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Is there a better way of exponentiating QED corrections?

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Abstract There is always a certain freedom in the practical prescription for the QED exponentiation. Restricting ourselves to the leading logarithmic approximation we are able to compare the result of the fixed (second) order exponentiated result with the exact (infinite order) solution calculated numerically with help of the Monte Carlo technique. Taking three types of the second order exponentiation recipes we conclude that the method related closely to the Yennie-Frautschi-Suura (YFS) classical work is more convergent than the other ones used by many authors.

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Exponentiation of the perturbative QED calculation is a very efficient method of summing up the higher order contributions (almost) without calculating them. There is always, however, a certain amount of freedom in the exponentiation prescription. Generally, the procedure should be such that the result is convergent as quickly as possible to the exact, infinite order answer. Since the infinite order result is not usually known the quality of a given exponentiation scheme may only be deduced by subtracting results from two consecutive perturbative orders. For example, Kuraev and Fadin¹ have proposed an improved way of exponentiating first and second order calculations by looking into the leading-logarithmic second order calculation. This type of exponentiation superseded the previous common style of exponentiation². There are many other exponentiation recipes which are in the $O(\alpha^2)$ equivalent to Kuraev-Fadin ansatz^{3,4}. In our paper we give arguments from the third and higher order calculations as to which of them is actually better.

Let us stress that in the present paper we discuss the popular inclusive "ad hoc" exponentiation which amounts to manipulating the analytical formulas for the inclusive distributions. The other "exclusive" type of exponentiation^{5,3,6}, following closely the Yennie, Frautschi and Suura work⁷ and relying on the Monte Carlo technique, has many theoretical and practical advantages⁸ and will not be used directly in this paper. Let us recall shortly what are the ingredients in the typical "ad hoc" exponentiation. The total cross section for the electron-positron annihilation process $e^-(p_1) + e^+(p_2) \rightarrow f(q_1) + \bar{f}(q_2) + n\gamma(k_i)$ with the initial state QED bremsstrahlung in the n -th order (no exponentiation),

$$\sigma^{(n)}(s) = \sigma_{soft}^{(n)}(s, \epsilon) + \sigma_{hard}^{(n)}(s, \epsilon), \quad s = (p_1 + p_2)^2, \quad (1)$$

includes the combined contribution from the virtual and soft real photons

$$\sigma_{soft}^{(n)}(s, \epsilon) = \sigma_{Born}(s)(1 + \gamma \ln \epsilon(1 + \delta_S^{(n)})) \quad (2)$$

where real photons are below a certain small energy cutoff $\sum_{i=1}^n k_i^0 < \frac{1}{2}\epsilon\sqrt{s}$ and the contribution from (up to n) hard real photons reads

$$\sigma_{hard}^{(n)}(s, \epsilon) = \int_{\epsilon}^1 dv \rho^{(n)}(v) \sigma_{Born}(s(1-v)), \quad (3)$$

where the variable $v = 1 - (q_1 + q_2)^2/(p_1 + p_2)^2$ is roughly proportional to the total energy carried away by the emitted real photons and we denote $\gamma = 2\frac{\alpha}{\pi}(\ln \frac{s}{m_e^2} - 1)$. The first step in the "ad hoc" exponentiation is the replacement of the n -th order formula for the soft cross section

$$\sigma_{soft}^{(n)}(s, \epsilon) \rightarrow \sigma_{Born}(s) F(\gamma)(1 + \delta_S^{(n)}) \int_{\epsilon}^1 \gamma v^{\gamma-1} dv. \quad (4)$$

with the new one which sums corrections from soft photons up to infinite order. In fact, it is possible to do it because soft real photons are emitted almost independently and the only correlation comes from the energy conservation. This phase space effect is embodied in the factor^{7,9,10,11}

$$F(\gamma) = \frac{e^{-C\gamma}}{\Gamma(1+\gamma)} \simeq 1 - \frac{\pi^2}{12}\gamma^2, \quad C = 0.5772156, \quad (5)$$

which shows up for the first time in the $O(\alpha^2)$ calculation^{1,12} and is often expanded and incorporated in $\delta_S^{(2)}$.

