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Primordial Nucleosynthesis Without a Computer

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ABSTRACT

The results of the standard theory of big bang nucleosynthesis are derived analytically, *i.e.* without the use of the Wagoner code, utilizing ideas of fixed points and freeze out.

1. Introduction

It has been nearly a quarter of a century since Peebles (1966) and Wagoner, Fowler and Hoyle (1967) performed their numerical calculations of the relic abundances of light elements in a hot big bang universe. The theory, which is now known as the standard theory of primordial nucleosynthesis, augmented by a wealth of observational data forms one of the cornerstones of modern cosmology. The theory is appealing because of its simplicity and predictive power: the only free parameter of the model, the baryon density of the universe, is constrained by the observations of abundances of light elements to contribute only a small fraction to the total energy density of the universe; hence, providing the only quantitative argument in favor of non-baryonic dark matter.

While most of the attention in recent years has been concentrated on refining the observational and experimental aspects of the field, some theoretical progress has also been made. A modern and user-friendly numerical program developed by Kawano (1988), based on the original code of Robert Wagoner (1967), has allowed researchers to study the effects of new physics on the element abundances. Such calculations for many years provided the best limit on the number of light neutrino species (Kolb *et al.* 1989). On other fronts, some non-standard scenarios of primordial nucleosynthesis have been recently proposed, mostly with the purpose of reconciling the abundance observations with a closed baryonic universe (Dimopoulos *et al.* 1988, Fuller *et al.* 1988, and Carlson *et al.* 1990).

Despite this extensive interest in the field of primordial nucleosynthesis, the abundances of deuterium, helium-3, helium-4 and lithium are still calculated only by sophisticated numerical programs, which do not provide any physical understanding of the results. While a detailed analytic argument gives a good estimate for the relic mass fraction of helium-4 (Bernstein *et al.*, 1989), there is not even an order-of-magnitude estimate of the other abundances. This is clearly an embarrassment. The physics is obscured by the numerical codes, and modifications due to new effects are difficult to make. The authors feel that there is a gap in the

literature and this paper is an attempt to fill that gap. We present a calculation which yields accurate results for all the relic element abundances, employing simple analytic methods.

The techniques of fixed points and freeze out used in this paper have been discussed before in a paper by Dimopoulos *et al.* (1988) to study the effects of decaying particles on primordial nucleosynthesis. (The authors thank Robert Wagoner for suggesting that we use the same techniques for the standard theory of nucleosynthesis.) The reader is encouraged to review the first appendix of Dimopoulos *et al.* (1988) for a complete discussion of the method. A short summary is presented here.

The equation governing the abundance of an element i , has the simple and general form

$$\dot{f}_i = J - \Gamma f_i, \quad (1.1)$$

where $J(t)$ and $\Gamma(t)$ are the time dependent source and sink terms, which generally depend on abundances of other elements. As long as the sink term is large, which in an expanding universe means $\Gamma \gg H$, where H is the Hubble parameter at that epoch, f_i follows a time dependent static solution

$$f_i = \frac{J(t)}{\Gamma(t)}, \quad (1.2)$$

where the source and the sink terms almost cancel each other such that $\dot{f}_i \simeq 0$. This state is dubbed as quasi-static equilibrium (QSE). The QSE solution approximates the true solution as long as $\Gamma \gg H$, which would also make numerical integration of the equation difficult.

As the universe expands, the reactions responsible for changing the abundances of elements slow down, both because there is a dilution of densities due to the general expansion, and because the Coulomb barriers become harder to overcome and inhibit most reactions. Generally the result is that the value of f_i freezes out

when $\Gamma \simeq H$. Let us call this time t_f . If QSE holds when we reach freeze-out, and J and Γ are sufficiently fast decreasing functions of time, then

$$f(\infty) \simeq \frac{J(t_f)}{\Gamma(t_f)}. \quad (1.3)$$

(In fact the condition for QSE is $|\dot{\Gamma} - \dot{J}| \ll \Gamma$, which is similar to $\Gamma \gg H$, since $\dot{\Gamma} \sim \dot{J} \sim H$, but can be more stringent. Also freezeout is actually given by $\int_{t_f}^{\infty} \dot{f} dt \leq f(t_f)$. However these conditions are more difficult to calculate. Fortunately, one can show that for most reasonable sources and sinks, in particular those which appear in the nucleosynthesis calculations, QSE fails precisely when freezeout occurs, and the two freezeout conditions give very nearly the same t_f .)

Our task is now obvious. We must identify the largest source and sink terms for each species, check for quasi-static equilibrium, write down the QSE solution, find the freeze-out time, and evaluate the QSE solution at this time to get the final abundance for that species. The final abundance of a species should only depend on the ratio of the largest source to the largest sink for that species at freeze-out.

This paper is organized as follows. In section 2, we introduce our notation and review some useful formulae. In section 3, we write down the equations governing the element abundances of deuterium, tritium, helium-3, helium-4, lithium-6, lithium-7 and beryllium-7, and explain why we kept some of the rates in the equations and ignored others. Section 4 contains our approximate solution of these equations for a particular value of baryon density equivalent to $\Omega_B h^2 = 0.01$. This example will help us to explain the techniques, while keeping the discussion and the formulae simple. In section 5, we discuss the dependence of the relic abundances on the value of baryon density and compute approximate values of the relic abundances for $0.001 < \Omega_B h^2 < 1$. In the last section we summarize our results, draw some conclusions and propose further applications of our techniques.

2. Preliminaries

The framework for understanding the era of nucleosynthesis is discussed in the textbooks of Weinberg (1972) and Kolb and Turner (1990). We shall try to keep our notation the same as that of Weinberg. A natural measure of cosmic time at this era, is the temperature T_9 , in units of 10^9 °K, which is equivalent to 86.171 keV. For three massless neutrino species, the time-temperature relation is independent of the baryon density and is given by

$$\frac{t}{\text{sec}} = \frac{\theta}{T_9^2}, \quad (2.1)$$

where

$$\theta = \begin{cases} 99.6, & T_9 > 5 \\ 178, & T_9 < 1. \end{cases}$$

$T_9 > 5$ is before, and $T_9 < 1$ is after e^+e^- annihilation. If there is need to know the time-temperature relationship during the annihilation era, we shall use table 15.5 of Weinberg. The rate of expansion of the universe at this era is given by

$$H = \frac{1}{2\theta} \text{sec}^{-1} T_9^2. \quad (2.2)$$

The rate equations, generally have the form of two body processes

$$\dot{n}_k = n_i n_j \langle \sigma v \rangle_{ij \rightarrow kl}, \quad (2.3)$$

where n_i is the number density of specie i , the dot represents derivative with respect to time, $\langle \sigma v \rangle$ is the Boltzmann-averaged cross section for the reaction $ij \rightarrow k$ (for more details see Wagoner), and so far we have ignored expansion. The dilution due to expansion can be factored out by using the reduced number density $Y_i = n_i/n_B$. The baryon number density is $n_B = \eta n_\gamma$, where $n_\gamma = 2 \times 10^{28} T_9^3 \text{cm}^{-3}$

is the number density of background photons, and after electrons and positrons have annihilated we expect η to be a constant

$$\eta = 2.8 \times 10^{-8} \Omega_B h^2. \quad (2.4)$$

Ω_B is the ratio of baryon density to the closure density today, and h is the Hubble parameter in units of 100 km/s/Mpc .

We can now write the equations as

$$\dot{Y}_k = Y_i Y_j [ij \rightarrow kl], \quad (2.5)$$

where,

$$[ij \rightarrow kl] = \frac{\rho_B}{g \text{ cm}^{-3}} \frac{N_A \langle \sigma v \rangle_{ij \rightarrow kl}}{\text{cm}^3 \text{ sec}^{-1} g^{-1}} \text{ sec}^{-1}, \quad (2.6)$$

and,

$$\frac{\rho_B}{g \text{ cm}^{-3}} = 0.93 \times 10^{-3} \Omega_B h^2 T_9^3. \quad (2.7)$$

It is customary to quote the value of $N_A \langle \sigma v \rangle$ in $\text{cm}^3 \text{ sec}^{-1} g^{-1}$ for the rates, where N_A is Avogadro's number per gram.

For the sake of clarity of notation, we shall adopt the following convention for the rates: n stands for a neutron, p for a proton, d a deuteron, 3 a helium-3, T a tritium, α a helium-4, 6 a lithium-6, 7 a lithium-7, B a beryllium-7, and γ a photon. Two body reactions are denoted by their initial and final products; for example, $[npd\gamma]$ is the reaction rate for a neutron and a proton to form a deuteron.

