A New Variable to Study Intermittency

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Abstract

It is proposed to study intermittency properties of particle spectra using the variables for which the single particle distribution is constant. The construction of such a set of variables is described. It is shown that this method drastically reduces distortions of intermittency due to a non-uniform single particle density. A method of systematic analysis of intermittency in 3-dimensions based on the new variable is suggested.

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Recently, several experimental groups presented data on "intermittency", i.e. the studies of factorial moments of multiplicity distributions as a function of the resolution [1]. To extract from these data the anomalous dimensions of the spectra, one needs to determine the slopes f_q in the $\log - \log$ plot [2]:

$$\log F_q(\delta x) = \log F_q(\Delta x) + f_q \log(\Delta x / \delta x), \quad (1)$$

where δx and Δx are two resolutions. The slopes f_q are related to the anomalous dimensions d_q by:

$$d_q = f_q / (q - 1). \quad (2)$$

In the theory of fractals Eq. (1) is called "anomalous scaling" [3] and it is expected to be valid in the limit $\Delta x \gg \delta x \rightarrow 0$. In this theoretical limit Eq. (1) has two important properties:

- the intermittency slopes f_q do not depend on the shape of the single particle density [2] and thus they measure directly the fractal properties of the spectrum,
- they are invariant with respect to any continuous change of variables:

$$x \rightarrow X = X(x), \quad (3)$$

which is non-singular, i.e. for which both dX/dx and dx/dX do not vanish in the considered region of x .¹ Consequently, in the ideal limit of vanishing resolution, the selection of a particular variable to study the fractal properties of the spectrum is, to a large extent, arbitrary.

In practical applications, however, it is necessary to consider the finite intervals δx . Indeed, the scaling law, Eq. (1), is never expected to be valid until the mathematical limit $\delta x \rightarrow 0$ but rather in a limited range of δx . Furthermore, in real experiments there is always some limit on experimental resolution which can be achieved. Hence the experimental results do depend on the shape of the single particle spectrum [4] and on the choice of variables used for analysis. This is rather unfortunate because it makes difficult to assess the real significance of the obtained results and to compare measurements from different experiments. The problem becomes even more acute when one studies distributions in more than one dimension (as shown by recent experimental [5] and theoretical [6,7] results this is now becoming a crucial step in studies of intermittency). In the present paper we propose a solution to this problem which reduces radically the ambiguities discussed above.

Let us first make a trivial observation that the dependence on the shape of the single particle spectrum and that on the choice of variables are closely related, because the shape of the single particle spectrum depends obviously on the choice of variables. It is thus possible to deal with two of them at the same time. Taking advantage of this

¹To see this we observe that Eq. (3) implies, for $\delta x \rightarrow 0$, $\delta x = \delta(x(Y)) \approx \frac{dx}{dY} \delta Y$ and thus $\log(\Delta x / \delta x) = \text{const}(\delta Y) + \log(\Delta Y / \delta Y)$.

possibility, we propose to study intermittency in terms of variables for which the single particle density is a constant. We show that this choice does indeed reduce drastically the distortions of the simple scaling law (1) caused by a non-uniform single-particle spectrum.

To define the new variables, let us first discuss the one-dimensional case. Assume that the single particle distribution in a variable x is measured and given by a (non-negative) function $\rho(x)$. Our new variable $X = X(x)$ is defined as:

$$X = \int_a^x \rho(x) dx / \int_a^b \rho(x) dx, \quad (4)$$

where a and b are the lower and upper phase space limits of the variable x .

This new "integral" variable has the following, very useful properties:

1. its value for a given particle does not depend on the choice of the original variable x , but is uniquely determined by the shape of the single particle spectrum,
2. the single particle distribution in the variable X is uniform in the interval from 0 to 1.

The property (1) gives a new way to compare meaningfully the data obtained in different experiments. The property (2) removes the dependence of the intermittency parameters on the shape of the single particle distribution.

To see how the proposed change of variables affects the behaviour of factorial moments, and thus determination of intermittency parameters, let us consider a two particle inclusive density of the form:

$$\rho(x_1, x_2) = \rho(x_1)\rho(x_2)/|x_1 - x_2|^{1/2}, \quad (5)$$

where $\rho(x)$ is the single particle inclusive density. This distribution exhibits a power law singularity at $x_1 = x_2$. Consequently, it implies the anomalous scaling, Eq. (1), of factorial moments in the limit $\delta x \rightarrow 0$.

