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General theory of propagators / Umezawa, H, Visconti, A

CERN-55-15

Published in Nuovo Cimento 1 (1955) pp 1079-1103

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H UMEZAWA and A VISCONTI
1955, Giugno
Il Nuovo Cimento
Serie X, Vol 1, pag 1079 1103

CERN 55 - 15
THEORETICAL STUDY DIVISION

General Theory of Propagators

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(ricevuto il 15 Marzo 1955)

Summary — A general theory of propagators is developed. All propagators are derived from a generating functional. Renormalization corresponds to the adjustment of the coupling constants of the external sources. As an example, renormalized equations for Compton and Møller scattering are derived.

1 - Introduction

SCHWINGER ⁽¹⁾ has proposed a theory of propagators ^(§) which has several patent advantages over previous ones based on perturbation theory. The purposes of the present paper ^(×) are: 1) to give a general scheme for this theory by starting from a new representation (source free representation) which permits one to obtain all propagators and their equations from a generating functional and a set of basic equations, 2) to derive new equations, different from those of Bethe-Salpeter, for Compton and Møller scattering. These equations make the renormalisation procedure automatic and they can be treated by a covariant procedure in which successively increasing numbers of virtual photons and pairs are taken into account. It seems interesting to

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(§) Propagators are sometimes called GREEN functions see, e.g., J SCHWINGER ⁽¹⁾

(×) An abstract of the present theory has been given in a series of publications in *Comptes Rendus* ⁽²⁾

⁽¹⁾ J SCHWINGER: *Proc Nat Acad*, **37**, 452, 455 (1951)

⁽²⁾ H UMEZAWA and A VISCONTI: *Compt Rend (Paris)*, **239**, 690, 749, 1466 (1954)

note that the solutions of both equations can be obtained by means of only one equation, namely for their common resolvent

The paper is divided into four parts: the first part, sections 2, 3, and 4, contains a concise formulation of the theory of propagators for a system of particles. Following SCHWINGER, we introduce external sources; but in addition we choose a representation in which the electron and electromagnetic fields obey the same equations as they would in Heisenberg representation without sources. The sources influence therefore the state vector only, not the field operators. This representation is useful because it enables us to define initial and final states as eigenstates of the Hamiltonian without sources. The time variation of the state vector is given by a unitary evolution operator U which is used as generating functional for all many body propagators. From the equations U has to obey, the functional equations for the propagators are derived.

In the second part, which comprises sections 4 and 5, it is shown that the many-body-propagators give physical information about the system (theorems I and II of section 4) and have an intuitive graphical interpretation.

Sections 6, 7 and 8 (the third part) deal, as an example, with the two-body-propagators. The connection between the present theory and the Bethe Salpeter equation is studied. The integral equations found for Møller and Compton scattering differ from the Bethe-Salpeter equation and have the important property of «renormalisation invariance». The solution of these integral equations can be represented in terms of «skeletons» (*), i.e. irreducible diagrams which include neither self energy nor vertex diagrams (+). In section 8, the external lines of these irreducible diagrams are shown to correspond to the «free dressed» particle: the particle with the experimental mass and charge.

In sections 9 and 10, we study the connection between the present theory and the conventional renormalisation. We shall show that our theory can be made to agree with the renormalisation theory.

The last part, section 11, is devoted to the study of the solutions of our equations for Møller and Compton scattering.

(*) A provisional approach to this problem has been given by S. TANAKA and H. UMEZAWA (3).

(+) It has been pointed out by H. S. GREEN (4) that elimination of all uncorrected quantities γ_μ , S_F , D_F , by means of quantities including radiative corrections might lead to a renormalized theory. This suggestion is followed up in sections 9 and 10.

(3) S. TANAKA and H. UMEZAWA: *Prog. Theor. Phys.*, **10**, 617 (1953).

(4) H. S. GREEN: *Proc. Phys. Soc.*, A **66**, 873 (1953).

2 – Formulation of the Theory

Let us consider an electron position field $\psi(x)$, $\bar{\psi}(x)$ interacting with an electromagnetic field $A_\mu(x)$

The external sources of these fields we will denote by η , $\bar{\eta}$, and J_μ respectively. In the representation chosen by us, which we shall call the source free representation, the field equations are

$$(2.1a) \quad \not{D}(x) \psi(x) \equiv (\gamma_\mu \partial_\mu + m - ie\gamma_\mu A_\mu(x)) \psi(x) = 0$$

$$(2.1b) \quad \bar{\psi}(x) \not{D}(x) \equiv \bar{\psi}(x) (\gamma_\mu \partial_\mu + m + ie\gamma_\mu A_\mu(x)) = 0$$

$$(2.1c) \quad \square A_\mu(x) = -j_\mu(x) \equiv -\frac{e}{2} \bar{\psi}(x) \gamma_\mu \psi(x) + \text{h.c.}$$

If we denote by ψ' , $\bar{\psi}'$, A_μ the same fields in Heisenberg representation their field equations are given by

$$(2.2a) \quad \not{D}(x) \psi'(x) = \lambda \eta(x)$$

$$(2.2b) \quad \bar{\psi}'(x) \not{D}(x) = \lambda \bar{\eta}(x)$$

$$(2.2c) \quad \square A'_\mu(x) = -j_\mu(x) - \lambda' J_\mu(x),$$

where λ and λ' are the coupling constants of the external sources. This corresponds with an interaction Hamiltonian due to the sources

$$(2.3) \quad \mathcal{H}'_s(x) = -\lambda \bar{\psi}'(x) \eta(x) - \lambda \bar{\eta}(x) \psi'(x) - \lambda' J_\mu(x) A'_\mu(x)$$

It is shown in section 3 that all meaningful results are obtained by taking $\eta = \bar{\eta} = J_\mu = 0$ at the end of the calculations. The system then becomes conservative.

Let us assume that the state vectors Ψ , Ψ' in these two representations are connected by the unitary transformation U and that Ψ and Ψ' agree on a surface σ_{II} . Then we have

$$(2.4) \quad \Psi[\sigma] = U[\sigma, \sigma_{II}, \eta, J] \Psi[\sigma_{II}],$$

(⁵) H. UMEZAWA and S. ONEDA: *Nuovo Cimento*, **12**, 566 (1954); H. UMEZAWA, J. PODOLANSKI and S. ONEDA: *Proc. Phys. Soc.*, A (in the press); T. KINOSHITA: *Phys. Rev.*, **96**, 199 (1954).

where the evolution operator U satisfies

$$(2.5) \quad i \frac{\delta}{\delta \sigma(x)} U[\sigma, \sigma_H, \eta, J] = U[\sigma, \sigma_H, \eta, J] \mathcal{H}'_s(x)$$

As shown by UMEZAWA *et al.* ⁽⁵⁾, each term in the interaction Hamiltonian density must include an even number of operators, which anticommute with any given field operator. By taking $\psi'(x)$, $A'_\mu(x)$, $\eta(x)$ and $J_\mu(x)$ successively as test operators, we deduce from (2.3)

$$(2.4a) \quad [\eta(x), \bar{\psi}'(x')]_+ = [\eta(x), \psi'(x')]_+ = [\eta(x), \eta(x')]_+ = 0$$

$$(2.4b) \quad [\eta(x), A'_\mu(x')]_- = [\eta(x), J_\mu(x')]_- = 0$$

$$(2.4c) \quad [J_\mu(x), \psi'(x')]_- = [J_\mu(x), \bar{\psi}'(x')]_- = \bar{0}$$

$$(2.4d) \quad [J_\mu(x), A'_\mu(x')]_- = [J_\mu(x), J_\mu(x')]_- = 0$$

These commutation relations are proved to be valid only when x and x' have a space-like distance. We can, however, assume them to obtain for any two points x, x' without introducing a contradiction to the field equations. This implies that the operators $\eta, \bar{\eta}, J_\mu$ are quite independent of the electron-photon system, since they commute with the Hamiltonian at arbitrary points; there is no reaction of the field on the sources.