For large temperatures $T_9 > 10$, the neutron abundance is given by an equilibrium solution (Weinberg)

$$Y_n = [1 + e^{Q_9/T_9}]^{-1}, \quad (2.8)$$

where $Q_9 = m_n - m_p = 15$. Equilibrium is maintained by weak interactions $n + \nu \leftrightarrow p + e^-$, $n + e^+ \leftrightarrow p + \bar{\nu}$ and $n \leftrightarrow p + e^- + \bar{\nu}$. Equation 15.7.14 of

Weinberg gives the rate for $n \rightarrow p$ as an integral

$$\Gamma(n \rightarrow p) = A \int (1 - \frac{m_{e9}^2}{(Q_9 + q)^2})^{1/2} (Q_9 + q)^2 (1 + e^{q/T_9})^{-1} (1 + e^{-(Q_9+q)/T_9})^{-1} q^2 dq,$$

where $A = 8.0 \times 10^{-8} \text{sec}^{-1}$. The integral runs from $-\infty$ to $+\infty$, leaving out a gap from $-Q_9 - m_{e9}$ to $-Q_9 + m_{e9}$. We can approximate this integral analytically if $T_9 \gg Q_9$, by setting Q_9 and m_e to zero. Then

$$\Gamma(n \rightarrow p) = 0.361 \text{sec}^{-1} (\frac{T_9}{10})^5.$$

Freeze out happens when $\Gamma = H$, which gives $T_{9f} = 11.2$, when the above approximation for Γ does not hold anymore. We can do better by integrating numerically; such a calculation gives $T_{9f} = 9.1$. Therefore at freeze-out, $Y_n = 0.16$. After weak interactions have frozen out, the neutron abundance changes only through the $n \rightarrow p + e + \nu$, and until nucleosynthesis is approximately

$$Y_n = 0.16 \exp[-t/\tau]. \quad (2.9)$$

Bernstein *et al.* have evaluated analytically the corrections to this calculation of the neutron number density and found $Y_n = 0.15e^{-\frac{t}{\tau}}$ for $T_9 \lesssim 2.5$. The level of agreement between the freezeout calculation and the more careful analytic treatment is entirely fortuitous. We expect the freezeout calculations to be accurate only to about a factor of two. This is sufficient for all the elements except He^4 , for which accuracies of a few percent are necessary if one wishes to compare with observations. In fact, numerical calculations show that $Y_n = 0.23$, at $T_9 = 9.1$ but $Y_n = 0.164 \exp[-t/\tau]$ before nucleosynthesis. We will use equation (2.9) to determine the number density of neutrons at the onset of nucleosynthesis ($T_9 \sim 1$).

3. Reactions

The very small ratio of baryons to photons delays the onset of nucleosynthesis and the formation of heavier nuclei until $T_9 \sim 1$, when nuclear reactions can compete with the photodissociation of elements by photons in the high-energy tail of the thermal distribution. With this in mind, we can try to identify the largest production and destruction rates for each element. Kawano's code includes almost one hundred reactions, and their back reactions; however, only a few of these are important. As we will show, deuterium is produced through $n - p$ fusion; mass-three elements are produced through $d - d$, $d - p$ and $d - n$ fusion; helium-four is produced through $d - T$, $d - 3$ and $p - T$ reactions. Lithium-6 is produced by $d - \alpha$ fusion, and mass-seven elements by $T - \alpha$ and $3 - \alpha$ fusion. For such low temperatures we can ignore all the back reactions, except for photodissociation reactions, such as $n + p \leftrightarrow d + \gamma$, and those reactions for which a proton is in the final state, such as $n + He^3 \leftrightarrow T + p$. The number densities of mass-six and seven elements are suppressed compared to mass-two and three elements, since there is no bound mass-five nucleus, and therefore these elements must be produced by scatterings of d , He_3 and T off He^4 which has a higher Coulomb barrier than a proton. Therefore, we can assume that there is less of the mass-six and seven elements present before freeze out, and remove these species as sources and sinks in the equations for n, d, He_3, T and He_4 . This assumption can be checked later. The equations for the light elements are

$$\begin{aligned}
\dot{Y}_n &= Y_d Y_d[ddn3] + Y_d Y_T[dTn\alpha] + Y_p Y_T[pTn3] + Y_d Y_\gamma[d\gamma np] - Y_n Y_p[npd\gamma] \\
&\quad - Y_n Y_3[n3Tp] - Y_n[n] \\
\dot{Y}_d &= Y_n Y_p[npd\gamma] - 2Y_d Y_d([ddpT] + [ddn3]) - Y_d Y_T[dTn\alpha] - Y_d Y_3[d3p\alpha] \\
&\quad - Y_d Y_\gamma[d\gamma np] - Y_d Y_p[dp3\gamma] \\
\dot{Y}_3 &= Y_d Y_p[pd3\gamma] + Y_T Y_p[ptn3] + Y_d Y_d[ddn3] - Y_d Y_3[d3p\alpha] - Y_n Y_3[n3pT] \\
\dot{Y}_T &= Y_n Y_3[n3pT] + Y_d Y_d[ddpT] - Y_d Y_T[dTn\alpha] - Y_p Y_T[pTn3] - Y_p Y_T[pT\gamma\alpha] \\
\dot{Y}_6 &= Y_d Y_\alpha[d\alpha 6\gamma] - Y_n Y_6[n6\alpha T] - Y_p Y_6[p6T\alpha] \\
\dot{Y}_7 &= Y_n Y_6[n67\gamma] + Y_n Y_B[nBp7] + Y_T Y_\alpha[T\alpha 7\gamma] - Y_p Y_7[p7\alpha\alpha] - Y_n Y_7[n78\gamma] \\
\dot{Y}_B &= Y_p Y_6[p6B\gamma] + Y_3 Y_\alpha[3\alpha B\gamma] - Y_\gamma Y_B[B\gamma 3\alpha] - Y_n Y_B[nBp7] \\
&\quad - Y_p Y_B[pB\gamma 8] - Y_d Y_B[dB\alpha\alpha p] \\
\dot{Y}_\alpha &= Y_d Y_3[d3p\alpha] + Y_n Y_3[n3\alpha\gamma] + Y_d Y_T[dTn\alpha] + Y_p Y_T[pT\gamma\alpha]
\end{aligned}$$

The rates are taken from Kawano's code and tabulated in the appendix. The recipe for computing the rates for the inverse reactions is given in Wagoner (1967). Generally, the inverse reaction is equal to the rate, multiplied by $\exp(-Q_9/T_9)$ and a factor of order one coming from the kinematics. For neutrons, we take a decay lifetime of 908sec, which is equivalent to the inverse of $[n]$.

Not all the rates tabulated in the appendix are important. In the neutron equation, we kept $d(\gamma, p)n$, $d(d, n)T$ and $T(d, n)\alpha$ as sources for neutrons. We ignored the inverse of $d(n, \gamma)T$, $He_3(n, \gamma)\alpha$ and $Li_6(n, \gamma)Li_7$, because the photodissociation thresholds for helium-4, tritium and lithium-7 are very high. The values of Q_9 for these reactions are 72.62, 238.81 and 84.17, while $Q_9 = 25.82$ for $p(n, \gamma)d$. Therefore, the rate of photodissociation of tritium, for instance, is suppressed by at least a factor of $e^{(-72.62+25.82)}$ for $T_9 = 1$, compared to photodissociation of deuterium, and would be only important if there were 2×10^{20} more tritium nuclei than deuterons. We assume this does not happen. We also ignored $Be_7(n, p)Li_7$, $Li_6(n, \alpha)T$, $Be_7(n, \alpha)\alpha$, $Li_7(d, n\alpha)\alpha$ and $Li_7(n, \gamma)Li_8$ in the neutron

equation, based on our assumption that not enough of mass-six and seven elements are produced to be significant for the mass one, two, three and four elements. We have kept the decay of neutrons, proton capture, and capture by helium-3 as sinks in the neutron equation, and ignored capture by deuterium, tritium and helium-4, simply because these processes have a lower cross section. The reader should also note that we have ignored all the weak interactions with leptons. Therefore, these equations are only valid once these reactions have frozen out.

