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Theory of Particle Orbits in the Alternating Gradient Synchrotron

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In this paper the theoretical aspects of the particle behaviour in the alternating gradient synchrotron shall be outlined. For a description of the situation in more physical terms the reader is referred to the paper of Dr ADAMS.

The considerations shall be confined in two respects,

1) Only the behaviour of particles of the proper momentum shall be analysed. So we shall only be concerned with the stability of deviations of the position of particles from the equilibrium orbit, i.e. deviations which are essentially perpendicular to this equilibrium orbit. Nothing shall be said about stability of the phase with respect to the accelerating radio frequency.

2) We shall only be concerned with linear effects, viz. we shall drop all non-linear terms in the kinematic part of the equations of motion (Coriolis forces, centrifugal terms) and neglect all intrinsic non-linearities of the magnetic field. This approximation means that the results can be valid only for small deviations from the equilibrium orbit and for a sufficiently high degree of linearity of the magnetic field. The reasons for the limitation to «linear theory» are twofold: no general theory for the non-linear behaviour is available, and the non-linear considerations made so far by the CERN Proton Synchrotron Group are in a rather preliminary stage.

In Part 1 of this paper the theory of the perfect machine will be sketched which treats the proton synchrotron proposed by COURANT, LIVINGSTON and SNYDER in their classical paper. In this machine, the magnet structure repeats itself in strictly identical periods along the circumference. This perfect machine can never be realized in practice and so one meets with several types

of imperfections the influence of which will be outlined in part 2. The theory of these imperfections has been developed at several places but not much has yet been published.

1 – Perfect Machine

In Fig. 1 a cross-section of the beam is shown, taken perpendicularly to the equilibrium orbit. The centre of the machine may be imagined far to the left. Vertical deviations of the particles from the equilibrium orbit may be expressed by a coordinate, z , horizontal ones by another coordinate, r . Measuring the arc length along the equilibrium orbit in a unit (angle) θ , which increases from 0 to 2π over one revolution and assuming symmetry of the magnetic field with respect to a horizontal plane, one derives the following equation of motion for, e.g., the vertical deviations

Fig. 1

$$(1.1) \quad z''(\theta) + n(\theta)z(\theta) = 0$$

In this equation, the dashes mean derivatives with respect to the arc length θ and the quantity $n(\theta)$ (field index) is defined by

$$(1.2) \quad n(\theta) = \begin{cases} -\frac{R}{B_z} \frac{\partial B_z(\theta)}{\partial r}, \\ -\frac{R^2}{\varrho B_z} \frac{\partial B_z(\theta)}{\partial r} \end{cases}$$

The first line is valid if the equilibrium orbit is simply a circle with radius R . The second one refers to the general case, the quantities R now being defined by

$$(1.3) \quad \text{circumference} = 2\pi R$$

and $\varrho(\theta)$ being the local radius of curvature (magnetic radius) of the equilibrium orbit (notice that the product ϱB_z is a function of the particle momentum only).

In Eq. (1.1) the acceleration and, consequently, the change of the magnetic guiding field B with time have been neglected. Taking them into account, one finds that the particle orbits following from Eq. (1.1) are then damped down in amplitude proportionally to the reciprocal square root of the particle momentum. So all these oscillations about the equilibrium orbit (which are called « betatron oscillations ») become smaller and smaller in the course of acceleration.

The following two mathematical statements are true for the solutions of Eq (1.1)

1) Provided a pair $z_a(\theta)$, $z_b(\theta)$ of linearly independent solutions of the Eq (1.1) has been found, the most general solution (particle orbit) can be expressed as a linear combination of those two basic solutions

2) For every pair of solutions, the Wronskian

$$(1.4) \quad W_{ab} = z_a(\theta)z'_b(\theta) - z'_a(\theta)z_b(\theta)$$

is independent of θ and is different from 0 if $z_a(\theta)$ and $z_b(\theta)$ are linearly independent