Denoting by H the Hamiltonian in the source-free representation, its eigenstates Ψ_n are defined by

$$(2.7) \quad H\Psi_n = E_n\Psi_n$$

It is an advantage of the present representation that the states Ψ_n are independent of the sources. The vacuum Ψ_0 is defined as the state of lowest energy E_0 . Then

$$(2.8a) \quad H\psi^\pm(x, E)\Psi_n = (\mp E + E_n)\psi^\pm(x, E)\Psi_n$$

$$(2.8b) \quad HA_\mu^\pm(x, E)\Psi_n = (\mp E + E_n)A_\mu^\pm(x, E)\Psi_n$$

$$(2.9) \quad \psi^\dagger(x)\Psi_0 = \bar{\psi}^\dagger(x)\Psi_0 = A_\mu^\dagger(x)\Psi_0 = 0,$$

where $\psi^\pm(x, E)$, $A_\mu^\pm(x, E)$, $\psi^\mp(x, E)$, $A_\mu^\mp(x, E)$ are the positive and negative frequency parts of the fields. The interpretation of these operators as anni-

hilation and creation operators is obtained in a similar way to the free field case. Equations (2.8a) and (2.8b) show that $\psi^\pm(x, E)\Psi_n$, $\bar{\psi}^\pm(x, E)\Psi_n$, $A_\mu^\pm(x, E)\Psi_n$ are eigenstates of the source-free Hamiltonian H ; therefore we shall define a state of n electrons, m positrons, and l photons by the state vector:

$$(2.10) \quad \psi^-(x_1) \dots \psi^-(x_n) \bar{\psi}^-(x'_1) \dots \bar{\psi}^-(x'_m) A_{\mu_1}^-(z_1) \dots A_{\mu_l}^-(z_l) \Psi_0$$

3 - Definition of Propagators

The equation (2.7) can be replaced by the integral equation

$$(3.1) \quad U[\sigma_I, \sigma_{II}, \eta, J] = 1 - i \int_{\sigma_{II}}^{\sigma_I} \mathcal{H}_s(x') U[\sigma', \sigma_{II}, \eta, J] d^4x',$$

where σ' passes through the point x' . We change $\mathcal{H}_s$ by an infinitesimal amount $\delta\mathcal{H}$ and ask for the corresponding change δU .

When U and U_1 satisfy the operator equations:

$$U = 1 + (K_1 + K_2)U$$

$$U_1 = 1 + K_1 U_1,$$

we can prove⁽⁶⁾:

$$U - U_1 = U_1 K_2 U$$

From this follows, neglecting higher order terms,

$$(3.2) \quad \delta U[\sigma_I, \sigma_{II}, \eta, J] = -i \int_{\sigma_{II}}^{\sigma_I} U[\sigma_I, \sigma', \eta, J] \delta\mathcal{H}(x') U[\sigma', \sigma_{II}, \eta, J] d^4x'$$

We now define $\delta\mathcal{H}$ as a change induced by a variation of the sources:

$$(3.3) \quad \delta\mathcal{H}(x) = -\lambda(\bar{\psi}(x)\delta\eta(x) + \delta\bar{\eta}(x)\psi(x)) - \lambda'\delta J_\mu(x)A_\mu(x),$$

where $\delta\eta$, $\delta\bar{\eta}$, δJ_μ have the same commutation properties as η , $\bar{\eta}$, J_μ .

⁽⁶⁾ A. VISCONTI: *Thèses (Paris)*, p. 75 (1953); *Journ. Phys. et Rad.*, **14**, 1951 (1953); *Compt. Rend. (Paris)*, **236**, 2489 (1953).

We define now the functional $G, \mathcal{G}_\mu$ by (*)

$$(3.4a) \quad G[\eta, J; x'] \equiv \frac{1}{i} \frac{\delta}{\delta \bar{\eta}(x')} U[\sigma_I, \sigma_{II}, \eta, J]$$

$$(3.4b) \quad G[\eta, J; x] \equiv \frac{1}{i} \frac{\delta}{\delta \bar{\eta}(x)} U[\sigma_I, \sigma_{II}, \eta, J]$$

$$(3.4c) \quad \mathcal{G}_\mu[\eta, J; z] \equiv \frac{1}{i} \frac{\delta}{\delta J_\mu(z)} U[\sigma_I, \sigma_{II}, \eta, J]$$

From (3.3) we find

$$(3.5a) \quad G[\eta, J; x'] = \lambda(\sigma_I, \bar{\psi}(x'), \sigma_{II})$$

$$(3.5b) \quad G[\eta, J; x] = \lambda(\sigma_I, \psi(x), \sigma_{II})$$

$$(3.5c) \quad \mathcal{G}_\mu[\eta, J; z] = \lambda'(\sigma_I, A_\mu(z), \sigma_{II}),$$

where

$$(\sigma_I, A(x)B(x'), \sigma_{II}) \equiv U[\sigma_I, \sigma, \eta, J] A(x) U[\sigma, \sigma', \eta, J] B(x') U[\sigma', \sigma_{II}, \eta, J]$$

for any operators $A(x), B(x')$. We generalise the definition (3.4) to many particle operators:

$$(3.6a) \quad G[\eta, J; x_1 \dots x_n, x'_1 \dots x'_m] \equiv \\ \equiv (i)^{-n-m} \frac{\delta^{n+m}}{\delta \bar{\eta}(x_1) \dots \delta \bar{\eta}(x_n) \delta \eta(x'_1) \dots \delta \eta(x'_m)} U[\sigma_I, \sigma_{II}, \eta, J],$$

$$(3.6b) \quad \mathcal{G}_{\mu_1 \dots \mu_l}[\eta, J; z_1 \dots z_l] \equiv (i)^{-l} \frac{\delta^l}{\delta J_{\mu_1}(z_1) \dots \delta J_{\mu_l}(z_l)} U[\sigma_I, \sigma_{II}, \eta, J],$$

$$(3.6c) \quad \mathcal{G}_{\mu_1 \dots \mu_l}[\eta, J; x_1 \dots x_n, x'_1 \dots x'_m, z_1 \dots z_l] \equiv \\ \equiv (i)^{-n-m-l} \frac{\delta^{n+m+l}}{\delta \bar{\eta}(x_1) \dots \delta \bar{\eta}(x_n) \delta \eta(x'_1) \dots \delta \eta(x'_m) \delta J_{\mu_1}(z_1) \dots \delta J_{\mu_l}(z_l)} U[\sigma_I, \sigma_{II}, \eta, J],$$

(*) Functional derivatives are defined by

$$F[\eta + \delta\eta, \bar{\eta} + \delta\bar{\eta}, J + \delta J] = F[\eta, \bar{\eta}, J] + \int \frac{\delta}{\delta \eta(\xi)} F[\eta, \bar{\eta}, J] \delta\eta(\xi) d^4\xi + \\ + \int \delta\bar{\eta}(\xi) \frac{\delta}{\delta \bar{\eta}(\xi)} F[\eta, \bar{\eta}, J] d^4\xi + \int \frac{\delta}{\delta J_\mu(\zeta)} F[\eta, \bar{\eta}, J] \delta J_\mu(\zeta) d^4\zeta,$$

with infinitesimal increments

we obtain

$$(3.7a) \quad \mathbf{G}[\eta, J; x_1 \dots x_n, x'_1 \dots x'_m] = \\ = \lambda^{n+m} \varepsilon(x_1 \dots x_n, x'_1 \dots x'_m) (\sigma_I, P[\psi(x_1) \dots \psi(x_n) \bar{\psi}(x'_1) \dots \bar{\psi}(x'_m)], \sigma_{II});$$

$$(3.7b) \quad \mathcal{G}_{\mu_1 \dots \mu_l}[\eta, J; z_1 \dots z_l] = \lambda'^l (\sigma_I, P[A_{\mu_1}(z_1) \dots A_{\mu_l}(z_l)], \sigma_{II});$$

$$(3.7c) \quad \mathbf{g}_{\mu_1 \dots \mu_l}[\eta, J; x_1 \dots x_n, x'_1 \dots x'_m, z_1 \dots z_l] = \\ = \lambda^{n+m} \lambda'^l \varepsilon(x_1 \dots x_n, x'_1 \dots x'_m) (\sigma_I, P[\psi(x_1) \dots \psi(x_n) \bar{\psi}(x'_1) \dots \bar{\psi}(x'_m) A_{\mu_1}(z_1) \dots A_{\mu_l}(z_l)], \sigma_{II}),$$

where P is the chronological operator (time increasing from right to left) and $\varepsilon(x_1 \dots x_n, x'_1 \dots x'_m)$ is the product of all $\varepsilon(x_i, x_j)$ ($i > j$), $\varepsilon(x'_i, x'_j)$ ($i > j$) and $\varepsilon(x_i, x'_j)$.