In the deuterium equation, the only significant source is $p(n, \gamma)d$. Photodissociation of tritium, helium-3 and helium-4 are suppressed by large Q s, and proton destruction of helium-3 in $d(d, n)He_3$, and helium-4 in $He^3(d, p)He_4$, are suppressed by large Coulomb barriers due to the charge two of helium. We have kept $d(d, p)He_3$, $d(d, p)T$, $T(d, n)\alpha$, $He_3(d, p)\alpha$ and photodissociation of deuterium as the main destruction mechanisms, ignoring processes such as $Li_7(d, n\alpha)\alpha$ and $Be_7(d, p\alpha)\alpha$.

In the helium-3 equation, the main sources are $d(d, p)He_3$, $d(p, \gamma)He_3$ and the inverse of $He_3(n, p)T$. We have ignored $Li_6(p, \alpha)He_3$, $He_3(\alpha, \gamma)Be_7$, $\alpha(p, d)He_3$ and $He_3(He_3, 2p)\alpha$, for reasons similar to those discussed above. The sinks are $He_3(n, p)T$ and $He_3(d, p)\alpha$, having ignored $He_3(n, \gamma)\alpha$ since it is an electromagnetic process (compared to the nuclear process of $He_3(n, p)T$), proton and helium-4 destruction of He^3 , because of the large Coulomb barriers, and photodissociation of helium-3, because of the large Q .

In the tritium equation, the main sources and sinks are $d(d, p)T$, $He_3(n, p)T$ and $d(T, n)\alpha$. $T(p, \gamma)\alpha$ and $T(\alpha, \gamma)Li_7$ are insignificant because they are electromagnetic processes, $T(\alpha, n)Li_6$ is suppressed by a large Coulomb barrier, and $T(p, d)d$ is always smaller than $T(p, n)He_3$.

Due to its stability, once formed, helium-4 is not efficiently destroyed. The main sources of helium-4 are $d(He_3, p)\alpha$, $d(T, n)\alpha$, $He_3(n, \gamma)\alpha$ and $p(T, n)\alpha$. We have ignored the following processes for reasons which by now should be apparent: $Be_7(n, \alpha)\alpha$, $Li_7(p, \alpha)\alpha$, $He_3(He_3, 2p)\alpha$, $Li_7(d, n\alpha)\alpha$ and $Be_7(d, p\alpha)\alpha$. As we

will shortly derive, helium-4 is produced very rapidly at the onset of nucleosynthesis when the abundance of deuterium is high, such that the rates for $T(d, n)\alpha$, $He_3(d, p)\alpha$ and $p(T, n)\alpha$ are high enough to quickly cook all the neutrons into helium.

The equations for the mass-six and seven elements are derived similarly. For Li_6 the largest source is $d(\alpha, \gamma)Li_6$. $T(\alpha, n)Li_6$, $He_3(\alpha, p)Li_6$ and photodissociation of lithium-7 and beryllium-7 are all suppressed by large Q 's. $Li_6(n, \alpha)T$ is a larger sink than $Li_6(n, \gamma)Li_7$, $Li_6(p, \alpha)He_3$ is larger than $Li_6(p, \gamma)Be_7$ and there are no other sinks of lithium-6.

Lithium-7 can be produced by $He_3 - \alpha$ fusion, and neutron capture of lithium-6 and beryllium-7. We have no reason to believe *a priori*, that any one of these rates should dominate. We have also kept neutron destruction and proton destruction in the equation for lithium-7. Similarly, for beryllium-7 the main sources are $He_3(\alpha, \gamma)Be_7$ and perhaps $Li_6(p, \gamma)Be_7$. We do not expect $\alpha(\alpha, n)Be_7$ to contribute due to the large Coulomb barrier. The main sinks for beryllium are neutron, proton and deuterium destruction.

4. Element Abundances From Fixed Points

The set of equations for n , d , He^3 and T separate from those of Li^6 , Li^7 and Be^7 . We shall begin by solving the first set of equations using quasi-static equilibrium, and then use the results to solve the second set. As an example, we shall first work out the relic abundances for $\Omega_B h^2 = 0.01$, and later in the next section, we will explore other values of baryon density.

If all the equations in the first set reach quasi-static equilibrium, one would imagine obtaining an approximate solution by setting all the time derivatives equal to zero. However, this perfectly static solution is inconsistent. The equations obey "neutron number conservation,"

$$\dot{Y}_n + \dot{Y}_d + \dot{Y}_3 + 2\dot{Y}_T + 2\dot{Y}_\alpha = -Y_n[n]. \quad (4.1)$$

If we set $\dot{Y}_n$, $\dot{Y}_d$, $\dot{Y}_3$ and $\dot{Y}_T$ to zero, equation (4.1) forces $\dot{Y}_\alpha$ to be negative and proportional to the rate of decay of neutrons. However, the equation for helium-4 has only production terms, which means $\dot{Y}_\alpha \geq 0$. This condition forces all the abundances to be zero. We therefore set all of $\dot{Y}_n$, $\dot{Y}_d$, $\dot{Y}_3$ and $\dot{Y}_T$ equal to zero, except for the largest one, which by (4.1) is equal to $-(2\dot{Y}_\alpha + Y_n[n])$.

We can now solve a differential equation for the time evolution of one element, and obtain all the other abundances, using their QSE solutions. Initially, the time derivative of the neutrons should dominate those of the other elements, simply because there are more neutrons. We can test this assumption later by comparing $\dot{Y}_n$ with the other rates, using their QSE solutions. Therefore, for temperatures less than the freeze-out temperature for weak interactions we have

$$\begin{aligned}\dot{Y}_n &= -2\dot{Y}_\alpha - Y_n[n] \\ \dot{Y}_d &= \dot{Y}_3 = \dot{Y}_T = 0 \\ \dot{Y}_\alpha &= Y_d Y_3 [d3p\alpha] + Y_d Y_T [dTn\alpha] + Y_3 Y_n [n3\alpha\gamma] + Y_p Y_T [pT\gamma\alpha]\end{aligned}\tag{4.2}$$

After weak interactions freeze out, neutrons decay freely, until the rate of production of helium-4 dominates the decay in (4.2). The neutron density is given by equation (2.9) up until this point. The abundance of deuterium follows its equilibrium value,

$$Y_d = 0.2 \times 10^{-12} \Omega_B h^2 Y_n Y_p T_9^{3/2} e^{25.82/T_9},\tag{4.3}$$

where $Y_p \simeq 1 - Y_n$. This is equivalent to a QSE solution for the deuterium equation where the largest sink is photodestruction,

$$Y_d = \frac{Y_p Y_n [npd\gamma]}{Y_\gamma [d\gamma np]},\tag{4.4}$$

with

$$Y_\gamma [d\gamma np] = 0.471 \times 10^{10} T_9^{3/2} e^{-Q_9/T_9} \frac{[npd\gamma]}{(\rho_B / \text{gcm}^{-3})}.$$

The abundance of deuterons increases as T_9 decreases to 1, building more complex nuclei. One can check for quasi-static equilibrium for the mass-three elements. The

rate of neutron capture for He_3 is

$$Y_n[n3pT] = 6.5 \times 10^4 \Omega_B h^2 \frac{Y_n}{0.1} T_9^3 (1 - .15T_9^{1/2} + .098T_9) (\text{sec}^{-1}),$$

which is much larger than the Hubble expansion rate (2.2) for $T_9 \sim 1$. For tritium, the proton destruction rate, $Y_p[pTn3] = \frac{Y_p}{Y_n} e^{-8.86/T_9} \times [n3pT]$, also larger than the expansion rate. The QSE solutions for helium-3 and tritium are

$$\begin{aligned} Y_3 &= \frac{Y_d Y_p [pd3\gamma] + Y_d Y_d [ddn3] + Y_T Y_p [ptn3]}{Y_d [d3p\alpha] + Y_n [n3pT]} \\ Y_T &= \frac{Y_n Y_3 [n3pT] + Y_d Y_d [ddn3]}{Y_d [dTn\alpha] + Y_p [pTn3]}. \end{aligned} \quad (4.5)$$

These form a set of two algebraic equations which can be solved for the two unknowns, Y_3 and Y_T .