In the soft limit $v \rightarrow 0$ the hard photon distribution has a well defined limit $\rho_h^{(n)} \rightarrow \gamma v^{-1}(1 + \delta_S^{(n-1)} + \gamma \ln v)$ such that the total cross section is insensitive to ϵ . In the second step of the "ad hoc" exponentiation the hard photon distribution $\rho_h^{(n)}(v)$ is modified (modulo higher order terms) in such a way that it matches/interpolates smoothly the integrand of eq. (4), i.e., $(1 + \delta_S^{(n)})\gamma v^{\gamma-1}$. The common prescription, due to Kuraev and Fadin¹, is to make the following modification

$$\rho_h^{(n)} \rightarrow \gamma v^{\gamma-1}(1 + \delta_S^{(n)}) + R^{(n)}(v) \quad (6)$$

such that the total cross section can be written as a simple elegant expression (independent of ϵ)

$$\sigma_{KF}^{(n)}(s) = \int_0^1 [(1 + \delta_S^{(n)})\gamma v^{\gamma-1} + R^{(n)}(v)] \sigma_{Born}(s(1-v)) dv. \quad (7)$$

From the inspection of the convergence rate of exponentiated and unexponentiated cross sections up to a second order¹² it is clear that the above exponentiation boosts convergence significantly. One may question the physical meaning of the above procedure⁸ but the most serious practical disadvantage (avoided in the "exclusive" type of exponentiation^{3,8}) is that it has to be repeated from the scratch for any other inclusive distribution different from $d\sigma/dv$. Nevertheless this type of inclusive exponentiation is extremely useful for the semi-analytical discussion of the QED effects^{4,13,14} in the total cross section and one would like to know whether the above prescription sums up the higher order contributions in an optimal way.

In the discussion of the numerical results from the YFS2 Monte Carlo program³ which features the exclusive Yennie-Frautschi-Suura type of exponentiation it was found that another type of an "ad hoc" inclusive exponentiation fits the numerical results very well. The corresponding formula reads

$$\begin{aligned} \sigma_{YFS}^{(n)}(s) &= \int_0^1 F(\gamma) e^{\delta_{YFS}} \gamma v^{\gamma-1} (1 + \delta_S^{(n)} + \Delta^{(n)}(v)) \\ &\quad \times \sigma_{Born}(s(1-v)) dv. \\ \delta_{YFS} &= \frac{1}{4}\gamma + \frac{\alpha}{\pi} \left(-\frac{1}{2} + \frac{1}{3}\pi^2 \right). \end{aligned} \quad (8)$$

The main difference with the previous prescription is that the factor $F(\gamma)\gamma v^{-1}e^{\delta_{YFS}}$ is now factorized off entirely, *also out of the hard part of the spectrum*. There is a simple argument why $F(\gamma)$ should be factorized off: this factor results from the competition among soft photons for the total energy and this effect is also present if there is one or more additional hard photons. Whether to factorize or not the other part is not clear. The aim of our paper is to answer this question.

We shall do it by considering the initial state bremsstrahlung in the strict leading-logarithmic approximation leaving out all subleading terms and light fermion pair production. In this well defined framework we may unfold the perturbative calculation order by order and at a given order exponentiate the distribution $d\sigma/dv$ with one or another of the methods described above. The very attractive feature of this "toy model" is the fact that the exact infinite order result may also be calculated numerically, by means of the Monte Carlo method, with an arbitrary precision¹⁵. In this framework one may find a definite answer to the question: which one of the "ad hoc" exponentiation prescriptions is better? Of course, another immediate question is: to what extent this result is relevant if the subleading terms are included? We shall comment on this question later in the paper.

Let us describe briefly the technique of the Monte Carlo calculation in the "toy model" and present the numerical results. We shall follow here closely the methods used by Skrzypek and Jadach¹⁵. The arbitrarily small lower limit ϵ on the energy of real photons (infrared cut-off) is introduced. The probability of losing a fraction v of the beam energy is expressed as a sum/integral over multiplicities/energies of many photons. Photon energy distributions are obtained by multiple convolution of the Altarelli-Parisi kernel^{1,16}, corresponding to one photon emission. The integration over photon phase space is performed by means of the (by now) well established^{17,3} Monte Carlo technique. The exact solution for the energy loss (so called electron structure function) of one beam particle is given by the following integral

$$D(v) = \exp\left(\frac{1}{2}\gamma \ln \frac{\epsilon}{1-v} + \frac{3}{4}\gamma\right) \times \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\gamma}{2}\right)^n \prod_{i=1}^n \int \frac{dx_i}{x_i} \delta(v - \sum_{i=1}^n x_i) \times \prod_{i=1}^n \chi\left(\frac{x_i}{1 - \sum_{j=1}^i x_j}\right) \theta\left(x_i - \frac{\epsilon}{1-v} \left(1 - \sum_{j=1}^i x_j\right)\right), \quad (9)$$

