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Perturbative expansions / Caianiello, E R

CERN-56-10

Published in Nuovo Cimento 2 (1955) pp 155-159

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To Jens Vigen <Jens.Vigen@cern.ch>

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Carmen Vasini

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1955, Luglio

Il Nuovo Cimento

Serie X, Vol 2, pag 155 159

CERN 56 - 10

THEORETICAL STUDY DIVISION

Perturbative Expansions

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(ricevuto il 23 Maggio 1955)

We have given, previously, the explicit perturbative expansion of any element of the U matrix pertaining to a field theory of the type usually envisaged, and, equivalently, that of the corresponding kernel ⁽¹⁾ We record here some additional results which complete those already reported, both for future reference and because they may be useful in expediting calculations The notation is that established in II

1 - It was shown in II that, for all purposes, it suffices to know the expansions of the purely fermionic kernels, which are given by:

$$(1) \quad K_{N_0} = K \begin{pmatrix} x_1 & x_{N_0} \\ y_1 & y_{N_0} \end{pmatrix} = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} \left(\sum \gamma^1 \quad \gamma^{2n} [\xi_1 \quad \xi_{2n}] \begin{pmatrix} x_1 & x_{N_0} \xi_1 & \xi_{2n} \\ y_1 & y_{N_0} \xi_1 & \xi_{2n} \end{pmatrix} \right);$$

or, if the orders of the integrations can be freely exchanged, more simply by:

$$(2) \quad K \begin{pmatrix} x_1 & x_{N_0} \\ y_1 & y_{N_0} \end{pmatrix} = \sum_{n=0}^{\infty} \left(\frac{\lambda^2}{2} \right)^n \frac{1}{n!} \left(\sum \gamma^1 \quad \gamma^{2n} [\xi_1 \xi_2] \quad [\xi_{2n-1} \xi_{2n}] \begin{pmatrix} x_1 & x_{N_0} \xi_1 & \xi_{2n} \\ y_1 & y_{N_0} \xi_1 & \xi_{2n} \end{pmatrix} \right)$$

Both expressions have the advantage of maximum formal simplicity, and the disadvantage of expressing kernels which are *not* normalized to unity vacuum vacuum transition amplitude This means, of course, that in actual computation one finds, on expanding, say, the determinants of (2), increasingly many uninteresting vacuum vacuum fluctuation terms, to be dropped on inspection It is desirable, therefore, to know some simple rule which may avoid automatically the appearance of such terms, yielding directly the expansion of any *normalized* kernel $G_{N_0} = K_{N_0}/K_{N_0=0}$ This task, which would be next to hopeless if one knew only the expansions (1) or (2), becomes easily feasible if one takes as starting point the branching equations (II 51) (or, equivalently, (II 52)) It follows then, indeed, that the branching equations

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(¹) E R CAIANIELLO: *Nuovo Cimento* **10** 1634 (1953) and **11** 492 (1954): referred to as I and II

for the normalized kernels can be written as:

$$(3) \quad G \begin{pmatrix} x_1 & x_{N_0} \\ y_1 & y_{N_0} \end{pmatrix} = \sum_{h=1}^{N_0} (-1)^{h-1} (x_1 y_h) G \begin{pmatrix} x_2 & x_h & x_{h+1} & x_{N_0} \\ y_1 & y_{h-1} & y_{h+1} & y_{N_0} \end{pmatrix} + \\ + (-1)^{N_0} \lambda^2 \int \int \sum_{1,2} d\xi_1 d\xi_2 \gamma^1 \gamma^2 (x_1 \xi_1) [\xi_1 \xi_2] G \begin{pmatrix} x_2 & x_{N_0} & \xi_1 \xi_2 \\ y_1 & y_{N_0-1} & y_{N_0} \xi_2 \end{pmatrix},$$

where now, of course, $G_0 = 1$. On setting

$$G_{N_0} = \sum_{p=0}^{\infty} G_{N_0}^{(p)} \lambda^{2p}; \quad G_{N_0}^{(p)} = G^{(p)} \begin{pmatrix} x_1 & x_{N_0} \\ y_1 & y_{N_0} \end{pmatrix},$$

one has for the coefficients $G_{N_0}^{(p)}$ the recursion formulae:

$$(4) \quad G^{(p)} \begin{pmatrix} x_1 & x_{N_0} \\ y_1 & y_{N_0} \end{pmatrix} = \sum_{h=1}^{N_0} (-1)^{h-1} (x_1 y_h) G^{(p)} \begin{pmatrix} x_2 & x_h & x_{h+1} & x_{N_0} \\ y_1 & y_{h-1} & y_{h+1} & y_{N_0} \end{pmatrix} + \\ + (-1)^{N_0} \int \int \sum_{1,2} d\xi_1 d\xi_2 \gamma^1 \gamma^2 (x_1 \xi_1) [\xi_1 \xi_2] G^{(p-1)} \begin{pmatrix} x_2 & x_{N_0} & \xi_1 \xi_2 \\ y_1 & y_{N_0-1} & y_{N_0} \xi_2 \end{pmatrix}$$

Introduce now the symbol:

$$(5) \quad \begin{pmatrix} x_1 & x_{N_0} & \xi_1 \xi_2 & \xi_3 \xi_4 \\ y_1 & y_{N_0} & \xi_1 \xi_2 & \xi_3 \xi_4 \end{pmatrix} \left| \begin{array}{c} \xi_{2p-1} \xi_{2p} \\ \xi_{2p-1} \xi_{2p} \end{array} \right| = \\ = \sum_{h=1}^{N_0} (-1)^{h-1} (x_1 y_h) \begin{pmatrix} x_2 & x_h & y_{h+1} & x_{N_0} \\ y_1 & y_{h-1} & y_{h+1} & y_{N_0} \end{pmatrix} \left| \begin{array}{c} \xi_1 \xi_2 \\ \xi_1 \xi_2 \end{array} \right| \left| \begin{array}{c} \xi_{2p-1} \xi_{2p} \\ \xi_{2p-1} \xi_{2p} \end{array} \right| + \\ + (-1)^{N_0} (x_1 \xi_1) \begin{pmatrix} x_2 & x_{N_0} & \xi_1 \xi_2 & \xi_3 \xi_4 \\ y_1 & y_{N_0-1} & y_{N_0} \xi_2 & \xi_3 \xi_4 \end{pmatrix} \left| \begin{array}{c} \xi_{2p-1} \xi_{2p} \\ \xi_{2p-1} \xi_{2p} \end{array} \right| \quad (N_0 \geq 2)$$

$$(5') \quad \begin{pmatrix} x_1 & \xi_1 \xi_2 & \xi_3 \xi_4 \\ y_1 & \xi_1 \xi_2 & \xi_3 \xi_4 \end{pmatrix} \left| \begin{array}{c} \xi_{2p-1} \xi_{2p} \\ \xi_{2p-1} \xi_{2p} \end{array} \right| = - (x_1 \xi_1) \begin{pmatrix} \xi_1 \xi_2 & \xi_3 \xi_4 \\ y_1 \xi_2 & \xi_3 \xi_4 \end{pmatrix} \left| \begin{array}{c} \xi_{2p-1} \xi_{2p} \\ \xi_{2p-1} \xi_{2p} \end{array} \right| \quad (N_0 = 1),$$

defined by (5) and (5'), the ultimate products of the reduction being determinants. It is then a straightforward matter to verify that the equations (4) are satisfied if, and only if:

$$(6) \quad G \begin{pmatrix} x_1 & x_{N_0} \\ y_1 & y_{N_0} \end{pmatrix} = \sum_{p=0}^{\infty} \lambda^{2p} \int \sum \gamma^1 \gamma^{2p} [\xi_1 \xi_2] \left| \begin{array}{c} \xi_{2p-1} \xi_{2p} \\ \xi_{2p-1} \xi_{2p} \end{array} \right| \begin{pmatrix} x_1 & x_{N_0} & \xi_1 \xi_2 \\ y_1 & y_{N_0} & \xi_1 \xi_2 \end{pmatrix} \left| \begin{array}{c} \xi_{2p-1} \xi_{2p} \\ \xi_{2p-1} \xi_{2p} \end{array} \right|$$

(6) is, thus, the desired expansion. The symbol (5), (5') which is, of course, expanded with greater ease than a determinant, since less terms are required, all other rules being the same—contains the prescription for the systematic dropping of the vacuum vacuum terms in the simple form exhibited by the difference between (5) and (5').