To study the factorial moments at a finite resolution we observe that they can be expressed as integrals of inclusive distributions [8]. For F_2 we have:

$$F_2 = \frac{< k(k-1) >}{< k >^2} = \frac{\int_x^{x+\delta x} dx_1 \int_x^{x+\delta x} dx_2 \rho(x_1, x_2)}{[\int_x^{x+\delta x} \rho(x) dx]^2}, \quad (6)$$

where k is the number of particles in the interval $(x, x + \delta x)$. This formula allows to calculate factorial moments of the distribution in any variable. In Figure 1 the horizontal averages:

$$\frac{1}{M} \sum_{m=1}^M < k_m(k_m - 1) > \frac{1}{[\frac{1}{M} \sum_{m=1}^M \rho(x) dx]^2}, \quad (7)$$

where $M = (b - a)/\delta x$ is the number of subintervals, are compared for distributions in variables x and X defined by Eq. (4). In Figure 1a the results are presented for $\rho(x) = x \exp(-6x)$ and $0 < x < 1$. One sees that the moment F_2 of the x -distribution

does show the effect of the non-uniform single particle density: it deviates substantially from the simple linear dependence, Eq. (1). One sees also that this non-linear behaviour is almost entirely removed when the moment is calculated from the X -distribution. In the same figure we also show the factorial moment of the x -distribution corrected according to the prescription of Fialkowski *et al.* [4]. This correction is not as effective in removing the non-linear behaviour as the change of variable we propose. In Figure 1b the results are presented for $\rho(x) = \exp(-x^2)$ and $0 < x < 4$, with similar conclusions.

We feel that these results are encouraging: indeed, for two rather different single particle spectra, our prescription turned out to be very effective in removing the distortion of the simple scaling law, Eq. (1), due to the non-uniform single particle spectrum. It is also important to note that, since the X -distribution is uniform, the "horizontal" and "vertical" averages of the moments are equivalent in the X variable.

We now consider 3-dimensional distributions. First, we observe that if the single-particle density $\rho(x, y, z)$ factorizes:

$$\rho(x, y, z) = \rho_x(x)\rho_y(y)\rho_z(z), \quad (8)$$

our one-dimensional definition, Eq. (4), can be applied separately to the three independent variables x, y and z . In this way one obtains three new variables X, Y and Z which satisfy the conditions (1) and (2) above. If the single particle density does not factorize, the change of variables which guarantees a constant single particle spectrum is not uniquely defined; any transformation with the Jacobian equal to $\rho(x, y, z)$ does the trick. Therefore a specific choice must be made and we propose to generalize the construction in the following way.

We first introduce the probability densities:

$$P(x, y, z) = \rho(x, y, z) / \int dx dy dz \rho(x, y, z), \quad (9)$$

$$P(x, y) = \int dz P(x, y, z), \quad (10)$$

$$P(x) = \int dy dz P(x, y, z), \quad (11)$$

where the integrals run over the full phase-space. The new set of variables is then defined in terms of these densities as:

$$X = \int_{a_x}^x P(x) dx, \quad (12)$$

$$Y = \int_{a_y}^y P(x, y) dy / P(x) = \int_{a_y}^y P(X, y) dy \quad (13)$$

$$Z = \int_{a_z}^z P(x, y, z) dz / P(x, y) = \int_{a_z}^z P(X, Y, z) dz \quad (14)$$

The second equalities in Eqs. (13) and (14) indicate the simple algorithm which realizes the transformation $(x, y, z) \rightarrow (X, Y, Z)$: one performs the change of variables successively, $(x, y, z) \rightarrow (X, y, z)$, $(X, y, z) \rightarrow (X, Y, z)$, $(X, Y, z) \rightarrow (X, Y, Z)$, each time using the density of particles in the corresponding set of variables. This prescription reduces to the one given by Eq. (4) if the factorization, Eq. (8), holds. Therefore it is natural to choose the original variables (x, y, z) so that the factorization condition is satisfied as closely as possible. The results of many high energy experiments suggest that a reasonable choice may be rapidity, transverse momentum, p_T , and azimuthal angle, ϕ , of the particle.

Thus the transformation described above changes a complicated 3-dimensional single particle distribution $\rho(x = rapidity, y = p_T, z = \phi)$ into a uniform distribution $\rho_0(X, Y, Z) = 1$, where variables, X, Y and Z are the "integral" variables generated from rapidity, p_T and ϕ by use of Eqs. (12,13,14); they range between 0 and 1.

This new uniformly populated phase space can be now divided into different subvolumes and the standard method can be used to estimate the intermittency parameters. Since we are using the "integral" variables X, Y, Z , we expect that, as explained in the one-dimensional case, the bias caused by a non-uniform single particle spectrum is practically removed.

This allows to study a physical dependence of these parameters on the position of the point (X, Y, Z) in phase-space and on the shape of the subvolumes. The position dependence can be easily studied by separate analysis in selected subspaces. A possible method to study the dependence of intermittency parameters on the shape of the subvolume is described below.

Let us describe the shape of a rectangular subvolume ($\delta V = \delta X \cdot \delta Y \cdot \delta Z$) or equivalently a direction in the X, Y, Z space by two parameters, A and B :

$$A = \delta Y / \delta X, \quad (15)$$

$$B = \delta Z / \delta X \quad (16)$$

We define the slope f_θ as calculated along direction (A, B) if for each division of the original volume V into parts of equal size Eqs. (15,16) are fulfilled for all subvolumes.