Once helium-4 production becomes efficient, *i.e.* $2\dot{Y}_4 \simeq Y_n[n]$, neutrons rapidly combine to form α -particles. For $\Omega_B = 0.01$, this happens when $T_9 = 0.89$, when $Y_d = 8.2 \times 10^{-5}$, $Y_3 = 1.1 \times 10^{-7}$, $Y_T = 3.5 \times 10^{-5}$, $[dTn\alpha] = 3.4 \times 10^3 (\text{sec}^{-1})$, $[pT\gamma\alpha] = 0.01 (\text{sec}^{-1})$, $[n3\alpha\gamma] = 0.035 (\text{sec}^{-1})$ and $[d3p\alpha] = 4.4 \times 10^2 (\text{sec}^{-1})$. These numbers tell us that the main process for forming He^4 is $d(T, n)\alpha$, which dominates over $d(He_3, p)\alpha$, $[n3\alpha\gamma]$ and $p(T, \gamma)\alpha$. The number density of neutrons at this temperature is $Y_n = 0.125$, which results in a mass fraction of 0.25 for He^4 . Hence, helium-4 has a density of approximately zero before, and a density equal to 0.063 after, $T_9 = 0.89$. Similarly for protons, we have $Y_p = 1 - Y_n$ before, and $Y_p = 0.75$ after, $T_9 = 0.89$.

For $T_9 < 0.89$, we can approximate the neutron conservation equation by

$$\dot{Y}_n = -2\dot{Y}_\alpha. \quad (4.6)$$

Our task is to find a solution for (4.6) with the initial condition $Y_n = 0.125$ at $T_9 = 0.89$, assuming QSE for all the other elements. Let us simplify our original

equations, keeping only the largest terms:

$$\begin{aligned}\dot{Y}_n &= -2Y_d Y_T [dTn\alpha] \\ \dot{Y}_d &= Y_n Y_p [npd\gamma] - Y_d Y_\gamma [d\gamma np] \\ \dot{Y}_3 &= Y_d Y_d [ddn3] - Y_d Y_3 [d3p\alpha] \\ \dot{Y}_T &= Y_d Y_d [ddpT] - Y_d Y_T [dTn\alpha].\end{aligned}$$

To obtain these equations, we assumed that neutrons do not play a role in the equations for helium-3 and tritium, and dropped $Y_d Y_p [pd3\gamma]$ and $Y_T Y_p [pTn3]$ in the helium-3 equation, and $Y_p Y_T [pTn3]$ in the tritium equation. These terms are negligible compared to the terms we kept. For instance, we have $Y_p [pd3\gamma] = 0.19 \times 10^{-2}(\text{sec}^{-1})$, $Y_p [pTn3] = 0.18(\text{sec}^{-1})$ and rapidly decreasing, while $Y_d [d3p\alpha] = 0.36(\text{sec}^{-1})$, $Y_d [dTn\alpha] = 0.28(\text{sec}^{-1})$ and rapidly increasing, at $T_9 = 0.89$.

The QSE solutions are

$$\begin{aligned}Y_d &= \frac{Y_n Y_p [npd\gamma]}{Y_\gamma [d\gamma np]} \\ Y_3 &= Y_d \frac{[ddn3]}{[d3p\alpha]} \\ Y_T &= Y_d \frac{[ddpT]}{[dTn\alpha]}.\end{aligned}\tag{4.7}$$

Substituting for Y_d and Y_T in the neutron equation, we get

$$\dot{Y}_n = -2Y_n Y_n \left(\frac{Y_p [npd\gamma]}{Y_\gamma [d\gamma np]} \right)^2 [ddpT].\tag{4.8}$$

A differential equation of the form $\dot{x} = -x^2 f(t)$, has a closed form solution

$$x(t) = \left(\frac{1}{x(t_i)} + \int_{t_i}^t f(t') dt' \right)^{-1}.$$

The integral is too complicated, if we use the exact form of the rate $[ddpT]$ (see the appendix). We shall use the approximate value of $1.4 \times 10^7 \times \rho_B$. Substituting

for ρ_B and changing the integration variable to T_9 , we have the integral

$$\int_{T_9}^{.89} T_9'^3 \exp(51.64/T_9' - 56.8) dT_9'. \quad (4.9)$$

Approximating the integral, we get

$$Y_n = 1/(\frac{1}{0.125} + \exp(-60.74)T_9^5[\exp(51.64/T_9) - \exp(58.02)]). \quad (4.10)$$

This solution is valid as long as the rate of change of neutrons dominates the rate of change of deuterium, tritium and helium-3. We can guess that the rate of change of deuterium would become larger than the rate of change of neutrons, when the number density of neutrons falls below that of the deuterons. From equation (4.3), this happens at a temperature

$$\begin{aligned} T_9 &= \frac{25.82}{34.12 - \frac{3}{2}\log(T_9)} \\ &= 0.75. \end{aligned}$$

At this temperature, deuterium has the abundance $Y_d = 1.25 \times 10^{-3}$. For temperatures below $T_9 = 0.75$, the time derivative of deuterium dominates the other time derivatives. Therefore, we shall set $\dot{Y}_d = -2\dot{Y}_\alpha$, and $\dot{Y}_n = \dot{Y}_3 = \dot{Y}_T = 0$. The differential equation we need to solve, becomes

$$\dot{Y}_d = -2Y_d Y_d [ddpT], \quad (4.11)$$

with the initial condition that at $T_9 = 0.75$, $Y_d = 1.25 \times 10^{-3}$. We know how to solve this equation. We shall approximate the rate $[ddpT]$ with

$6 \times 10^8 T_9^{-2/3} \exp(-4.258/T_9^{1/3}) \times \rho_B$. The solution takes the form

$$Y_d = \left(\frac{1}{Y_{di}} + \int_{0.75}^{T_9} T_9^{-2/3} \exp(-4.258/T_9^{1/3} + 14.50) \right)^{-1}. \quad (4.12)$$

Approximating the integral, we get

$$Y_d = (8 \times 10^2 + \exp(9.46) - \exp(14.15 - 4.258/T_9^{1/3}))^{-1}. \quad (4.13)$$

We can obtain the final abundance of deuterium by setting the temperature equal to zero in (4.13), which gives $Y_d = 0.7 \times 10^{-4}$.

Next we determine the freeze-out values of helium-3 and tritium. Let us assume the deuterium abundance is approximately equal to its final value when helium-3 and tritium freeze-out. For He^3 , the freeze-out condition, $Y_d[d3p\alpha] = H$, can be solved for T_{9f} : assuming $[d3p\alpha] = 4.36 \times 10^8 / T_9^{1/2} \exp(-1.72/T_9) \times \rho_B$,

$$T_{9f} = \frac{1.72}{14.2 + \frac{1}{2} \log(T_{9f}) + \log(Y_d)},$$

which gives $T_{9f} = 0.40$, for $Y_d = 0.7 \times 10^{-4}$. Near $T_9 = 0.4$, the source term for helium-3 is approximately $[ddn3] = 5.3 \times 10^8 T_9^{-2/3} \exp(-4.258/T_9^{1/3}) \times \rho_B$. Once we substitute for these rates in the QSE solution of helium-3 in (4.7), the abundance of helium-3 is given by

$$Y_3 = 1.2 Y_d T_9^{-1/6} \exp(-4.258/T_9^{1/3} + 1.72/T_9), \quad (4.14)$$

which evaluated at $T_{9f} = 0.40$ gives $Y_3 = 0.2 \times 10^{-4}$.

The calculation of freeze-out temperature for tritium is similar to helium-3. The rate $[dTn\alpha]$ can be well approximated by $1.5 \times 10^9 T_9^{-2/3} \exp(-0.523/T_9) \times \rho_B$,

which gives

$$T_{9f} = \frac{0.523}{15.49 + \frac{1}{3}\log(T_{9f}) + \log(Y_d)} = 0.11.$$

The rate of production of tritium is governed by $[ddpT]$, which is approximately equal to $5 \times 10^8 T_9^{-2/3} \exp(-4.258/T_9^{1/3}) \times \rho_B$. The abundance of tritium is given by

$$Y_T = 0.3Y_d \exp(-4.258/T_9^{1/3} + 0.523/T_9). \quad (4.15)$$

This corresponds to a freeze-out abundance of $Y_T = 0.3 \times 10^{-6}$ for $T_{9f} = 0.11$.

Once we have the solutions for deuterium, helium-3 and tritium, calculating the freeze-out abundances of lithium-6, lithium-7 and beryllium-7 are almost trivial. For lithium-6, the QSE solution is

$$Y_6 = \frac{Y_d Y_\alpha [d\alpha 6\gamma]}{Y_p [p6T\alpha]},$$

where we have ignored neutron capture, which is an assumption we shall justify shortly. The freeze-out temperature can be computed using the approximation $[p6T\alpha] = 3.73 \times 10^{10} T_9^{-3/2} \exp(-8.143/T_9^{1/3}) \times \rho_B$,

$$T_{9f} = \left(\frac{8.143}{18.34 - \frac{1}{2}\log(T_{9f})} \right)^3 = 0.07$$

Therefore, the freeze-out abundance of Li^6 is $Y_6 = 0.4 \times 10^{-14}$.