We shall denote the limit of a functional $F[\eta, J; x]$ for $\bar{\eta} = \eta = 0$ or $J_\mu = 0$ by

$$F[J; x] \equiv F[\eta = 0, J; x]$$

$$F[\eta; x] \equiv F[\eta, J = 0; x]$$

$$F(x) \equiv F[\eta = 0, J = 0; x]$$

The functional Taylor expansion of U can now be written:

$$(3.7) \quad U[\sigma_I, \sigma_{II}, \eta, J] = 1 + \\ + \sum \frac{(i)^{n+m}}{n! m!} \int \bar{\eta}(\xi_1) \dots \bar{\eta}(\xi_n) \mathbf{G}(\xi_1 \dots \xi_n, \xi'_1 \dots \xi'_m) \eta(\xi'_1) \dots \eta(\xi'_m) + \\ + \sum \frac{(i)^l}{l!} \int \mathcal{G}_{\mu_1 \dots \mu_l}(\xi_1 \dots \xi_l) J_{\mu_1}(\xi_1) \dots J_{\mu_l}(\xi_l) + \sum \frac{(i)^{n+m+l}}{n! m! l!} \\ \int \bar{\eta}(\xi_1) \dots \bar{\eta}(\xi_n) \mathbf{g}_{\mu_1 \dots \mu_l}(\xi_1 \dots \xi_n, \xi'_1 \dots \xi'_m, \xi_1 \dots \xi_l) \eta(\xi'_1) \dots \eta(\xi'_m) J_{\mu_1}(\xi_1) \dots J_{\mu_l}(\xi_l)$$

Here and in the following we integrate over all Greek variable (ξ, ζ) .

We give some properties of the vacuum expectation values of the derivatives defined in (3.7):

$$(3.8a) \quad G[\eta, J; x_1 \dots x_n, x'_1 \dots x'_m] \equiv \langle 0 | \mathbf{G}[\eta, J; x_1 \dots x_n, x'_1 \dots x'_m] | 0 \rangle$$

$$(3.8b) \quad \mathcal{G}_{\mu_1 \dots \mu_l}[\eta, J; z_1 \dots z_l] \equiv \langle 0 | \mathcal{G}_{\mu_1 \dots \mu_l}[\eta, J; z_1 \dots z_l] | 0 \rangle$$

$$(3.8c) \quad \mathbf{g}_{\mu_1 \dots \mu_l}[\eta, J; x_1 \dots x_n, x'_1 \dots x'_m, z_1 \dots z_l] \equiv \\ \equiv \langle 0 | \mathbf{g}_{\mu_1 \dots \mu_l}[\eta, J; x_1 \dots x_n, x'_1 \dots x'_m, z_1 \dots z_l] | 0 \rangle$$

Since the Lagrangian with $\eta = \bar{\eta} = 0$ is invariant under the transformation $\psi \rightarrow e^{i\alpha}\psi$, the functionals G and g must also be invariant under this transformation, therefore $G[\eta=0, J; \]$ and $g_{\mu_i}[\eta=0, J; \]$ with different numbers of ψ 's and $\bar{\psi}$'s, i.e. for $m \neq n$, are zero.

The Lagrangian is furthermore (for $\bar{\eta} = \eta = J_\mu = 0$) invariant under charge conjugation of the ψ field together with $A_\mu \rightarrow -A_\mu$. Therefore the function $\mathcal{G}_{\mu_i}(\)$ made up of an odd number of A_μ 's must be zero. The following theorem permits us to call $G(x_1 \dots x_n, x'_1 \dots x'_n)$ and $\mathcal{G}_{\mu_1 \dots \mu_l}(z_1 \dots z_l)$ (with l even) propagators of electrons and photons, since they show the intimate relations between G , $\mathcal{G}$, and the probability-amplitudes.

Theorem I The electron (photon) propagators

$$G(x_1 \dots x_n, x'_1 \dots x'_n)$$

and

$$\mathcal{G}_{\mu_1 \dots \mu_l \mu_{l+1} \dots \mu_{l+l'}}(z_1 \dots z_l, z'_1 \dots z'_{l'}) \quad (l + l' = \text{even})$$

(where the undashed points are on σ_I , the dashed ones on σ_{II}) are proportional to the transition matrix elements between the state of m electrons (l photons) on σ_I , and n electrons (l' photons) on σ_{II} . Correspondingly, theorem II gives the physical interpretation of the mixed propagator.

Theorem II The mixed propagators

$$g_{\mu_1 \dots \mu_l \mu_{l+1} \dots \mu_{l+l'}}(x_1 \dots x_n, x'_1 \dots x'_m, z_1 \dots z_l, z'_1 \dots z'_{l'})$$

where $x_1 \dots x_n, z_1 \dots z_l$ are on σ_I ; $x'_1 \dots x'_m, z'_1 \dots z'_{l'}$ on σ_{II} are proportional to the transition matrix elements between n electrons on σ_I and m electrons on σ_{II} , accompanied by the emission of l photons on σ_I and the absorption of l' photons on σ_{II} .

We shall only prove the theorem I for $G(x, x')$. Generalization and the proof of II follow immediately. By picking out the Fourier amplitude corresponding to electron states with energies E' and E at the surfaces σ_{II} and σ_I respectively and taking into account the time order of σ_I, σ_{II} , we can write

$$G(x, x') \Big|_{\substack{x \text{ on } \sigma_I \\ x' \text{ on } \sigma_{II}}} = \lambda^2 \langle 0, \sigma_I | \bar{\psi}(x, E) \psi(x', E') | \sigma_{II}, 0 \rangle \exp[iEt - iE't']$$

We note further that the evolution operator, U , is the generating operator of the propagators as shown by the equations (3.6)

4 – Equations for the Propagators Normalized Propagators

Quite generally, the source part of the Hamiltonian has the form

$$\mathcal{H}_s(x) = Q(x) \alpha(x)$$

where $Q(x)$ is one of the field-quantities and $\alpha(x)$ represents its source. By generalising (3.5) we can write:

$$(4.1) \quad \frac{\delta}{\delta \alpha(x)} U[\sigma_I, \sigma_{II}, \alpha] = -i U[\sigma_I, \sigma, \alpha] Q(x) U[\sigma, \sigma_{II}, \alpha],$$

and find

$$(4.2) \quad \frac{\partial}{\partial x_\mu} \frac{\delta}{\delta \alpha(x)} U[\sigma_I, \sigma_{II}, \alpha] = \frac{\delta}{\delta \sigma(x)} \int_{\sigma(x)} \frac{\delta}{\delta \alpha(x')} U[\sigma_I, \sigma_{II}, \alpha] d\sigma'_\mu =$$

$$= -i \frac{\delta}{\delta \sigma(x)} \int_{\sigma(x)} U[\sigma_I, \sigma, \alpha] Q(x') U[\sigma, \sigma_{II}, \alpha] d\sigma'_\mu,$$

$$(4.3) \quad = -i U[\sigma_I, \sigma, \alpha] \frac{\partial}{\partial x_\mu} Q(x) U[\sigma, \sigma_{II}, \alpha] -$$

$$- i \int_{\sigma(x)} \left(\frac{\delta}{\delta \sigma(x)} U[\sigma_I, \sigma, \alpha] Q(x') U[\sigma, \sigma_{II}, \alpha] + U[\sigma_I, \sigma, \alpha] Q(x') \frac{\delta}{\delta \sigma(x)} U[\sigma, \sigma_{II}, \alpha] \right) d\sigma'_\mu$$