In practical calculations we propose the following algorithm:

- The full intervals in X, Y and Z ($\Delta X = \Delta Y = \Delta Z = 1$) are divided onto M_X, M_Y and M_Z intervals (each of M_X, M_Y, M_Z are powers of 2). Then according to Eqs. (15,16): $A = M_Y / M_X$ and $B = M_Z / M_X$
- The factorial moments are calculated by standard procedure averaging over all subvolumes $M_X \cdot M_Y \cdot M_Z$ ($\delta V = 1 / (M_X \cdot M_Y \cdot M_Z)$).
- The numbers of intervals are reduced to $M_X/2, M_Y/2$ and $M_Z/2$ and the factorial moments are calculated by averaging over the new subvolumes ($\delta V_{new} = 8 \cdot \delta V_{old}$). This continues until one of the M_X, M_Y or M_Z values equals unity.

The procedure allows to study in a systematic way the behaviour of intermittency parameters in 3-dimensional phase space.

It is not difficult to convince oneself that, if the singularity in the two particle density distribution has the form:

$$\rho(X_1, X_2, Y_1, Y_2, Z_1, Z_2) = \frac{1}{[c_x(X_1 - X_2)^2 + c_y(Y_1 - Y_2)^2 + c_z(Z_1 - Z_2)^2]^{1/2}}, \quad (17)$$

which is a natural generalization of Eq. (5) for the 3-dimensional case [7], then the slope f_2 is independent of the direction and of the position in phase space. It would be interesting to verify this experimentally.

In summary, we propose to study intermittency in terms of the new "integral" variables, which automatically guarantee a flat single particle spectrum. We showed that this method reduces very effectively a well-known bias in the determination of intermittency parameters. It gives also a natural framework for intermittency studies in more than one dimension.

We close the paper with the following two comments.

- The use of the "integral" variables leads to an undistorted scaling law (1) for a large class of models which contains, in particular, the random cascading models of the type discussed in [2]. However, this result can only be approximate for models which contain non-Markovian processes [9].
- The normalization of the factorial moments of the spectra in the "integral" variables at high resolution does depend on the shape of the single particle spectrum and on the type of singularity of the multiparticle distribution. With $\rho_2(x_1, x_2)$ given by (5) one obtains

$$F_2 = (\Delta/\delta)^{f_2} \frac{2 \int \rho(x)^{1+f_2} dx}{(2 - f_2)(1 - f_2) \int \rho(x) dx^{1+f_2}}. \quad (18)$$

Thus the measurements of the normalization may serve as an independent information about the nature of the singularity.

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References

- [1] For a recent review of the experimental data, see ,e.g., A.Bialas, "Intermittency '90", CERN preprint CERN-TH-5791/90, to be Published in Nucl.Phys.B (Proc.Suppl.).
- [2] A.Bialas and R.Peschanski, Nucl.Phys. B273 (1986) 703.
- [3] See, e.g., G.Paladin and A.Vulpiani, Phys.Rep. 154 (1987) 247.
- [4] K.Fialkowski, B.Wosiek and J.Wosiek, Acta Phys.Pol. B20 (1989) 639.
- [5] KLM coll., R.Holynski et al., Phys.Rev.C40 (1989) R2449. NA22 coll., W.Kittel, Nijmegen preprint, to be published in the Proceedings of the Santa Fe workshop (March 1990). DELPHI coll. , private communication from A.De Angelis and N.Demaria.
- [6] W.Ochs, "Intermittency and cascade processes" Munich preprint MPI-PAE/PTh 35/90, to be published in Proc. of the Santa Fe workshop (March 1990). W.Ochs "The importance of phase space dimension in the intermittency analysis of multiparticle production", Munich preprint MPI-PAE/PTh 26/90, to be published.
- [7] A.Bialas and J.Seixas, "Strong Intermittency in Momentum Space", CERN preprint, CERN-TH-5757/90, to be published. A.Bialas, "Three topics in intermittency", to be published in the Proceedings of the Marburg CAMP workshop (May 1990).
- [8] See, e.g., A.H.Mueller, Phys.Rev. D4 (1971) 150.
- [9] J.Seixas, private communication.

Figure Caption

The $\log F_2$ dependence on $1/M$ where M is the number of intervals. The double-solid lines show slope assumed in the anomalous scaling law ($f_2 = 0.2$). The dotted lines were calculated according to Eq. (7), whereas the dashed-dotted lines according to prescription given in Ref. [4]. The solid lines show results obtained by use of the "integral" variable introduced in this paper.

In a) the results are presented for $\rho(x) = e^{-x^2}$ and $0 < x < 4$, whereas in b) for $\rho(x) = xe^{-6x}$ and $0 < x < 1.5$.

Fig. 1