (1979) 1; Abstracts of the Baku Topological Conference 1987.
- [2] M.F. Atiyah, *New Invariants of Three- and Four-Dimensional Models*, in *The Mathematical Heritage of Hermann Weyl*, Proc. Symp. Pure Math. 48, ed. R. Wells (Amer. Math. Soc., 1988).
- [3] E. Witten, Comm. Math. Phys. 121 (1989) 351.
- [4] V.F.R. Jones, Bull. Amer. Math. Soc. 12 (1985) 103; Ann. Math. 126 (1987) 335;
- P. Freyd, D. Yetter, J. Hoste, W.B.R. Lickorish, K. Millet and A. Ocneanu, Bull. Amer. Math. Soc. 12 (1985) 239;
- J.H. Przytycki and P. Traczyk, Kobe J. Math. 4 (1987) 115.
- [5] E. Guadagnini, M. Martellini and M. Mintchev, Nucl. Phys. B330 (1990) 575.
- [6] V.G. Drinfeld, *Quasi-Hopf Algebras and Knizhnik-Zamolodchikov Equations*, in *Problems of Modern Quantum Field Theory*, ed. A.A. Belavin, A.V. Khnyk and A.B. Zamolodchikov (Springer-Verlag, 1990).
- [7] E. Guadagnini, M. Martellini and M. Mintchev, Nucl. Phys. B336 (1990) 581.
- [8] F. Delduc, O. Piguet, C. Lucchesi and S.P. Sorella, *Exact Scale Invariance of the Chern-Simons Theory in the Landau Gauge*, University of Geneva preprint UGVA-DPT 1990/2-653, to appear in Nucl. Phys. B.
- [9] T. Kohno, Ann. Inst. Fourier, Grenoble, 37 (1987) 139; Adv. Studies in Pure Math. 16 (1988) 255; Nagoya University preprints No 3 and No 6, 1989.
- [10] V.G. Turaev, Invent. Math. 92 (1988) 527; *The Yang-Baxter Equation and Invariants of Links*, LOMI preprint E-3-87, Steklov Institute, Leningrad 1988.
- [11] E. Guadagnini, *The Universal Link Polynomial*, CERN preprint CERN-TH. 5826/90.