Using (2.1) we obtain

$$(4.4) \quad \frac{\partial}{\partial x_\mu} \frac{\delta}{\delta \alpha(x)} U[\sigma_I, \sigma_{II}, \alpha] = -i \left(\sigma_I, \left[T_\mu + \int_{\sigma(x)} \mathcal{H}_s(x') d\sigma'_\mu, Q(x) \right], \sigma_{II} \right),$$

where T_μ is the energy momentum vector in the source free representation

When $Q(x)$ describes the electron field, i.e. $Q(x) = \psi(x)$ and $\alpha(x) = \lambda \bar{\eta}(x)$, we have, for $\mu = 4$,

$$\gamma_4 \frac{\partial}{\partial x_4} \frac{\delta}{\delta \bar{\eta}(x)} U[\sigma_I, \sigma_{II}, \eta, J] = i \lambda \gamma_4 (\sigma_I [H + H_s, \psi(x)], \sigma_{II}),$$

with

$$[H + H_s, \psi(x)] = \gamma_4 (-\boldsymbol{\gamma} \cdot \nabla - m + ie \gamma_\mu A_\mu(x)) \psi(x) + \lambda \eta(x)$$

Since $\boldsymbol{\gamma} \cdot \nabla + m$ commutes with $U[\sigma, \sigma', \eta, J]$, we have

$$\gamma_4 \frac{\partial}{\partial x_4} \frac{\delta}{\delta \bar{\eta}(x)} U[\sigma_I, \sigma_{II}, \eta, J] = -i\lambda(\boldsymbol{\gamma} \cdot \nabla + m)(\sigma_I, \psi(x), \sigma_{II}) - \\ - e\lambda\gamma_\mu(\sigma_I, A_\mu(x)\psi(x), \sigma_{II}) + i\lambda^2\eta(x) U[\sigma_I, \sigma_{II}, \eta, J]$$

Taking into account formula (3.6), we finally obtain (*)

$$(4.5a) \quad \left(\gamma_\mu \partial_\mu + m - e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(x)} \right) \frac{\delta}{\delta \bar{\eta}(x)} U[\sigma_I, \sigma_{II}, \eta, J] = i\lambda^2\eta(x) U[\sigma_I, \sigma_{II}, \eta, J]$$

with

$$(4.6) \quad e_1 = e/\lambda'$$

In a similar way we obtain, by taking $Q(x) = A_\mu(x)$ and $\alpha(x) = \lambda' J_\mu(x)$,

$$(4.5b) \quad \square \frac{\delta}{\delta J_\mu(z)} U[\sigma_I, \sigma_{II}, \eta, J] = i\lambda'^2 J_\mu(z) U[\sigma_I, \sigma_{II}, \eta, J] + \\ + i(\lambda'/\lambda)^2 e_1 \text{Sp}(\mathbf{G}[J; z, z] \gamma_\mu)$$

Equations (4.5a) and (4.5b) are the generating equations for the many-body propagators; by operating successively with $\delta/\delta\eta$, $\delta/\delta\bar{\eta}$ and $\delta/\delta J_\mu$ we can derive equations for all possible propagators. This is done in the following for the one electron, one-photon, and two electron propagators.

Taking the vacuum expectation value of the derivative of (4.5a) with respect to $\eta(x')$, we obtain the equation for $G[\eta, J; x, x']$:

$$(4.7) \quad i \left(\gamma_\mu \partial_\mu + m - e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(x)} \right) G[\eta, J; x, x'] = \\ = \lambda^2 \delta(x - x') \langle 0 | U[\sigma_I, \sigma_{II}, \eta, J] | 0 \rangle + i\lambda^2 \eta(x) \frac{\delta}{\delta \eta(x')} \langle 0 | U[\sigma_I, \sigma_{II}, \eta, J] | 0 \rangle$$

Let us define the normalized propagators by dividing them by the vacuum expectation value of U , e.g.

$$(4.8a) \quad G_N[\eta, J; x, x'] \equiv \frac{G[\eta, J; x, x']}{\langle 0 | U[\sigma_I, \sigma_{II}, \eta, J] | 0 \rangle}$$

It must be noted that the propagators are not changed by the above normal

(*) A similar formula has been derived by SYMANZIK⁽⁷⁾. We received his publication only after completion of this paper.

⁽⁷⁾ K. SYMANZIK: *Zeits. Naturf.*, **9**, 809 (1954).

isation when η and J are zero, because of

$$\langle 0 | U[\sigma_I, \sigma_{II}, \eta=0, J=0] | 0 \rangle = 1$$

Putting $\eta=0$ in (4.7) we obtain the equation for the normalised propagator

$$(4.9) \quad i \left(\gamma_\mu \partial_\mu + m - i e_1 \gamma_\mu \mathcal{G}_{\mu N}[J; x] - e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(x)} \right) G_N[J; x, x'] = \lambda^2 \delta(x - x')$$

Application of $\delta^3/\delta\bar{\eta}(x_2)\delta\eta(x'_1)\delta\eta(x'_2)$ in (4.5a) gives the equation for the normalised two-electron propagator (with $\eta=0$):

$$(4.10) \quad i \left(\gamma_\mu \frac{\partial}{\partial x_{1\mu}} + m - i e_1 \gamma_\mu \mathcal{G}_{\mu N}[J; x_1] - e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(x_1)} \right) G_N[J; x_1 x_2 x'_1 x'_2] = \\ = \lambda'^2 \delta(x_1 - x'_1) G_N[J; x_2, x'_2] - \lambda'^2 \delta(x_1 - x'_2) G_N[J; x_2, x'_1]$$

In the same manner, we can deduce equations for the one photon propagator from (4.5b):

$$(4.11) \quad i \square \mathcal{G}_{N\mu\nu}(zz') = \lambda'^2 \delta(z - z') \delta_{\mu\nu} + (\lambda'/\lambda)^2 e_1 \text{Sp} \left\{ \gamma_\mu \frac{\delta}{\delta J_\nu(z')} G_N[J; z, z'] \right\}_{J=0}$$

5 - Vertex-Part, Self-Energy and Free Dressed Particle (*)

For the derivative of a composite functional we have

$$(5.1) \quad -i \frac{\delta}{\delta J_\mu(z)} G[J; x, x'] = -i \int \frac{\delta}{\delta \mathcal{G}_\nu[J; \zeta]} G[J; x, x'] \frac{\delta}{\delta J_\mu(z)} \mathcal{G}_\nu[J; \zeta] = \\ = \int \frac{\delta}{\delta \mathcal{G}_\nu[J; \zeta]} G[J; x, x'] \bar{\mathcal{G}}_{\nu\mu}[J; \zeta, z],$$

where

$$\bar{\mathcal{G}}_{\nu\mu}[J; z, z'] \equiv \mathcal{G}_{\nu\mu}[J; z, z'] - \mathcal{G}_\nu[J; z] \mathcal{G}_\mu[J; z']$$

In order to express $\delta G/\delta \mathcal{G}_\nu$ by the vertex part Γ_ν , we assume the existence

(*) Since we always consider normalized propagators in the rest of the paper, we shall drop the index N

of the inverse of the propagator $G[J; x, x']$ satisfying

$$(5.2) \quad \int G^{-1}[J; x, \xi] G[J; \xi, x'] = \delta(x - x')$$

and define the vertex-part by

$$(5.3) \quad \Gamma_v[J; z, x, x'] = \frac{1}{e_1} \frac{\delta}{\delta \mathcal{G}_v[J; z]} G^{-1}[J; x, x'],$$

$$(5.4) \quad = - \int G^{-1}[J; x, \xi] \frac{1}{e_1} \frac{\delta}{\delta \mathcal{G}_v[J; z]} G[J; \xi, \xi'] G^{-1}[J; \xi', x']$$

Equation (5.4) leads to

$$(5.5) \quad \frac{1}{e_1} \frac{\delta}{\delta \mathcal{G}_v[J; z]} G[J; x, x'] = - \int G[J; x, \xi] \Gamma_v[J; z, \xi, \xi'] G[J; \xi', x']$$

Finally, we have from (5.1)

$$(5.6) \quad i \frac{\delta}{\delta J_\mu(z)} G[J; x, x'] = e_1 \int G[J; x, \xi] \Gamma_v[J; z, \xi, \xi'] G[J; \xi', x'] \bar{\mathcal{G}}_{\nu\mu}[J; z, \xi]$$

Equations (5.5) and (5.6) have the graphical interpretations given in Fig. 1, where solid and wavy lines correspond to one electron and one-photon propagators, respectively