where $\chi(x) = \frac{1}{2}(1 + (1-x)^2)$. (It is rather straightforward to verify that the above distribution fulfills the Altarelli-Parisi/Lipatov¹⁸ evolution equation.) In our case we are going to calculate the convolution describing simultaneous fragmentation of

two incoming beams

$$\rho^{(\infty)}(v) = \int_0^1 \int_0^1 dv_1 dv_2 \delta(1-v-(1-v_1)(1-v_2)) D(v_1) D(v_2). \quad (10)$$

In the work of Skrzypek and Jadach¹⁵ the single beam fragmentation was considered and a simple M.C. program¹⁷ was enough for the phase space integration. In our case (two-sided fragmentation) we employ the one dimensional version of the Monte Carlo algorithm used in the YFS2 Monte Carlo³. In the first step of the Monte Carlo integration/simulation the (poissonian) multiplicities n and n' for each of the two fragmentation branches are chosen, then, photon momenta x_i and x'_i are independently generated in the (ϵ, v) range according to the $1/x_i$ distribution and finally the constraint $1-v = (1 - \sum_{i=1}^n x_i)(1 - \sum_{j=1}^{n'} x'_j)$ is imposed by appropriate dilatation. The products of χ -functions and θ 's are introduced at the very end by weighting events. The weight variance is rather small and the precision 10^{-4} may be reached with about $4 \cdot 10^6$ events.

The Monte Carlo infinite order result is compared with the $O(\alpha^2)$ exponentiation formulas¹⁴. The raw material for the $O(\alpha^2)$ exponentiation is the $\rho^{(2)}(v)$ distribution which may be taken from Kuraev and Fadin¹ or obtained from the Eq. (10). (One has to truncate it at $O(\gamma^2)$ and integrate over momenta of two photons). The two exponentiation recipes described earlier in the paper provide the corresponding two formulas

$$\begin{aligned} \rho_{KF}^{(2)} &= F(\gamma)\gamma v^{\gamma-1} \left\{ 1 + \frac{3}{4}\gamma + \frac{1}{2}\left(\frac{3}{4}\gamma\right)^2 \right\} \\ &+ \gamma \left(-1 + \frac{1}{2}v \right) (1 + \gamma \ln v) \\ &+ \gamma^2 \left[-\frac{3}{4} + \frac{1}{8}v - \frac{1+3(1-v)^2}{8v} \ln(1-v) \right], \end{aligned} \quad (11)$$

$$\begin{aligned} \rho_{FES}^{(2)} &= F(\gamma)\gamma v^{\gamma-1} e^{\frac{1}{2}\gamma} \left\{ 1 + \frac{\gamma}{2} + \frac{1}{2}\left(\frac{\gamma}{2}\right)^2 + v \left(-1 + \frac{v}{2} \right) \right. \\ &\left. + \gamma \left[-\frac{1}{2}v - \frac{1+3(1-v)^2}{8} \ln(1-v) \right] \right\}. \end{aligned} \quad (12)$$

Note that replacing expansions with explicit exponentials $e^{\frac{1}{2}\gamma}$ and $e^{\frac{1}{2}\gamma}$ to reproduce manifestly known Gribov⁹ leading-logarithmic soft limit does not change the following numerical results and conclusions.

In Fig. 1. we compare the Monte Carlo result $\rho^{(\infty)}$ with the two above $O(\alpha^2)$ approximate solutions. As we see the YFS type exponentiation procedure is clearly

superior: for $v < 0.1$ it is closer to the exact one by an order of magnitude than the Kuraev-Fadin type and for higher v by factor 3 or more. We also included the leading logarithmic version of the formula recommended in the recent LEP workshop⁴ $\rho_{LW}^{(2)} = F(\gamma)\gamma v^{-1}e^{\frac{3}{2}\gamma}[1 + (1-v)^2] + \gamma^2[-\frac{1}{4}v - \frac{1}{8v}(1+3(1-v)^2)\ln(1-v)]$. It turns out to be better than that of Kuraev-Fadin but still worse than that of YFS type. At $v = 0.1$ the workshop formula deviates from the exact solution by 0.1% while the YFS solution agrees within statistical error.