Let us estimate the number of neutrons at such a low temperature. The largest sink in the neutron equation for $T_9 < 0.13$ is the decay, with a rate much larger than the expansion rate. We can estimate the number of neutrons by calculating the QSE value at this temperature

$$Y_n = \frac{Y_d Y_\alpha [ddn3] + Y_d Y_T [dTn\alpha]}{[n]},$$

which for $Y_d = 0.7 \times 10^{-4}$, $Y_T = 0.3 \times 10^{-6}$ and $[n] = 1/908 \text{sec}^{-1}$, gives $Y_n = 0.4 \times 10^{-9}$. The neutron capture rate by lithium-6 can be computed for

$T_f = 0.07$. We get $[n6T\alpha] + [n6\gamma7] = 0.2 \times 10^{-9}$, much smaller than the Hubble rate, $H = 0.14 \times 10^{-4}$, justifying our assumption to ignore neutrons in the freeze-out of lithium-6.

The QSE solution for lithium-7 is

$$Y_7 = \frac{Y_T Y_\alpha [T\alpha7\gamma] + Y_n Y_B [nBp7]}{Y_p [p7\alpha\alpha]},$$

where we again assume neutrons do not play a role in the freeze-out. The proton destruction rate is given approximately by

$Y_p [p7\alpha\alpha] = 0.76 \times 9.4 \times 10^8 T_9^{-2/3} \exp(-8.473/T_9^{1/3})$, which freezes-out at

$$\begin{aligned} T_{9f} &= \left(\frac{8.473}{14.65 + \frac{1}{3} \log(T_{9f})} \right)^3 \\ &= 0.215 \end{aligned}$$

This is before tritium has frozen out, therefore, we need to compute the abundance of tritium at $T_9 = 0.215$. Using the QSE solution for tritium, we get

$$\begin{aligned} Y_T &= Y_d \left. \frac{[ddpT]}{[dTn\alpha]} \right|_{(T_f=0.215)} \\ &= 0.37 \times 10^{-6}. \end{aligned}$$

To find the freeze-out value of lithium-7, we first need to find the abundance of beryllium. The QSE solution for beryllium is

$$Y_B = \frac{Y_3 Y_\alpha [3\alpha B\gamma]}{Y_n [nBp7]},$$

where we ignored destruction by p and d , which can be shown to be insignificant. The QSE solution for neutrons is

$$Y_n = Y_d \frac{Y_d [ddn3] + Y_T [dTn\alpha]}{Y_p [npd\gamma] + Y_3 [n3pT] + [n]},$$

that of tritium is given in equation (4.7). Using the deuterium abundance of 0.7×10^{-4} , and helium-3 abundance of 0.2×10^{-4} , beryllium freezes out at $T_{9f} = 0.23$, with a value $Y_B = 0.1 \times 10^{-10}$. Evaluating the QSE solution of lithium-7 for $T_9 = 0.215$, we get the freeze-out abundance of Li^7 , $Y_7 = 0.7 \times 10^{-10}$.

5. Varying The Baryon Density

Figures (1) and (2) present the relic abundances of light elements as a function of η , the ratio of baryon density to photon density. Some features of these plots are important in determining the allowed range of baryon densities. It is currently believed that the primordial value of helium-4 mass fraction X_4 , is about 0.22 to 0.24. It is also known that the primordial number density of deuterium relative to hydrogen is larger than 10^{-5} , and that of lithium-7 is about 10^{-10} (or possibly 10^{-9}). Extracting primordial abundances from observations of astrophysical objects is an art and the interested reader is referred to any of the excellent reviews of the subject, for example Bosegaard and Steigman (1985).

The observations indeed allow only a small range of baryon densities. A universe with large baryon density has too much helium-4, too little deuterium, and too much lithium. There are other features of the plots that are interesting. There is a “lithium dip”, where lithium has a number density relative to hydrogen of about 10^{-10} for only a very small range of baryon densities. It is also interesting that while the deuterium number density drops sharply with increasing baryon density, mass-three and mass-seven elements have appreciable relic concentrations for large baryon densities.

In this section we shall attempt to understand the dependence of the element abundances on the baryon density using our analytic tools. We will concentrate on $0.001 \leq \Omega_B h^2 \leq 1$. For $\Omega_B h^2 \leq 0.001$ additional terms must be considered in the QSE solutions; since this range for $\Omega_B h^2$ is observationally excluded anyhow, we will not complicate the discussion by considering these values.

Let us first calculate the abundance of He^4 as a function of $\Omega_B h^2$. Since we are now familiar with the techniques involved, we shall get to work immediately. We first need to find the time when the rate of formation of helium-4 dominates the rate of decay of free neutrons. The He^4 equation is

$$\dot{Y}_\alpha = Y_d Y_T [dT n_\alpha] + Y_p Y_T [pT \gamma \alpha] + Y_d Y_3 [d3\alpha p]. \quad (5.1)$$

During this era, He^4 production is dominated by $[dTn\alpha]$ and $[pT\gamma\alpha]$ reactions:

$$\dot{Y}_\alpha = Y_d Y_T [dTn\alpha] + Y_p Y_T [pT\gamma\alpha].$$

Substituting for Y_T and Y_d their QSE solutions,

$$\begin{aligned} Y_d &= \frac{Y_n Y_p [npd\gamma]}{Y_\gamma [d\gamma np]} \\ Y_T &= \frac{Y_d Y_d [ddpT]}{Y_d [dTn\alpha] + Y_p [pT\alpha\gamma]}, \end{aligned} \quad (5.2)$$

we get

$$\dot{Y}_\alpha = (0.2 \times 10^{-12} \Omega_B h^2 Y_n Y_p e^{25.82/T_9})^2 [ddpT]. \quad (5.3)$$

We next solve for the temperature, T_9^α , when $2\dot{Y}_\alpha = Y_n/\tau$. We shall use an approximate value of $7.5 \times 10^5 T_9^{7/3} \exp(-4.258/T_9^{1/3}) \Omega_B h^2 (\text{sec}^{-1})$ for the rate $[ddpT]$. We shall also need equations (2.1) and (2.9). Taking the log of both sides of

$$\frac{1}{\tau} = 6 \times 10^{-20} T_9^{7/3} (\Omega_B h^2)^3 (0.16 e^{-t/\tau}) Y_p^2 e^{(51.64/T_9 - 4.258/T_9^{1/3})},$$

and solving for T_9 , we get

$$T_9^\alpha = \frac{51.64}{39.6 - 3\log(\Omega_B h^2) - \frac{0.196}{T_9^2} + \frac{4.258}{T_9^{1/3}} - \frac{7}{3}\log(T_9)},$$

which can be well approximated by

$$T_9^\alpha = \frac{51.64}{43.7 - 3\log(\Omega_B h^2)}, \quad (5.4)$$

in the range of interest, $T_9 \sim 1$. Using (2.9) to get the number of neutrons at

temperature T_9^α ,

$$Y_n^\alpha = 0.16 \exp(-0.197/(T_9^\alpha)^2) , \quad (5.5)$$

the relic mass fraction of helium-4 as a function of the baryon density is

$$X_4 = 2Y_n(T_9^\alpha) \simeq 0.32 \exp(-0.142(1 - \frac{3 \log(\Omega_B h^2)}{44})^2). \quad (5.6)$$

For temperatures below T_9^α , the evolution of the abundances is governed by $\dot{Y}_n = -2\dot{Y}_\alpha$, where $\dot{Y}_\alpha$ is given in equation (5.3). Therefore

$$Y_n = \frac{1}{\frac{1}{Y_n^\alpha} + (\Omega_B h^2)^3 e^{-46.92 T_9^5} (e^{(51.64/T_9)} - e^{(51.64/T_9^\alpha)})}. \quad (5.7)$$