Fig. 1

Fig. 2

If $z=x$, this reduces to Fig. 2. Corresponding to Fig. 2, we introduce the mass-operator by

$$(5.7) \quad \mathcal{M}[J; x, x'] = ie_1^2 \int \gamma_\mu G[J; x, \xi] \Gamma_v[J; z, \xi, \xi'] \bar{\mathcal{G}}_{\nu\mu}[J; x, \xi]$$

Equation (5.6) goes over into

$$(5.8) \quad -e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(x)} G[J; x, x'] = \int \mathcal{M}[J; x, \xi] G[J; \xi, x']$$

Then (4.9) can be written as follows:

$$(5.9) \quad i(\gamma_\mu \partial_\mu - ie_1 \gamma_\mu \mathcal{G}_\mu[J; x] + m) G[J; x, x'] + i \int \mathcal{M}[J; x, \xi] G[J; \xi, x'] = \lambda^2 \delta(x - x')$$

For $J_\mu =$ and $x \neq x'$, equation (4.5) goes over in

$$(5.10) \quad i(\gamma_\mu \partial_\mu + m) G(x, x') + i \int \mathcal{M}(x, \xi) G(\xi, x') = 0$$

Since the theory is invariant under an inhomogeneous Lorentz transformation, the propagator $G(x, x')$ and the mass operator $\mathcal{M}(x, x')$ must be functions of $x - x'$. Their Fourier components depend on one momentum, p , only. In particular, the Fourier component of $\mathcal{M}$ — this being a scalar — must be a function of $\gamma_\mu p_\mu$, we write it $\mathcal{M}(-i\gamma p)$.

Equation (5.10) reads then

$$[i\gamma_\mu p_\mu + m + \mathcal{M}(-i\gamma p)] G(p) = 0$$

This is the equation of a particle with mass M , where $M = m^{(1)}, m^{(2)}$, is a root of the equation

$$(5.11) \quad \mathcal{M}(M) = M - m$$

Taking in $G(x, x')$ the point x' on the initial surface σ_{II} , it follows that the incoming particles must have a spectrum $m^{(1)}, m^{(2)}$, of observable masses. We call these the free dressed particle. This is true also for outgoing particles (i.e. particles arriving on the surface σ_I).

6 - Two-Electron Propagator I Bethe-Salpeter Equation

As in the case of the one-electron propagator, we introduce the inverse of the two-electron propagator and the two electron vertex part by

$$(6.1a) \quad \int G[J; x_1 x_2, \xi_1 \xi_2] G^{-1}[J; \xi_1 \xi_2, x'_1 x'_2] = \\ = \frac{1}{2} \delta(x_1 - x'_1) \delta(x_2 - x'_2) - \frac{1}{2} \delta(x_1 - x'_2) \delta(x_2 - x'_1),$$

$$(6.1b) \quad \Gamma_\nu[J; z, x_1 x_2, x'_1 x'_2] = -e_1 \frac{\delta}{\delta \mathcal{G}_\nu[J; z]} G^{-1}[J; x_1 x_2, x'_1 x'_2]$$

From (6.1) we deduce as in section 5

$$(6.2) \quad e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(z)} G[J; x_1 x_2, x'_1 x'_2] = \\ = e_1^2 \gamma_\mu \int G[J; x_1 x_2, \xi_1 \xi_2] I_1[J; \zeta, \xi_1 \xi_2, \xi'_1 \xi'_2] G[J; \xi'_1 \xi'_2, x'_1 x'_2] \bar{\mathcal{G}}_{1,\mu}[J; \zeta z]$$

Further, we introduce a two electron mass operator in the following way:

$$(6.3) \quad \int G[J; x_2, \xi] \mathcal{M} J; x_1 \xi, x'_1 x'_2] = -\frac{1}{2} \{ \mathcal{M}[J; x_1, x'_1] \delta(x_2 - x'_2) - \\ - \mathcal{M}[J; x_1, x'_2] \delta(x_2 - x'_1) \} + e_1^2 \gamma_\mu \int G[J; x_1 x_2, \xi_1 \xi_2] I_1[J; \zeta, \xi_1 \xi_2, x'_1 x'_2] \bar{\mathcal{G}}_{1,\mu}[J; x_1 \zeta],$$

which leads to

$$(6.4) \quad -e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(x_1)} G[J; x_1 x_2, x'_1 x'_2] = \int \mathcal{M}[J; x_1, \xi] G[J; \xi x_2, x'_1 x'_2] + \\ + \int G[J; x_2, \xi_2] \mathcal{M}[J; x_1 \xi_2, \xi'_1 \xi'_2] G[J; \xi'_1 \xi'_2, x'_1 x'_2]$$

By substituting (6.4) into (4.10) we can derive the Bethe-Salpeter equation:

$$(6.5) \quad G(x_1 x_2, x'_1 x'_1) = G(x_1, x'_1) G(x_2, x'_2) - G(x_1, x'_2) G(x_2, x'_1) - \\ - \int G[x_1, \xi_1] G(x_2, \xi_2) \mathcal{M}(\xi_1 \xi_2, \xi'_1 \xi'_2) G(\xi'_1 \xi'_2, x'_1 x'_2)$$

7 - Two-Electron Propagator II The Renormalized Equations

In this section, we shall show that the equation (4.10) of the two electron propagator is equivalent to the following set:

$$(7.1a) \quad G(x_1 x_2, x'_1 x'_2) = G(x_1, x'_1) G(x_2, x'_2) - \\ - i e_1 \int G(x_1, \xi) v_e(\zeta, \xi, \xi') I_e(\zeta, \xi' x_2, x'_1 x'_2) - (x'_1 \rightarrow x'_2),$$

where

$$(7.1b) \quad v_\mu[J; z, x, x'] = i I_\rho[J; \zeta, x, \xi] G[J; \xi, x'] \bar{\mathcal{G}}_{\rho\mu}[J; \zeta z]$$

and I_μ has to satisfy the equation

$$(7.1c) \quad I_\mu[J; z, x_1 x_2, x'_1 x'_2] = \delta(x_1 - x'_1) \frac{\delta}{\delta \mathcal{G}_\mu[J; z]} G[J; x_2, x'_2] + \\ + e_1 \int V_{\mu e}[J; z, x_1, \zeta, \xi_1] I_e[J; \zeta, \xi_1 x_2, x'_1 x'_2],$$

with

$$(7.1d) \quad V_{\mu e}[J; z, x, z', x'] \equiv v_e[J; z', x, x'] \frac{\delta}{\delta \mathcal{G}_\mu[J; z]}$$

Here the bracket $(x'_1 \rightarrow x'_2)$ means the addition of all the terms obtained by exchanging x'_1 and x'_2 . We call (7.1) the «renormalised equations» because it can be shown (cf. sections 8 and 9) that they can yield the renormalised solutions.

Before deducing (7.1), let us consider its physical meaning. Successive approximation of (7.1b) leads to

$$(7.2) \quad I_\mu[J; z, x_1 x_2, x'_1 x'_2] = \\ = -e_1 \delta(x_1 - x'_1) \int G[J; x_2, \xi_2] I'_\mu[J; z, \xi_2, \xi'_2] G[J; \xi'_2, x'_2] + \\ + \sum_1 (-e_1)^{n+1} \int V_{\mu e_n}[J; z, x_1, \zeta_n, \xi_n] V_{e_n e_{n-1}}[J; \zeta_n, \xi_n, \zeta_{n-1}, \xi_{n-1}] \\ V_{e_{n-1} e_1}[J; \zeta_2, \xi_2, \zeta_1, x'_1] G[J; x_2, \xi'_1] I'_{e_1}[J; \zeta_1, \xi'_1, \xi] G[J; \xi, x'_2]$$

By substituting this expansion into (7.1a) we obtain the two electron propagator which can be graphically expressed by Fig. 3

Fig. 3

The n th term of this expansion can be represented by Fig. 4, in which a photon created at a vertex I'_n can be absorbed by electron lines, photon lines and vertices only in the region above the boundary represented by the dotted curve $n-2$. It can be seen from Fig. 4 that we have no self-energy and connected vertex diagrams. Therefore, diagram 4 is made up of all possible irreducible diagrams (i.e. skeletons).