According to COURANT, LIVINGSTON and SNYDER, the field index $n(\theta)$ (Eq (1.2)) has to be chosen as a strictly periodical function, viz

$$(1.5) \quad n(\theta + \theta_0) = n(\theta)$$

where the period θ_0 has to be an integral fraction of 2π ,

$$(1.6) \quad \theta_0 = 2\pi/M,$$

(M = number of periods of the magnet structure) Assumption (1.5) means that the differential Eq (1.1) is of the so called Floquet type. According to Floquet's theorem, one can normally choose two solutions (Floquet solutions) which somehow reflect the periodicity of the coefficient $n(\theta)$

$$(1.7) \quad z_1(\theta + \theta_0) = \lambda_1 z_1(\theta), \quad z_2(\theta + \theta_0) = \lambda_2 z_2(\theta),$$

where λ_1 , λ_2 are constants independent of θ

For the Floquet multipliers λ_1 , λ_2 one finds: what follows:

1) since $n(\theta)$ is a real quantity, λ_1 and λ_2 are both real or they form a complex conjugate pair;

Fig. 2

2) by application of the constancy of the Wronskian to θ and $\theta + \theta_0$ one sees that

$$(1.8) \quad \lambda_1 \lambda_2 = 1;$$

consequently, in the complex λ plane (Fig. 2), the locus of λ_1 and λ_2 is given by the real axis and the unit circle, and the position of the pair λ_1 , λ_2 is either that indicated by $\times$ or that indicated by $\circ$. In the first case ($\lambda_{1,2}$ on the real axis), one of the multipliers has a modulus bigger than one. Assume this to be the case for λ_1 ; then $z_1(\theta)$ will formally grow in amplitude indefinitely in the course of the time and one therefore has instability of the betatron oscillations in this case. If λ_1

and λ_2 both lie on the unit circle their modulus y is equal to one and one has stability

Let us assume now that the distribution of the field index (i.e. the function $n(\theta)$) is changed continuously everywhere. Then, if we have stable working conditions in the beginning (if, consequently, λ_1 and λ_2 are situated on the unit circle) these points will move continuously on the circle in some direction and finally approach the points ± 1 where the two branches of the locus intersect. Then normally they will no longer continue on the circle but move for a while on the real axis. This shows that, in the alternating gradient scheme, a sufficiently big change of $n(\theta)$ brings one from stable to unstable working conditions and that, on the border between stability and instability, there is an orbit which is periodical either over one period ($\lambda = +1$) or over two periods ($\lambda = -1$) of the field index.

Normally one puts

$$(19) \quad \lambda_{1,2} = \exp[\pm i\mu]$$

and the «mode» μ is a real quantity under stable conditions. Although the foregoing discussion was confined to one direction of oscillation only, one should keep in mind that for the successful performance of the synchrotron actually stability of all kinds of betatron oscillations is needed or, what is the same, stability both of vertical and horizontal oscillations.

2 - Influence of Imperfections

There are, within the frame of linear theory, three types of imperfections which were already mentioned in DI ADAMS' paper

1) Imperfections of the *guiding field* $B_z(\theta)$ on the equilibrium orbit (due to geometrical displacements of magnets perpendicular to the equilibrium orbit, remanent fields, etc.)

2) Imperfections of the *field index* $n(\theta)$ (individual differences of the magnets, different lengths, different saturation behaviour)

3) *Coupling* between vertical and horizontal oscillations (geometrical twists about the equilibrium orbit)

All these imperfections do, of course, act simultaneously on the particles but for systematical reasons they shall be treated separately.

The *guiding field* determines the equilibrium orbit. If it deviates from its proper value, a particle can no longer travel along the original equilibrium path. But one can show that, also in this case, there is one orbit in the ma-

chine which is closed in itself after one revolution and that all particles starting off this closed orbit just perform ordinary betatron oscillations about it

Mathematically, the equation of motion (1.1) is now modified by an inhomogeneous term

$$(2.1) \quad z''(\theta) + n(\theta)z(\theta) = \delta f(\theta)$$

In the case that the imperfection is caused by the vertical displacement $\delta\zeta(\theta)$ of the magnets, it is almost evident that

$$(2.2) \quad \delta f(\theta) = n(\theta) \delta\zeta(\theta)$$

In the more general case that there is some small horizontal component $\delta B_r(\theta)$ of the guiding field on the original equilibrium orbit, one can show that

$$(2.3) \quad \delta f(\theta) = \frac{R^2}{B_z Q} \delta B_r(\theta)$$

The « closed orbit » $\hat{z}(\theta)$

$$(2.4) \quad \hat{z}(\theta + 2\pi) = \hat{z}(\theta)$$

can be expressed in a closed form in terms of $\delta f(\theta)$ and the Floquet solutions for the perfect machine as follows:

$$(2.5) \quad \hat{z}(\theta) = \int_{\theta-2\pi}^{\theta} d\theta' \delta f(\theta') \left\{ \frac{z_1(\theta')z_2(\theta)}{W(1 - \exp[-2\pi iQ])} - \frac{z_2(\theta')z_1(\theta)}{W(1 - \exp[+2\pi iQ])} \right\}$$