We next calculate the deuterium abundance as a function of $\Omega_B h^2$. Once again, as long as the number density of neutrons is larger than that of deuterons, we use equation (5.7) to estimate the number of neutrons, and solve for deuterons in QSE, equation (4.3). The deuteron abundance becomes equal to the neutron abundance at

$$T_9^d = \frac{25.82}{29.53 - \log(\Omega_B h^2) - \frac{3}{2} \log(T_9^d)}, \quad (5.8)$$

with a value $Y_d^i \equiv Y_d(T_9^d) \equiv Y_n(T_9^d)$, where $Y_n(T_9^d)$ is given by (5.7). The rate of change of deuterium now dominates that of neutrons, so that the neutron-conservation equation becomes $\dot{Y}_d = -2\dot{Y}_\alpha$, where $\dot{Y}_\alpha$ is given in equation (5.1). The QSE solution for Y_T is

$$Y_T = \frac{Y_n Y_3 [n3pT] + Y_d Y_d [ddpT]}{Y_p [pT\alpha\gamma] + Y_d [dTn\alpha]}, \quad (5.9)$$

and that of Y_3 is

$$Y_3 = \frac{Y_p Y_d [pd3\gamma] + Y_d Y_d [ddn3]}{Y_n [n3pT] + Y_d [d3p\alpha]}. \quad (5.10)$$

Substituting into (5.1), we find

$$\dot{Y}_d = -2(Y_d)^2 ([ddpT] + [ddn3]) - 2Y_p Y_d [pd3\gamma]. \quad (5.11)$$

Initially the $(Y_d)^2$ term dominates the equation. Taking only this term, the solu-

tion is

$$Y_d^{approx} = \frac{1}{\frac{1}{Y_d} + \Omega_B h^2 e^{18.75} (e^{-4.258/(T_9^d)^{1/3}} - e^{-4.258/T_9^{1/3}})}. \quad (5.12)$$

However, once Y_d begins to drop, the second term becomes important. We can account for this by defining

$$Y_d \equiv Y_d^{approx} e^{-\delta}. \quad (5.13)$$

The equation for δ is

$$\dot{\delta}(t) = -f(t)(1 - e^{-\delta}) + g(t), \quad (5.14)$$

where $f(t) = 2Y_d^{approx}([ddpt] + [ddn3])$, and $g(t) = 2Y_p[pd3\gamma]$. Since initially $\delta = 0$, this equation has the solution

$$\begin{aligned} \delta &\simeq \int_{t_i}^t g(t') dt' \\ &= 1.48 \times 10^3 \Omega_B h^2 Y_p \int_{T_9}^{T_9^d} e^{-3.72/X_9^{1/3}} X_9^{-2/3} (1 + .112X_9^{1/3} + 3.38X_9^{2/3} + 2.65X_9) dX_9 \end{aligned} \quad (5.15)$$

so long as

$$\exp\left(-\int_{t_i}^t g(t') dt'\right) > 1 - \frac{g}{f}. \quad (5.16)$$

We must calculate $\int g$ accurately (for $\delta \gtrsim 1$), since it appears in the exponent; this can only be done numerically (although can be expressed in terms of exponential integral functions). Similarly, verification of condition (5.16) must be done numerically. In fact this condition fails for $\Omega_B h^2 \lesssim 0.01$, when $\delta \ll 1$ and can therefore be ignored. However, to proceed further with our analytic equations, we

shall approximate δ by

$$\delta(T_9) \simeq 2.22 \times 10^3 \Omega_B h^2 (e^{-3.72/(T_9^d)^{1/3}} - e^{-3.72/T_9^{1/3}}), \quad (5.17)$$

for all values of $\Omega_B h^2$. Substituting for δ and Y_d^{approx} in (5.13), we derive an expression for the deuterium abundance as a function of temperature

$$Y_d = \frac{\exp(-2.22 \times 10^3 \Omega_B h^2 (e^{-3.72/(T_9^d)^{1/3}} - e^{-3.72/T_9^{1/3}}))}{\frac{1}{Y_d} + \Omega_B h^2 e^{18.75} (e^{-4.258/(T_9^d)^{1/3}} - e^{-4.258/T_9^{1/3}})}. \quad (5.18)$$

Therefore, the relic abundance of deuterium is

$$Y_d^f = \frac{1}{\Omega_B h^2} \exp(18.75 - \frac{4.258}{(T_9^d)^{1/3}} - 2.22 \times 10^3 \Omega_B h^2 e^{-3.72/(T_9^d)^{1/3}}), \quad (5.19)$$

with T_9^d given by (5.8).

The QSE solution of He^3 is

$$Y_3 = \frac{Y_d Y_d [ddn3] + Y_p Y_d [pd3\gamma]}{Y_d [d3p\alpha] + Y_n [n3pT]}, \quad (5.20)$$

and that of neutrons is

$$Y_n = \frac{Y_d Y_d [ddn3] + Y_d Y_T [dTn\alpha] + Y_\gamma Y_d [d\gamma np]}{Y_p [npd\gamma]}, \quad (5.21)$$

where we ignored $Y_3 [n3pT]$ compared to $Y_p [npd\gamma]$, which is true so long as $Y_3 < 0.5 \times 10^{-4}$. For tritium in QSE, we have

$$Y_T = \frac{Y_d Y_d [ddpT] + Y_n Y_3 [n3pT]}{Y_d [dTn\alpha] + Y_p [pT\alpha\gamma]}. \quad (5.22)$$

Therefore, the neutron abundance is approximately given as

$$Y_n = \frac{Y_d Y_d [ddn3] + Y_d Y_d [ddpT] + Y_\gamma Y_d [d\gamma np]}{Y_p [npd\gamma]}, \quad (5.23)$$

where we assumed $Y_d [dTn\alpha] > Y_p [pT\alpha\gamma]$. If this condition does not hold, the contribution $Y_d Y_d [ddpT]$ in (5.23) is suppressed by a factor of $Y_d [dTn\alpha]/Y_p [pT\alpha\gamma]$. Since $[ddn3]$ and $[ddpT]$ are of the same order, this introduces an uncertainty of at most a factor of two in the neutron number density.

Substituting for neutrons in (5.20), helium-3 freezes-out when

$$Y_d[d3p\alpha] + \frac{Y_d Y_d[ddn3] + Y_d Y_d[ddpT] + Y_\gamma Y_d[\gamma dnp]}{Y_p[npd\gamma]}[n3pT] = H,$$

which we shall approximate as $S_3(T_9^f, \Omega_B) = 1$, where

$$S_3(T_9) = 1.4 \times 10^8 \Omega_B h^2 T_9^{1/2} e^{-1.72/T_9} Y_d + 1.7 \times 10^{13} \Omega_B h^2 T_9^{1/3} e^{-4.258/T_9^{1/3}} Y_d^2 \\ + 1.6 \times 10^{21} T_9^{-1/2} e^{-25.82/T_9} Y_d$$

This condition can be evaluated numerically after substitution of (5.18). The abundance of helium-3 as a function of temperature is given by

$$Y_3 = \frac{2.6 \times 10^8 \Omega_B h^2 Y_d T_9^{1/3} (Y_d e^{-4.258/T_9^{1/3}} + 1.2 \times 10^{-5} e^{-3.72/T_9^{1/3}})}{S_3(T_9, \Omega_B)}, \quad (5.24)$$

and the relic abundance of helium-3 is obtained by letting T_9 be the freeze-out temperature of helium-3 and evaluating Y_d at this temperature from (5.18).

The calculation of the relic abundance of tritium is similar to helium-3. Tritium freezes-out when $S_T(T_9^f, \Omega_B) = 1$, where

$$S_T(T_9, \Omega_B) = \Omega_B h^2 T_9^{1/3} (2.9 \times 10^8 Y_d e^{-0.523/T_9} + 6.4 \times 10^{10} Y_d e^{-4.524/T_9^{1/3} - (T_9/.12)^2} \\ + 1.1 \times 10^4 e^{-3.869/T_9^{1/3}}),$$

with an abundance as a function of temperature

$$Y_T = (1.8 \times 10^8 \Omega_B h^2 T_9^{1/3} Y_d^2 e^{-4.258/T_9^{1/3}} (1 + 5.6 \times 10^4 Y_3) \\ + Y_d Y_3 1.7 \times 10^{21} T_9^{-1/2} e^{-25.82/T_9}) / S_T(T_9, \Omega_B). \quad (5.25)$$

The relic abundance of tritium can be obtained by setting T_9 equal to the freeze-out temperature of tritium, and using the freeze-out abundance of He^3 for Y_3 .