Alternatively, the Fredholm method can be applied⁽⁸⁾ for solving (7.1c). Let us denote by K the integral operator with the kernel (7.1d) and define the integral operators $\mathcal{D}_n$ by the following formula:

$$(7.3) \quad \mathcal{D}_{n+1} = K D_n - n K \mathcal{D}_n$$

with

$$(7.4) \quad D_n \equiv \text{Sp } \mathcal{D}_n = \int d\xi |\mathcal{D}_n| \xi \rangle d^4\xi, \quad D_0 \equiv 1$$

The solution I_μ of equation (7.1c) is

$$(7.5) \quad I_\mu(z, x_1 x_2, x'_1 x'_2) = \left[1 + \frac{\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \mathcal{D}_{n+1}}{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} D_n} \right] \delta(x_1 - x'_1) \frac{\delta}{\delta \mathcal{G}_\mu[J; z]} G[J; x_2, x'_2]_{J=0}$$

It can be shown that the term $\mathcal{D}_{n+1}$ in the numerator of (7.5) includes all the diagrams with a number of vertices up to n as shown by equation (7.3). The denominator of (7.5) is made up of closed diagrams (i.e. vacuum diagrams) given in Fig. 5.

Fig. 5

Thus, by substituting renormalised one-body propagators and renormalised vertices in the equations (7.1) we obtain the renormalised solution automatically. This is the essential advantage of the equations (7.1).

The derivation of equation (7.1) is as follows: Equation (4.9) gives

$$\left(\gamma_e \partial_e + m + i e_1 \gamma_e \mathcal{G}_e[J; x] - e_1 \gamma_e \frac{\delta}{\delta J_e(x)} \right) \frac{\delta}{\delta J_\mu(z)} G[J; x, x'] = e_1 \gamma_e \bar{\mathcal{G}}_{e\mu}[J; xz] G[J; x, x'],$$

which leads, using equations (5.6), to

$$(7.6) \quad e_1 \gamma_e \bar{\mathcal{G}}_{e\mu}[J; xz] G[J; x, x'] = -\lambda^2 e_1 \int \Gamma_e[J; \zeta, x, \xi] \bar{\mathcal{G}}_{e\mu}[J; \zeta z] G[J; \xi, x'] + i e_1^2 \gamma_e \int G[J; x, \xi] \frac{\delta}{\delta J_e(x)} \{ \Gamma_e[J; \zeta, \xi, \xi'] G[J; \xi', x'] \bar{\mathcal{G}}_{e\mu}[J; \zeta z] \}$$

We can show by means of (7.1c) and (7.6) that (7.1a) satisfies (4.10)

⁽⁸⁾ A. VISCONI: *Journ. Phys. et Rad.*, **16**, 1 (1955)

Fig. 4

This method can easily be extended to other many body propagators. For instance, the propagator for the one-electron one-photon system (Compton scattering) can be derived from (4.5a) as follows:

$$(7.7) \quad i \left(\gamma_e \partial_e + m - i e_1 \gamma_e \mathcal{G}_e[J; x] - e_1 \gamma_e \frac{\delta}{\delta J_e(x)} \right) g_{\mu\nu}[J; x, x', z_1 z_2] = \\ = \lambda^2 \delta(x - x') (\bar{\mathcal{G}}_{\mu\nu}[J; z_1 z_2] - \mathcal{G}_{\mu\nu}[J; z_1] \mathcal{G}_{\nu\mu}[J; z_2])$$

We can rewrite (7.7) by means of (7.6) in the following way:

$$(7.8a) \quad g_{\mu\nu}(x, x', z_1 z_2) = G(x, x') \bar{\mathcal{G}}_{\mu\nu}(z_1 z_2) + e_1 \int G(x, \xi') v_\sigma(\zeta, \xi', \xi) I_{\sigma\mu}(\zeta, \xi, x', z_1 z_2),$$

where $I_{\sigma\mu}[\]$ is a solution of

$$(7.8c) \quad I_{\sigma\mu}[J; z, x, x', z_1 z_2] = \delta(x - x') \left\{ \frac{\delta}{\delta \mathcal{G}_\sigma[J; z]} \bar{\mathcal{G}}_{\mu\nu}[J; z_1 z_2] + \right. \\ \left. + \delta(z - z_1) \delta_{\sigma\mu} \mathcal{G}_\nu[J; z_2] + \delta(z - z_2) \delta_{\sigma\nu} \mathcal{G}_\mu[J; z_1] \right\} + \\ + e_1 \int V_{\sigma\theta}[J; z, x, \zeta, \xi] I_{\theta\mu}[J; \zeta, \xi, x', z_1 z_2]$$

These equations give directly the renormalised matrix elements of the Compton scattering. This is an essential difference between equation (7.8) and the Bethe-Salpeter equation, in which the ladder approximation gives rise to new infinities.

8 - Renormalization Invariance

The transformation

$$(8.1) \quad \lambda \rightarrow c\lambda, \quad \lambda' \rightarrow c'\lambda', \quad e_1 \rightarrow e_1 c'$$

induces the following changes

$$(8.2) \quad G(x, x') \rightarrow c^2 G(x, x'), \quad \bar{\mathcal{G}}_{\mu\nu}(zz') \rightarrow c'^2 \bar{\mathcal{G}}_{\mu\nu}(zz')$$

and analogous changes for the many body propagators. We call (8.1) the *renormalisation transformation* because it changes the normalisation factors of the propagators.

We note that equations (7.1) and (7.8) for Møller and Compton scattering

contain neither λ and λ' and are therefore invariant under the transformation (8.1); they are *renormalisation invariant*. It is physically reasonable that the value of the charge depends on the normalisation of the propagators; indeed if the strength of the electromagnetic field is multiplied by e' the charge must be divided by the same constant. Then, there remains one condition to be fulfilled by the renormalisation constants: the eigenvalues of the number operator for the photon must be integers.

9 - The Normalization Constants

Let us consider the equation (5.9) of the one electron propagators, for $J=0$. Its Fourier transform has the form

$$(9.1) \quad i(\gamma_\mu p_\mu + m + \mathcal{M}(-i\gamma p))G(p) = \lambda^2$$

The causality principle requires

$$G(x, x') = \begin{cases} G^-(x - x') & \text{for } t < t' \\ G^+(x - x') & \text{for } t > t' \end{cases}$$

Such a propagator is obtained from (9.1) as follows:

$$(9.2) \quad G(x, x') = \frac{\lambda^2}{(2\pi)^4} \int_L d^4p \exp[i(p, x - x')] \frac{1}{i\gamma_\mu p_\mu + m + \mathcal{M}(-i\gamma p)}$$

The integration path L is that used in the definition of the S_F function. From (5.11) we see that the denominator can be written in the form

$$(9.3) \quad a(-i\gamma p) \prod_{j=1} (i\gamma_\mu p_\mu + m^{(j)}),$$

where $m^{(1)} \leq m^{(2)} \leq \dots$ and a is a function without zero. To each eigenvalue $m^{(j)}$ of the mass we define a constant $Z_2^{(j)}$ by

$$(9.4) \quad \frac{1}{Z_2^{(j)}} \equiv \frac{\partial}{\partial M} (M - \mathcal{M}(M)) \Big|_{M=m^{(j)}} = a(m^{(j)}) \prod_{k \neq j}' (m^{(k)} - m^{(j)})$$

Then (9.2) can be written in the form

$$(9.6) \quad G(x, x') = - \frac{i\lambda^2}{(2\pi)^4} \sum_{j=1} Z_2^{(j)} \int_L d^4p \exp[i(p, x - x')] \frac{1}{i\gamma_\mu p_\mu + m^{(j)}} \frac{a(m^{(j)})}{a(-i\gamma p)} =$$

$$= - \frac{i\lambda^2}{(2\pi)^4} \sum_{j=1} Z_2^{(j)} \int_{\underline{L}} d^4p \exp[i(p, x-x')] \left[P \frac{1}{i\gamma_\mu p_\mu + m^{(j)}} \frac{a(m^{(j)})}{a(-i\gamma p)} - \right. \\ \left. - i\pi(i\gamma_\mu p_\mu - m^{(j)}) \delta(p_\nu p_\nu + m^{(j)2}) \right]$$

The second term in the bracket of (9.6) shows that the number operator of the particle with mass $m^{(j)}$ appears with the normalization factor $\lambda^2 Z_2^{(j)}$ and has therefore the eigenvalues

$$\lambda^2 Z_2^{(j)}, \quad 2\lambda^2 Z_2^{(j)},$$

Equation (9.5) shows that some of the $Z_2^{(j)}$ may be negative. That is certainly the case if $a(-i\gamma p)$ is an holomorphic function; its sign must then be definite, since it has no zero. The sign of the product in (9.5) and so of $Z_2^{(j)}$ alternates with increasing j .