(Notice that in the arguments of $z_{1,2}$ there appears θ as well as the variable of integration θ' !) In this equation, W represents the Wronskian constructed from the Floquet solutions ((1.4) for $z_a = z_1$, $z_b = z_2$) and Q is the number of (vertical) betatron oscillations along the circumference. This quantity is, in terms of M (1.6) and μ (1.9), given by

$$(2.6) \quad Q = \frac{M\mu}{2\pi}$$

Fig. 3

From (2.5) one sees that the closed orbit deviates more and more from the original equilibrium orbit the nearer one approaches working conditions which lead to integral Q . One has a typical phenomenon of resonance. In Fig. 3 the behavior of the peak deviation of the closed orbit (for given space) is shown qualitatively. The different values of Q on the abscissa are thought to be produced by some modification of $n(\theta)$ inside each period. Q_0 represents some integral value of Q .

Since the whole beam follows the closed orbit and has to stay inside the

vacuum chamber everywhere one has to make sure that this closed orbit does never deviate too much from the original equilibrium orbit. That means

- 1) $\delta f(\theta)$ has to be sufficiently small;
- 2) one may not work too close to a resonance

If there are imperfections of the *field index*, the condition of periodicity (1.5) will normally not be valid any longer. Instead, one has only periodicity over one revolution, i.e.

$$(2.7) \quad n(\theta + 2\pi) = n(\theta)$$

The orbits are therefore still given by an equation (1.1) of the Floquet type. But now, Floquet's theorem has to be applied to one revolution 2π . In general, there is a pair of Floquet solutions with the properties

$$(2.8) \quad \tilde{z}_1(\theta + 2\pi) = \tilde{\lambda}_1 \tilde{z}_1(\theta), \quad \tilde{z}_2(\theta + 2\pi) = \tilde{\lambda}_2 \tilde{z}_2(\theta),$$

and all that has been said about the properties of λ_1 and λ_2 holds unchanged for $\tilde{\lambda}_1, \tilde{\lambda}_2$. If there were no imperfections, one would simply have

$$(2.9) \quad \tilde{\lambda}_1 = \lambda_1^M = \exp[2i\pi Q], \quad \tilde{\lambda}_2 = \lambda_2^M = \exp[-2\pi i Q]$$

and, in the complex plane, the points $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ would, under a change of the working conditions, pass over the critical points $\tilde{\lambda} = \pm 1$ without noticing it. If, however, only the weaker periodicity condition (2.7) holds true, they will in general not pass over those critical points without deviating for a while onto the real axis. So, in the presence of imperfections of the field index, the region of stable working conditions is split up by unstable bands which are called « stopbands ». These stopbands lie close to those working conditions where, for a perfect machine, one would have

$$(2.10) \quad Q = \text{integer} \quad (\tilde{\lambda} = +1) \quad \text{or} \quad Q = \text{half an integer} \quad (\tilde{\lambda} = -1)$$

A mathematical solution of the problem in a closed form is not possible. In most cases of practical interest, one can apply a perturbation treatment. For this purpose one splits $n(\theta)$ into a perfect part and imperfections

$$(2.11) \quad n(\theta) = n_{\text{perf}}(\theta) + \delta n(\theta)$$

with

$$(2.12) \quad n_{\text{perf}}(\theta + \theta_0) = n_{\text{perf}}(\theta)$$

(That this splitting is not uniquely defined does not affect the result). Having performed the splitting (2.11) one finds **for** the stopband width (distance between the edges in terms of the value of Q **for** the corresponding perfect ma-

chine)

$$(2.13) \quad \delta Q_{\text{stop}} = \left| \frac{1}{\pi W} \int_0^{2\pi} d\theta \delta n(\theta) (z_1(\theta))^2 \right|$$

Here, $z_1(\theta)$ is one of the Floquet solutions for the corresponding perfect machine and W is the Wronskian formed from $z_1(\theta)$ and $z_2(\theta) = z_1^*(\theta)$

The centre of a stopband does normally not coincide exactly with an integral value of Q for the perfect machine, but neighbouring stopbands are shifted by practically the same amount. So, the position of the stopband centre is not a quantity of practical interest.