The QSE solution of beryllium is

$$Y_B = \frac{Y_3 Y_\alpha [3\alpha\gamma B]}{Y_n [nBp7]}. \quad (5.26)$$

Using (5.23) for Y_n , the freeze-out temperature of beryllium is determined by $S_B(T_9^1, \Omega_B) = 1$, where

$$S_B(T_9, \Omega_B) = 7.6 \times 10^{13} \Omega_B h^2 Y_d^2 T_9^{1/3} e^{-4.258/T_9^{1/3}} + 5.6 \times 10^{21} Y_d T_9^{-1/2} e^{-25.82/T_9} \\ + 2.3 \times 10^{19} T_9^{-7/6} e^{-18.407/T_9 - 12.826/T_9^{1/3}}.$$

The approximate solution for beryllium is

$$Y_B = \frac{1.2 \times 10^5 \Omega_B h^2 Y_3 T_9^{1/3} e^{-12.826/T_9^{1/3}}}{S_B(T_9, \Omega_B)}, \quad (5.27)$$

which we can evaluate at freeze-out to find the relic abundance of Be^7 .

Lithium has a similar story. The QSE solution is

$$Y_7 = \frac{Y_T Y_\alpha [T\alpha 7\gamma] + Y_n Y_B [nBp7]}{Y_p [p7\alpha\alpha]}. \quad (5.28)$$

The freeze-out temperature is when $S_7(T_9^f, \Omega_B) = 1$, where

$$S_7(T_9, \Omega_B) = 3.0 \times 10^8 \Omega_B h^2 T_9^{1/3} e^{-8.473/T_9^{1/3}}.$$

The relic abundance of lithium-7 is

$$Y_7 = (2 \times 10^4 \Omega_B h^2 Y_T T_9^{1/3} e^{-8.08/T_9^{1/3}} + 7.6 \times 10^{13} \Omega_B h^2 Y_B Y_d^2 T_9^{1/3} e^{-4.258/T_9^{1/3}} \\ + 5.6 \times 10^{21} Y_B Y_d T_9^{-1/2} e^{-25.82/T_9}) / S_7(T_9, \Omega_B), \quad (5.29)$$

where Y_T, Y_d, Y_B and T_9 are all evaluated for the freeze-out temperature of lithium-7 (equations (5.22), (5.23) and (5.26)).

6. Conclusion

We have calculated analytically the abundances of deuterium, tritium, helium-3, helium-4, lithium-7 and beryllium-7 as functions of both time and η . These calculations are summarized schematically in Figure 3. In order to evaluate the success of our analytic calculations we must compare them to the numerical results of the Wagoner-Kawano code. In Figure 1, the dashed line represents our analytic approximation (equation (5.6)) to the numerical solution for the He^4 mass fraction as a function of η (solid line). Our equation is accurate to better than 5% for baryon densities in the range $0.01 < \Omega_B h^2 < 1$. Part of the error is due to our assumption that once the rate of production of helium dominates, all free neutrons form helium nuclei. This is not quite true, since some neutrons decay before all the helium-4 nuclei form. Nevertheless, the agreement is better than one would have expected.

In figure 2, we have plotted our analytic results for the other light elements (dashed lines), as functions of η . The solid lines are results of computer runs using Kawano's code. The helium-3 value in this graph is from equations (5.24) and (5.25), computed at the freeze-out temperatures of He^3 and T respectively, and then combined to give the primordial relic density of helium-3. Similarly, the lithium value is from equations (5.27) and (5.29), computed at the freeze-out temperatures of Be^7 and Li^7 , and then combined to give the relic abundance of lithium. The deuterium relic density is from equation (5.19).

The success of our analytic approximation is indeed remarkable. We are able to predict the relic abundances of d , He^3 and Li^7 , to better than a factor of three over three decades of baryon densities. In fact, we were able to predict not only the freeze-out values of the abundances, but also the evolution of the elements as a function of temperature. Figure (4) compares our analytic solution for $\Omega_B h^2 = 0.01$ to the numerical integration of the equations. Again our QSE and freeze-out calculation provides an excellent approximation to the solution of the rate equations.

In conclusion, we have argued that it is no longer necessary to treat the results

of the theory of primordial big bang nucleosynthesis as coming from a numerical program only. The element abundances follow quasi-static equilibrium and eventually freeze out near their final values. The calculations are relatively simple and provide a better understanding of the physics than is possible from a numerical code containing nearly a hundred reactions. Because they indicate the relative contributions of each nuclear reaction to the final element abundances, it is also possible to use this analysis to identify the relative contributions to the errors in these abundances, and thus indicate which reactions require further study. These techniques could also be used to study the effects of new processes or other modifications to the standard nucleosynthesis model, as was done for the decaying particle scenario of Dimopoulos *et al.* (1988), complementing numerical work. We also propose the applications of these techniques to other fields of Astrophysics where solutions are sought to rate equations with two or more very different timescales.

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APPENDIX

1. $n \rightarrow p$

$$\tau_n = 908 \text{ sec}$$

For the following reactions, the value of $N_A \langle \sigma v \rangle$ (in $\text{cm}^3 \text{sec}^{-1} \text{g}^{-1}$) is given.

The number in the parentheses is the value of Q_9 . T_{9a} and T_{9b} are given as

$$T_{9a} = T_9 / (1 - 9.69 \times 10^{-2} T_9 + 2.84 \times 10^{-2} T_9^{5/3} / (1 - 9.69 \times 10^{-2} T_9)^{2/3})$$

$$T_{9B} = T_9 / (1 + 4.95 \times 10^{-2} T_9)$$

2. $p + n \rightarrow \gamma + d(25.82)$

$$4.40 \times 10^4 (1 - 0.860 T_9^{1/2} + 0.429 T_9)$$

3. $d + n \rightarrow \gamma + H_3(72.62)$

$$66.2(1 + 18.9 T_9)$$

4. $He_3 + n \rightarrow \gamma + He_4(238.81)$

$$6.62(1 + 905 T_9)$$

5. $Li_6 + n \rightarrow \gamma + Li_7(84.17)$

$$5.96 \times 10^3$$

6. $He_3 + n \rightarrow p + H_3(8.86)$

$$7.06 \times 10^8 (1 - .15 T_9^{1/2} + .098 T_9)$$

7. $Be_7 + n \rightarrow p + Li_7(19.08)$

$$6.76 \times 10^9 (1 - .903 T_9^{1/2} + .218 T_9)$$

8. $Li_6 + n \rightarrow He_4 + H_3(55.53)$

$$2.54 \times 10^9 T_9^{-3/2} \exp(-2.53/T_9) \\ + 1.68 \times 10^8 (1 - .261(1 + 49.18 T_9)^{-3/2})$$

9. $Be_7 + n \rightarrow He_4 + He_4(220.39)$

$$2.05 \times 10^4 (1 + 3760 T_9)$$

10. $d + p \rightarrow \gamma + He_3(63.753)$

$$2.24 \times 10^3 T_9^{-2/3} \exp(-3.720/T_9^{1/3}) (1 + .112 T_9^{1/3} + 3.38 T_9^{2/3} + 2.65 T_9)$$