In the case of a field with more than one mass level (*) we are faced with a difficulty. It is impossible to make all normalisation factors $\lambda^2 Z_2^{(j)}$ equal to 1 by adjusting the one constant, λ .

In the following discussion, we shall assume that the electron has only one mass level $m^{(1)}$. Then, for the renormalization of the number operator of the dressed electron we have to take (with $Z_2 = Z_2^{(1)}$)

$$(9.7) \quad \lambda^2 = 1/Z_2$$

From (9.3) and (9.5) we obtain

$$(9.8) \quad i\gamma_\mu p_\mu + m + \mathcal{M}(-i\gamma p) = (1/Z_2)(i\gamma_\mu p_\mu + m^{(1)})(1 + \mathcal{M}_1(-i\gamma p)),$$

where, according (9.3),

$$(9.9) \quad \mathcal{M}_1(-i\gamma p) = \frac{a(-i\gamma p)}{a(m^{(1)})} - 1,$$

so that

$$\mathcal{M}_1(m^{(1)}) = 0$$

Therefore

$$(9.10) \quad G(x, x') = \frac{1}{(2\pi)^4} \int_{\underline{L}} d^4p \exp[i(p, x-x')] \frac{1}{i\gamma_\mu p_\mu + m^{(1)}} \frac{1}{1 + \mathcal{M}_1(-i\gamma p)}$$

(*) LEE (9) gave a simple model of interacting fields, which leads to two mass levels

(9) T. D. LEE: *Phys. Rev.*, **95**, 1329 (1954)

The one-photon propagator can be discussed in a similar way. Gauge invariance admits of only one mass, namely zero. The normalization constant λ' can be determined in a similar way to λ .

10 - Renormalized Theory

The aim of this section is to prove that the present theory of propagators is a renormalized theory. For this purpose we shall calculate

$$\frac{1}{e_1} \frac{\delta}{\delta \mathcal{G}_\mu[J; z]} F[J; z_1, z_l, x_l, x_n, x'_1, x'_n]$$

for the limit of vanishing momentum of the photon at z . (As usual z and x denote photon and electron coordinate respectively.)

Let

$$F[J; z_1, z_l, p_1, p_n, p'_1, p'_n]$$

be the Fourier component of the functional

$$F[J; z_1, z_l, x_l, x_n, x'_1, x'_n],$$

i.e.

$$\begin{aligned} (10.1) \quad F[J; z_1, z_l, x_l, x_n, x'_1, x'_n] &= \\ &= (2\pi)^{-sn} \int F[J; z_1, z_l, p_1, p_n, p'_1, p'_n] \exp \left[i \left\{ \sum_{j=1}^n (p_j x_j - p'_j x'_j) \right\} \right], \end{aligned}$$

and be

$$F_\mu[J; z_1, z_l, p_1, p_n, p'_1, p'_n]$$

the Fourier component of

$$\frac{1}{e_1} \frac{\delta}{\delta \mathcal{G}_\mu[J; z]} F[J; z_1, z_l, x_l, x_n, x'_1, x'_n]$$

for the case of vanishing momentum of the photon at z . Then:

$$\begin{aligned} (10.2) \quad \frac{1}{e_1} F_\mu[J; z_1, z_l, p_1, p_n, p'_1, p'_n] &= \\ &= (2\pi)^{-4} \sum_{j=1}^n \left\{ \frac{\partial}{\partial p_{j\mu}} - \frac{\partial}{\partial p'_{j\mu}} \right\} F[J; z_1, z_l, p_1, p_n, p'_1, p'_n] \end{aligned}$$

The formula (10.2) can be proved by taking into account the fact that $\mathcal{G}_\mu[J; Z]$ appears in the form $\partial_\mu - ie_1 \mathcal{G}_\mu$ in the equations for the propagators, and that the derivatives $\partial_i \mathcal{G}_\mu$, $\partial_\partial \partial_i \mathcal{G}_\mu$, can be neglected in the limiting case of vanishing momentum of the photon at z

From (9.1), (9.3) and (9.10), we can derive

$$(10.3) \quad (i\gamma_\mu p_\mu + m^{(1)})(1 + \mathcal{M}_1(-i\gamma p))G(p) = 1,$$

where $m^{(1)}$ is the experimental mass. Now (9.8) shows that $\mathcal{M}_1(-i\gamma p)$ and Z_2 correspond respectively to the finite part of the mass operator Σ^* , and to Z_2 , in the Dyson theory. Therefore, we see from (10.3) that $G(p)$ is already renormalized.

Formulae (5.3) and (10.2) give

$$(10.4) \quad \Gamma_\mu(l=0, p) = \gamma_\mu + \frac{\partial}{\partial p_\mu} \{ (\gamma^\mu p_\mu + m^{(1)}) \mathcal{M}_1(-i\gamma p) \},$$

where l and p are the energy-momentum vectors of photon and electron. This relation shows that Γ_μ is the renormalised vertex part, because Dyson proved that $\Gamma_\mu(l, p)$ is renormalised for any value of l when it is so for $l=0$. Since we showed in section 7 that equations (7.1) give only *irreducible diagrams* made up of connected lines and vertices and the observable charge e_1 , we may conclude that the theory is renormalized.

11 - A Method of Solution

Since the integral equations (7.1) and (7.8) which describe respectively Møller and Compton scattering, have the same kernel $V_{\mu\nu}$, they can be solved by using the common resolvent operator $R_\mu[\mathcal{G}, \delta/\delta\mathcal{G}, z, x, z', x']$. Both I_μ and $I_{\lambda\mu}$ can be expressed by R_μ :

$$(11.1) \quad I_\mu(z, x_1 x_2, x'_1 x'_2) = -e_1 \delta(x_1 - x'_1) \left(G(x_2, \xi) \Gamma_\mu(z, \xi, \xi') G(\xi', x'_2) - \right. \\ \left. - e_1 \left[R_{\mu e} \left[\mathcal{G}, \frac{\partial}{\partial \mathcal{G}}, z, x_1, \zeta, x'_1 \right] G[J; x_2, \xi] \Gamma_e[J; \zeta, \xi, \xi'] G[J; \xi', x'_2] \right] \right)_{J=0}$$

$$(11.2) \quad I_{\lambda\mu}(z, x, x', z_1 z_2) = \int R_{\lambda\theta} \left[\mathcal{G}, \frac{\delta}{\delta \mathcal{G}_\theta}, z, x, \zeta, x' \right] \delta \mathcal{G}_\theta[J; \zeta] \mathcal{G}_\mu[J; z_1 z_2] \Big|_{J=0} + \\ + \int R_{\lambda\mu} \left[\mathcal{G}, \frac{\delta}{\delta \mathcal{G}}, z, x, z_1, x' \right] \mathcal{G}_\lambda[J; z_2] \Big|_{J=0} + \\ + \int R_{\lambda\nu} \left[\mathcal{G}, \frac{\delta}{\delta \mathcal{G}} z, x, z_2, x' \right] \mathcal{G}_\mu[J; z_1] \Big|_{J=0}$$

$R_{\mu\nu}$ satisfies the following integro-differential equation corresponding to the resolvent equation:

$$(11.3) \quad R_{\mu\nu} \left[\mathcal{G}, \frac{\delta}{\delta \mathcal{G}}, z, x, z', x' \right] = V_{\mu\nu}[J; z, x, z', x'] + \\ + \int V_{\mu\nu}[J; z, x, \zeta, \xi] R_{\sigma\tau} \left[\mathcal{G}, \frac{\delta}{\delta \mathcal{G}}, \zeta, \xi, z', x' \right]$$