Fig. 4

The stopbands, occurring at integral and half integral values of Q , are shown in Fig. 4. We emphasize that, inside a stopband, the beam shows a monotonical blow up whereas imperfections of the guiding field lead to a displacement of the beam as a whole. The blow up inside the stopbands is anticipated by beatings in amplitude inside the « windows » but close to a stopband border. There

any orbit increases in amplitude over a number of revolutions, decreases again and so forth. The physical reason is that, near to a stopband, the orbits are almost periodical (or antiperiodical) over one revolution. So, imperfections of the field index which initially may act as to blow up the beam, finally come out of phase and act in the opposite direction, come again out of phase and so on. Fortunately, these « shoulders » of the stopbands are rather narrow.

In the presence of imperfections both of the guiding field and of the field index, the peak deviation of the closed orbit goes to infinity at both edges of those stopbands which lie near to integral numbers of Q . The behaviour of the betatron oscillations about the closed orbit is not influenced by imperfections of the guiding field and, therefore, is described correctly by the foregoing considerations.

Two requirements must be fulfilled to have the machine still working successfully in presence of an imperfect field index.

1) The stopbands must be sufficiently narrow to have reasonably wide windows between the stopbands.

2) The working point must not be too close to a stopband to avoid troubles from the beatings in amplitude (and, of course, from a big displacement of the closed orbit).

In the case of *coupling* the considerations can no longer be confined to one component of oscillation. Now the orbits are described by a system of two

coupled equations

$$(2.14) \quad z''(\theta) + n_{zz}(\theta) z(\theta) + n_{zr}(\theta) r(\theta) = 0, \quad r''(\theta) + n_{rz}(\theta) z(\theta) + n_{rr}(\theta) r(\theta) = 0$$

Geometrical twists of the magnets also lead to a horizontal component of the guiding field but this affects only the closed orbit and not the betatron oscillations about it which are correctly described by (2.14)

Every solution of (2.14) is given by a pair of functions, $z(\theta)$ and $r(\theta)$. Further, any solution (orbit) is a linear combination of four linearly independent ones

By expressing the coefficients $n_{jk}(\theta)$ explicitly in terms of derivatives of the guiding field, one can show that

$$(2.15) \quad n_{rz}(\theta) = n_{zr}(\theta)$$

From this it follows that for any pair of solutions, $z_a(\theta)$, $r_a(\theta)$ and $z_b(\theta)$, $r_b(\theta)$, the following bilinear expression

$$(2.16) \quad W_{ab} = z_a(\theta) z'_b(\theta) - z'_a(\theta) z_b(\theta) + r_a(\theta) r'_b(\theta) - r'_a(\theta) r_b(\theta)$$

does not depend upon θ

The coefficients $n_{jk}(\theta)$ are obviously periodical over one revolution

$$(2.17) \quad n_{jk}(\theta + 2\pi) = n_{jk}(\theta)$$

This means that again we have a mathematical problem of the Floquet type. Normally, one therefore can find four Floquet solutions which after one revolution simply take up a numerical factor

$$(2.18) \quad \begin{pmatrix} \tilde{z}_1(\theta + 2\pi) \\ \tilde{r}_1(\theta + 2\pi) \end{pmatrix} = \tilde{\lambda}_1 \begin{pmatrix} \tilde{z}_1(\theta) \\ \tilde{r}_1(\theta) \end{pmatrix}$$

(and in an analogous manner for the other solutions). The following statements are true for the Floquet multipliers $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{\lambda}_4$

1) Since $n_{jk}(\theta)$ are real quantities, either all four multipliers are real or, for a complex multiplier the complex conjugate number is also among these multipliers

2) By applying (2.16) to the Floquet solutions one finds that the four multipliers can be ordered in pairs so that the product of the two members of each pair is equal to one

If we are interested in only those working conditions which would be stable in the absence of coupling, one has to distinguish between two entirely dif-

ferent configurations of the Floquet multipliers in the complex plane (Fig 5) The pairs of complex conjugate multipliers may either be identical with the pairs of those giving product one (configuration 1) or they may not (configuration 2) In the first case one obviously has stability, in the second one instability, since there are multipliers with a modulus bigger than one The limiting case between both configurations where, in the presence of coupling, one possibly might pass from stability to instability, appears whenever two multipliers approach each other Then, two overall phaseshifts become equal So one might expect instability due to coupling, whenever in the corresponding perfect machine (with no coupling) one is near to working conditions with