11. $H_3 + p \rightarrow \gamma + He_4(229.947)$

- $$2.20 \times 10^4 T_9^{-2/3} \exp(-3.869/T_9^{1/3})$$
- $$\times (1 + .108 T_9^{1/3} + 1.68 T_9^{2/3} + 1.26 T_9 + .551 T_9^{4/3} + 1.06 T_9^{5/3})$$
12. $Li_6 + p \rightarrow \gamma + Be_7(65.062)$
- $$6.69 + 5 T_{9a}^{5/6} T_9^{-3/2} \exp(-8.413/T_{9a}^{1/3})$$
13. $Li_6 + p \rightarrow He_4 + He_3(46.653)$
- $$3.73 \times 10^{10} T_9^{-3/2} \exp(-8.413/T_9^{1/3} - (T_9/5.50)^2)$$
- $$\times (1 + .050 T_9^{1/3} - .061 T_9^{2/3} - .021 T_9 + .006 T_9^{4/3} + .005 T_9^{5/3})$$
- $$+ 1.33 \times 10^{10} T_9^{-3/2} \exp(-17.763/T_9) + 1.29 \times 10^9 T_9^{-1} \exp(-21.820/T_9)$$
14. $Li_7 + p \rightarrow He_4 + He_4(201.321)$
- $$6.13 \times 10^8 T_9^{-2/3} \exp(-8.473/T_9^{1/3} - (T_9/30.068)^2)$$
- $$\times (1 + .0492 T_9^{1/3} + 1.56 T_9^{2/3} + .539 T_9 - .966 T_9^{4/3} - .845 T_9^{5/3})$$
- $$+ 1.07 \times 10^{10} T_9^{-3/2} \exp(-30.443/T_9) + 1.54 \times 10^6 T_9^{-3/2} \exp(-4.479/T_9)$$
15. $d + He_4 \rightarrow \gamma + Li_6(17.100)$
- $$30.1 T_9^{-2/3} \exp(-7.423/T_9^{1/3})$$
- $$\times (1 + .056 T_9^{1/3} - 4.85 T_9^{2/3} + 8.85 T_9 - .585 T_9^{4/3} - .584 T_9^{5/3})$$
- $$+ 85.5 T_9^{-3/2} \exp(-8.228/T_9)$$
16. $H_3 + He_4 \rightarrow \gamma + Li_7(28.63)$
- $$1.10 \times 10^6 T_9^{-2/3} \exp(-8.08/T_9^{1/3})$$
- $$\times (1 + .0516 T_9^{1/3} - .408 T_9^{2/3} - .148 T_9 + .102 T_9^{4/3} + .0940 T_9^{5/3})$$
17. $He_3 + He_4 \rightarrow \gamma + Be_7(18.407)$
- $$5.79 \times 10^6 T_{9B}^{5/6} T_9^{-3/2} \exp(-12.826/T_{9B}^{1/3})$$
18. $d + d \rightarrow n + He_3(37.938)$
- $$3.97 \times 10^8 T_9^{-2/3} \exp(-4.258/T_9^{1/3})$$
- $$\times (1 + .098 T_9^{1/3} + .876 T_9^{2/3} + .600 T_9 - .041 T_9^{4/3} - .071 T_9^{5/3})$$
19. $d + d \rightarrow p + H_3(46.802)$
- $$4.17 \times 10^8 T_9^{-2/3} \exp(-4.258/T_9^{1/3})$$
- $$\times (1 + .098 T_9^{1/3} + .518 T_9^{2/3} + .355 T_9 - .010 T_9^{4/3} - .018 T_9^{5/3})$$
20. $H_3 + d \rightarrow n + He_4(204.13)$
- $$8.09 \times 10^{10} T_9^{-2/3} \exp(-4.524/T_9^{1/3} - (T_9/.12)^2)$$
- $$\times (1 + .092 T_9^{1/3} + 1.80 T_9^{2/3} + 1.16 T_9 + 10.5 T_9^{4/3} + 17.24 T_9^{5/3})$$

- $+8.73 \times 10^8 T_9^{-2/3} \exp(-0.523/T_9)$
21. $He_3 + d \rightarrow p + He_4(212.994)$
 $6.67 \times 10^{10} T_9^{-2/3} \exp(-7.181/T_9^{1/3} - (T_9/.315)^2)$
 $\times (1 + .058 T_9^{1/3} - 1.14 T_9^{2/3} - .464 T_9 + 3.08 T_9^{4/3} + 3.18 T_9^{5/3})$
 $+4.36 \times 10^8 / T_9^{1/2} \exp(-1.72/T_9)$
22. $He_3 + He_3 \rightarrow 2p + He_4(149.24)$
 $5.96 \times 10^{10} T_9^{-2/3} \exp(-12.276/T_9^{1/3})$
 $\times (1 + .034 T_9^{1/3} - .199 T_9^{2/3} - .047 T_9 + .032 T_9^{4/3} + .019 T_9^{5/3})$
23. $Li_7 + d \rightarrow n + He_4 He_4(175.504)$
 $2.92 \times 10^{11} T_9^{-2/3} \exp(-10.259/T_9^{1/3})$
24. $Be_7 + d \rightarrow p + He_4 He_4(194.57)$
 $1.07 \times 10^{12} T_9^{-2/3} \exp(-12.43/T_9^{1/3})$
25. $Li_7 + n \rightarrow \gamma + Li_8(23.59)$
 $4.90 \times 10^3 + 9.96 \times 10^3 T_9^{-3/2} \exp(-2.62/T_9)$
26. $Be_7 + p \rightarrow \gamma + B_8(1.593)$
 $3.09 \times 10^5 T_9^{-2/3} \exp(-10.262/T_9^{1/3}) + 2.53 \times 10^3 T_9^{-3/2} \exp(-7.306/T_9)$
27. $Li_6 + He_4 \rightarrow \gamma + B_{10}(51.764)$
 $4.06 \times 10^6 T_9^{-2/3} \exp(-18.790/T_9^{1/3} - (T_9/1.326)^2)$
 $\times (1 + .022 T_9^{1/3} + 1.54 T_9^{2/3} + .239 T_9 + 2.20 T_9^{4/3} + .869 T_9^{5/3})$
 $+1.91 \times 10^3 T_9^{-3/2} \exp(-3.484/T_9) + 1.01 \times 10^4 T_9^{-1} \exp(-7.269/T_9)$

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FIGURE CAPTIONS

1. The relic mass fraction of helium-4 to baryons, as a function of the ratio of the baryon density to the photon density. The solid line is from computer runs, and the dashed line is our analytic approximation.
2. The relic abundance of light elements, relative to hydrogen, as a function of the ratio of the baryon density to the photon density. The solid lines are from computer runs, and the dashed lines are our analytic approximations.
3. Schematic representation of the analytic calculation of the abundances of the light element abundances.
4. The evolution of the abundances of the light elements as a function of the temperature for $\Omega_B h^2 = 0.01$. The dotted lines are the approximate solutions derived using fixed points and freeze-out, and the solid lines are obtained by numerical integration of the rate equations on a computer.

Figure 1

Figure 2

Figure 3

Weak interactions

$$T_9^W - \text{weak-interaction freezeout; } T_9^W \simeq 10 ; Y_n(T_9^W) \simeq 0.16$$

Free neutron decay

$$\begin{cases} \dot{Y}_n = -Y_n[n] \\ \dot{Y}_d = \dot{Y}_3 = \dot{Y}_T = 0 \\ \dot{Y}_\alpha = Y_d Y_3 [d3p\alpha] + Y_p Y_T [pT\alpha\gamma] + Y_d Y_T [dTn\alpha] \end{cases}$$

$$\Rightarrow Y_n(T_9) \Rightarrow Y_d(T_9) \Rightarrow Y_T(T_9), Y_3(T_9) \Rightarrow \dot{Y}_\alpha(T_9)$$

$$T_9^\alpha - Y_\alpha = Y_n[n]; T_9^\alpha \simeq \frac{51.64}{43.7 - 3 \log \Omega_b h^2} \Rightarrow Y_n(T_9^\alpha) \Rightarrow X_4 \simeq 2Y_n(T_9^\alpha)$$

Neutron cooking

$$\begin{cases} \dot{Y}_n = -2\dot{Y}_\alpha \\ \dot{Y}_d = \dot{Y}_3 = \dot{Y}_T = 0 \\ \dot{Y}_\alpha = Y_d Y_3 [d3p\alpha] + Y_p Y_T [pT\alpha\gamma] + Y_d Y_T [dTn\alpha] \end{cases}$$

$$\Rightarrow Y_d(Y_n) \Rightarrow Y_T(Y_d(Y_n)), Y_3(Y_n, Y_d(Y_n))$$

$$\Rightarrow \dot{Y}_n = -2\dot{Y}_\alpha[Y_n] \Rightarrow Y_n(T_9) \Rightarrow Y_d(T_9), Y_T(T_9), Y_3(T_9)$$

$$T_9^d - Y_n = Y_d; T_9^d \simeq \frac{25.82}{25.93 - \log \Omega_B h^2 - \frac{3}{2} \log T_9^d}$$

Deuterium cooking

$$\begin{cases} \dot{Y}_d = -2\dot{Y}_\alpha \\ \dot{Y}_n = \dot{Y}_3 = \dot{Y}_T = 0 \\ \dot{Y}_\alpha = Y_d Y_3 [d3p\alpha] + Y_p Y_T [pT\alpha\gamma] + Y_d Y_T [dTn\alpha] \end{cases}$$

$$\Rightarrow Y_n(Y_d, Y_T), Y_T(Y_d, Y_n, Y_3), Y_3(Y_d, Y_n)$$

$$\Rightarrow \dot{Y}_d(Y_d) \Rightarrow Y_d(T_9) \Rightarrow Y_d^f$$

Freezeout

$$\Rightarrow Y_3(T_9) \Rightarrow T_9^f(3), Y_3^f$$

$$\Rightarrow Y_T(T_9) \Rightarrow T_9^f(T), Y_T^f$$

$$\dot{Y}_B = \dot{Y}_7 = 0 \Rightarrow Y_B(T_9), Y_7(T_9) \Rightarrow T_9^f(B), Y_B^f; T_9^f(7), Y_7^f$$

Figure 4