We now give a solution of (11.3) by limiting the number of bosons in flight. Since according to (11.3) $R_{\mu\nu}$ is a functional of $\mathcal{G}$ and $\delta/\delta \mathcal{G}$, we can make a functional Taylor expansion with respect to $\delta/\delta \mathcal{G}$:

$$(11.4) \quad R_{\mu\nu} \left[\mathcal{G}, \frac{\delta}{\delta \mathcal{G}}, z, x, z', x' \right] = \\ = \sum_n \int R_{\mu\nu\sigma_1 \dots \sigma_n}^{(n)}[\mathcal{G}, z, x, z', x'; \zeta_1 \dots \zeta_n] \frac{\delta}{\delta \mathcal{G}_{\sigma_1}[J; \zeta_1]} \dots \frac{\delta}{\delta \mathcal{G}_{\sigma_n}[J; \zeta_n]}$$

By substituting (11.4) into (11.3) and by picking up terms of the same order in $\delta/\delta \mathcal{G}$, we obtain a recurrent set of equations

$$(11.5) \quad R_{\mu\nu\sigma_1 \dots \sigma_n}^{(n)}[\mathcal{G}, z, x, z', x'; z_1 \dots z_n] = \\ = i \int v_\sigma[J; \zeta, x, \xi] \frac{\delta}{\delta \mathcal{G}_\mu[J; z]} R_{\sigma\nu\sigma_1 \dots \sigma_n}^{(n)}[\mathcal{G}, \zeta, \xi, z', x'; z_1 \dots z_n] + \\ + i \int v_\sigma[J; \zeta, x, \xi] R_{\sigma\mu\sigma_1 \dots \sigma_{n-1}}^{(n-1)}[\mathcal{G}, \zeta, \xi, z', x'; z_1 \dots z_{n-1}] \delta_{\mu\sigma_n} \delta(z - z_n)$$

with v_σ from (7.1b). In the case $n=1$ an extra term has to be added (see (11.7)). For $n=0, 1$ we obtain so

$$(11.6) \quad R_{\mu\nu}^{(0)}[\mathcal{G}, z, x, z', x'] = i \int v_\sigma[J; \zeta, x, \xi] \frac{\delta}{\delta \mathcal{G}_\mu[J; z]} R_{\sigma\nu}^{(0)}[\mathcal{G}, \zeta, \xi, z', x'],$$

$$(11.7) \quad R_{\lambda\mu}^{(1)}[\mathcal{G}, z, x, z', x'] = i \int v_\sigma[J; \zeta, x, \xi] \frac{\delta}{\delta \mathcal{G}_\lambda[J; z]} R_{\sigma\mu}^{(1)}[\mathcal{G}, \zeta, \xi, z', x'; z_1] + \\ + i \delta_{\lambda\nu} \delta(z - z_1) \int v_\sigma[J; \zeta, x, \xi] R_{\sigma\mu}^{(0)}[\mathcal{G}, \zeta, \xi, z', x'] + \\ + i \delta_{\lambda\nu} \delta(z - z_1) v_\mu[J; z', x, x']$$

Since $R^{(0)}$ satisfies the homogeneous integral equation (11.6), $R^{(0)} = 0$ is a

possible solution. For scattering problems, this is the correct solution. One obtains namely, in this case, by successive approximations all possible irreducible diagrams by starting from $R^{(0)} = 0$.

To calculate $R^{(1)}$ from (11.7) for the case of scattering, we expand $R_{\lambda\mu\nu}^{(1)}$ and v_μ in powers of $\mathcal{G}_\mu$:

$$(11.8) \quad v_\mu[J; z, x, x'] = \sum_m v_{\mu q_1}^{(m)} e_m(z, x, x'; \zeta_1, \zeta_m) \mathcal{G}_{q_1}[J; \zeta_1] \mathcal{G}_{q_m}[J; \zeta_m]$$

$$(11.9) \quad R_{\lambda\mu\nu}^{(1)}[\mathcal{G}; z, x, z', x'; z_1] = \\ = \sum_m \left(R_{\lambda\mu\nu q_1}^{(1, m)} e_m(z, x, z', x'; z_1; \zeta_1, \zeta_m) \mathcal{G}_{q_1}[J; \zeta_1] \mathcal{G}_{q_m}[J; \zeta_m] \right)$$

Since the appearance of a functional derivative $\delta/\delta\mathcal{G}$ corresponds to the insertion of a photon line, we may limit the number of virtual photons by cutting off the development. As a simple example of this approximation method, we shall consider the case where we take into account only one such photon. That corresponds to neglecting all $R^{(m, n)}$ and $v^{(m)}$ with exception of $R^{(1, 0)}$, $R^{(1, 1)}$, $v^{(0)}$, $v^{(1)}$. Then equation (11.7) leads to

$$(11.10) \quad R_{\lambda\mu\nu}^{(1, 0)}(z, x, z', x'; z_1) = i\delta_{\lambda\nu}\delta(z - z_1)v_\mu^{(0)}(z', x, x') + \\ + i \int v_q^{(0)}(\zeta, x, \xi) R_{q\mu\lambda}^{(1, 1)}(\zeta, \xi, z', x'; z_1; z);$$

$$(11.11) \quad R_{\lambda\mu\nu q}^{(1, 1)}(z, x, z', x'; z_1; z_2) = i\delta_{\lambda\nu}\delta(z - z_1)v_{\mu q}^{(1)}(z', x, x'; z_2) - \\ - \int v_{\sigma q}^{(1)}(\zeta, x, \xi; z_2) R_{\sigma\mu\lambda}^{(1, 1)}(\zeta, \xi, z', x'; z_1; z)$$

Furthermore, we assume for simplicity

$$(11.12) \quad \begin{cases} G(x, x') = S_F(x - x') \\ \bar{\mathcal{G}}_{\mu\nu}(zz') = \delta_{\mu\nu} D_F(z - z') \\ \Gamma_\mu(z, x, x') = \gamma_\mu \delta(z - x) \delta(z - x') \end{cases}$$

Then

$$(11.13) \quad v_\mu^{(0)}(z, x, x') = \gamma_\mu D_F(x - z) S_F(x - x')$$

$$(11.14) \quad v_{\mu\nu}^{(1)}(z, x, x'; z_1) = -e_1 \gamma_\mu D_F(x - z) S_F(x - z_1) \gamma_\nu S_F(z_1 - x')$$

By substituting in (11.11) we have

$$\begin{aligned}
 (11.15) \quad R_{\lambda\mu\nu\rho}^{(1.1)}(z, x, z', x'; z_1; z_2) = \\
 = ie_1 \delta_{\lambda i} \delta(z - z_1) D_F(x - z') \gamma_\mu S_F(x - z_2) \gamma_\rho S_F(z_2 - x') + \\
 + ie_1 \int D_F(x - \zeta) \gamma_\sigma S_F(x - z_2) \gamma_\rho S_F(z_2 - \xi) R_{\sigma\mu i \lambda}^{(1.1)}(\zeta, \xi, z', x'; z_1; z)
 \end{aligned}$$

This is an ordinary integral equation for $R^{(1.1)}$. By substituting the solution in (11.10) we obtain $R^{(1.0)}$ which finally gives I_μ and $I_{\lambda\mu\nu}$ according to (11.1) and (11.2).

It can be shown that successive approximations of (11.15) lead to the diagrams 6 for $G(x_1 x_2, x'_1 x'_2)$ and $\mathcal{G}_{\mu\nu}(x, x', z_1 z_2)$. The solution can be improved in two ways:

Fig. 6

(a) by using more accurate expressions for the one body propagators and the vertex Γ_μ ,

(b) by taking into account higher orders of $\delta/\delta\mathcal{G}$ and $\mathcal{G}$

The writers would like to thank Professor ROSENFELD for inspiring discussions and his kind hospitality. Thanks are also due to Dr. J. PODOLANSKI for many stimulating comments.

RIASSUNTO (*)

Si sviluppa una teoria generale dei propagatori. Tutti i propagatori si derivano da un funzionale generatore. La rinormalizzazione corrisponde all'adattamento delle costanti di accoppiamento delle sorgenti esterne. Come esempio si derivano delle equazioni per gli scattering di Compton e di Møller.

(*) *Traduzione a cura della Redazione*