Fig 5

$$(2.19) \quad Q_z + Q_r = \text{integer} \quad \text{or} \quad Q_z - Q_r = \text{integer}$$

Fortunately, a closer analysis shows that actually only one of these two possibilities does lead to instability Under conditions normally considered one has instability only if $Q_z + Q_r$ is an integer but not if $Q_z - Q_r$ is

The situation shall be explained in somewhat more detail by considering the working diamond in the $n_1 - n_2$ plane (cf the paper by Dr ADAMS) This working diamond is shown in Fig 6 The borders of the working diamond

Fig 6

are given by stopbands due to an imperfect field index which are only slightly modified by the coupling These stopbands lie in the vicinity of the lines, $Q_z, Q_r = \text{integral or half integral number}$, in the $n_1 - n_2$ plot for the corresponding perfect machine At the line $Q_z + Q_r = \text{integer}$, which is outside the working diamond, one has a coupling stopband Inside such a stopband, one again has a monotical blow up of the beam but now as a two dimensional effect rather than as a one dimensional one as in the case of stopbands due to an imperfect field index A line

$Q_z - Q_r = \text{integer}$ crosses the diamond as a diagonal Also on this line there is a strong interaction between vertical and horizontal oscillations but this does not lead to instability but to some exchange between the amplitudes of vertical and horizontal motion One may describe this fact by saying that one here has beatings, not in amplitude but in direction of oscillation

The theory outlined so far allows us to predict, for known imperfections, their influence on the orbits But this is not the situation one meets in

practice One rather wants to obtain predictions for the behaviour of a machine which has not yet been built and the imperfections of which one does not know This leads one to try some kind of statistical approach in the sense that one imagines an ensemble of machines being built as actual realizations of the same perfect machine by identical procedures These machines will all be slightly different from each other and one of them — but we do not know which one — will be the machine we are actually building To construct such an ensemble mathematically one has to make assumptions (which have to be justified by considering the actual procedure of building the machine) about the probability distributions of the several types of imperfections and especially of the statistical interdependence of imperfections in different parts of the machine

Fig 7

The general procedure of this *statistical treatment* shall be shown in the case of the stopband width given by equation (1.13). For the sake of simplicity it shall be assumed that the imperfections $\delta n(\theta)$ are constant inside each sector and are statistically independent of each other in different sectors. Further, it shall be assumed that positive and negative values of δn are equally likely. Then, the integral appearing in the cited equation, shall be split into a sum with each term giving the integral over just one period θ_0 . This sum can be represented by a chain of lines in a complex plane (Fig. 7). The stopband width is then given, apart from a constant factor, by the vector connecting beginning and end of that chain. Between two subsequent lines there is an angle of 2μ as easily follows from

$$(2.20) \quad (z_1(\theta + \theta_0))^2 = \exp[2i\mu](z_1(\theta))^2$$

Under the assumptions made one just has a random walk problem with M steps in two dimensions and it is well known that the root mean square value of the distance between start and end of the walk is proportional to the square root of the number, M , of steps. Quantitatively one has in this case, as can easily be deduced from (2.13)

$$(2.21) \quad \langle \delta Q_{\text{stop}} \rangle_{\text{rms}} = \sqrt{M} \langle \delta n \rangle_{\text{rms}} \left| \frac{1}{\pi W} \int_0^{\theta_0} d\theta' (z_1(\theta'))^2 \right|,$$

where, by definition, for any quantity η

$$(2.22) \quad \langle \eta \rangle_{\text{rms}} = \sqrt{\langle \eta^2 \rangle}$$

Further, the probability distribution of the length of the vector in question ($\propto \delta Q_{\text{stop}}$) is well known for such a random walk problem. Writing x instead

of δQ_{stop} one has for the probability $p(x)$ of finding an x between x and $x + dx$ the following Rayleigh distribution

$$(2.23) \quad p(x) dx = \frac{2}{\langle x^2 \rangle} \exp \left[-\frac{x^2}{\langle x^2 \rangle} \right] x dx$$

Statistical considerations of this kind have been used for determining sets of possible machine parameters