The above discussion was restricted to the leading-log regime. The dominant new effect in Fig. 1 is of the third order origin. How does it compare with subleading contributions? As illustrated in Fig. 2, our new leading $O(\gamma^3)$ correction is as large as the next-to-leading $O(\alpha\gamma)$ correction and much larger than the pure nonlogarithmic $O(\alpha^2)$ correction.

In the comparisons of Fig. 2 we use the $O(\alpha^2)$ exponentiated distribution of the recent LEP workshop⁴ based on the exact $O(\alpha^2)$ calculation of Berends et al.¹² It represents the up to date state of art for the initial state radiation QED effects at LEP/SLC. The more precise analog of this formula with proper YFS exponentiation and explicit $O(\gamma^3)$ contribution¹⁵, but dropping negligible nonlogarithmic $O(\alpha^2)$ terms, reads

$$G_{YFS}^{(3)}(z) = \gamma(1-z)^{\gamma-1} \frac{1}{\Gamma(1+\gamma)} \exp\left(-C\gamma + \frac{3}{4}\gamma + \frac{\alpha}{\pi}\left(2\zeta(2) - \frac{1}{2}\right)\right) \left\{ \frac{1}{2}(1+z^2) + \frac{\alpha}{\pi}\gamma\left[\frac{3}{32} - \frac{3}{4}\zeta(2) + \frac{3}{2}\zeta(3)\right] + \frac{1}{8}\gamma\left[-(1+3z^2)\ln z - 2(1-z)^2\right] + \frac{\alpha}{\pi}\frac{1}{8}\left[4(1+z^2)(\text{Li}_2(1-z) + \ln(1-z)\ln z) - (1+3z^2)\ln^2 z + 2(3+2z+z^2)\ln z + 2(1-z)(3-2z)\right] + \gamma^2\frac{1}{8}\left[\frac{1}{12}(1+7z^2)\ln^2 z + \frac{1}{2}(1-z)(1-3z)\ln z + (1-z)^2\right] + (1-z^2)\text{Li}_2(1-z) \right\}, \quad z = 1-v. \quad (13)$$

In Fig. 2 we use the above expression to estimate the precision of the LEP Workshop formula⁴. It is generally below 0.1% level except high v region where it deteriorates down to 0.5%. Note that fermion pair production is left out in both of them. The precision of the distribution (13) could be estimated provided we include terms of $O(\gamma^4)$, $O(\alpha\gamma^2)$ and $O(\alpha^2)$. Only the last one is available¹².

To conclude let us recall that in the exclusive YFS exponentiation^{3,8,19,20} the numerical difference between the result of the $O(\alpha^2)$ and $O(\alpha)$ calculations for σ and $d\sigma/dv$ was found surprisingly small. This result is true both for initial and final state

bremstrahlung. When taken alone it could be regarded as a numerical accident but when combined with the result of the present leading-logarithmic exercise it strongly indicates that the exponentiation of the YFS type, whether exclusive or inclusive, is *very efficient* in summing up higher order effects. This may have important practical consequences, for example in the Bhabha scattering²¹.

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Fig. 1

FIG. 1. The ratio of the second order exponentiated distribution $\rho^{(2)}(v)$ and the infinite order solution $\rho^{(\infty)}$ calculated by means of the Monte Carlo ($10^6 - 4 \cdot 10^6$ M.C. events per point). Statistical error is below $2 \cdot 10^{-4}$. Numerical input was set for e^+e^- beams at $\sqrt{s} = 92 \text{ GeV}$. Three curves represent exponentiation of the type (a) Kuraev-Fadin, see eq. (11), (b) YFS, see eq. (12) and (c) LEP workshop⁴.

Fig. 2

FIG. 2. In this figure we take the second order exponentiated formula for $G(z) = G(1-v)$ of eq. (3.31) in LEP workshop proceedings⁴ at $\sqrt{s} = 92\text{GeV}$ and (a) we plot $G'(z)/G(z) - 1$ where $G'(z)$ is obtained from $G(z)$ by switching off pure $(\alpha/\pi)^2$ terms in δ_D^{V+S} and δ_E^H . Next, (b) we plot $G''(z)/G'(z) - 1$ where $G''(z)$ is obtained from $G'(z)$ by additionally switching off next-to-leading terms of order $(\alpha/\pi)\gamma$. Finally, (c) we plot $G(z)/G_{YFS}^{(3)}(z) - 1$ where $G_{YFS}^{(3)}(z)$ is the analog of $G(z)$ with the proper YFS exponentiation and third order leading correction, see eq. (13).