Interesting relations among the coefficients of (6) can be inferred from other

equations of II; consider, for instance, eq (II 55):

$$\frac{d}{d(\lambda^2)} K \begin{pmatrix} x_1 & y_{N_0} \\ x_1 & y_{N_0} \end{pmatrix} = \frac{1}{2} \int \sum_{1,2} d\xi_1 d\xi_2 \gamma^1 \gamma^2 [\xi_1 \xi_2] K \begin{pmatrix} x_1 & x_{N_0} \xi_1 \xi_2 \\ y_1 & y_{N_0} \xi_1 \xi_2 \end{pmatrix},$$

which becomes

$$(7) \quad \frac{d}{d(\lambda^2)} G \begin{pmatrix} x_1 & x_{N_0} \\ y_1 & y_{N_0} \end{pmatrix} = \\ = \frac{1}{2} \int \sum_{1,2} d\xi_1 d\xi_2 \gamma^1 \gamma^2 [\xi_1 \xi_2] \left\{ G \begin{pmatrix} x_1 & x_{N_0} \xi_1 \xi_2 \\ y_1 & y_{N_0} \xi_1 \xi_2 \end{pmatrix} - G \begin{pmatrix} x_1 & x_{N_0} \\ y_1 & y_{N_0} \end{pmatrix} G \begin{pmatrix} \xi_1 \xi_2 \\ \xi_1 \xi_2 \end{pmatrix} \right\}$$

On expanding according to (6), one finds:

$$(8) \quad 2p G^{(p)} \begin{pmatrix} x_1 & x_{N_0} \\ y_1 & y_{N_0} \end{pmatrix} = \\ = \int \sum_{1,2} d\xi_1 d\xi_2 \gamma^1 \gamma^2 [\xi_1 \xi_2] \left\{ G^{(p-1)} \begin{pmatrix} x_1 & x_{N_0} \xi_1 \xi_2 \\ y_1 & y_{N_0} \xi_1 \xi_2 \end{pmatrix} - \sum_{h+k=p-1} G^{(h)} \begin{pmatrix} x_1 & x_{N_0} \\ y_1 & y_{N_0} \end{pmatrix} G^{(k)} \begin{pmatrix} \xi_1 \xi_2 \\ \xi_1 \xi_2 \end{pmatrix} \right\}$$

This identity, which is by no means obvious a priori, can, of course, be verified by direct evaluation of both sides; it can be iterated at will, most simply by using the same procedure on higher derivatives of K_{N_0} and G_{N_0} .

As has been stated before, (6) is the counterpart of (2); in its derivation it is assumed implicitly that it is legitimate to exchange freely the orders in which the integrations are to be performed. This is certainly allowed in a theory where every thing is convergent; it is, conceivably, also true after a renormalization has been carried out with the Dyson-Salam prescriptions: it is important, however, to realize that no ambiguities can ever occur for this reason if one works with the expansions (1). It is well known, indeed, that careless exchange of the order of integration, when dealing with improper or principal value integrals, leads to wrong results *as a rule and not as an exception*, as is exemplified abundantly in the literature (1), that is the expression one gets from straightforward perturbation theory, cannot give rise to troubles of this sort, because all the integrands which appear in it are symmetric functions of their arguments, so that exchanges make no difference. There is no difficulty, though, in passing from (6) to its symmetrized version, which will correspond to (1): it suffices, simply, to replace each non symmetric integrand with its symmetrized form, before the integrations are carried out. This is quite obvious, if one reminds the methods followed to derive the results of I and II. The same caution, of course, must apply as well, in any quantitative treatment, to the equations given in II: a point, as relevant as it is simple, which shows that there is no need of postulating rules to resolve such ambiguities, since, in any case, reference to the correct expansions (1) will remove them. Proceeding in this manner one would find, for instance, that the linear divergent contributions to the electron self energy cancel one another exactly to all orders, or that the photon self energy vanishes (both well known results), in a direct and unambiguous way.

A quantity of particular interest is $\lg K_0$, because variational differentiations with respect to external fields and sources (which appear, in this formulation, in the

functions (xy) and $[xy]$ permit to derive from it kernels (different from those considered by us) which may find useful applications. Its perturbative expansion is immediately desumed from (II 55), on writing:

$$\frac{1}{K_0} \frac{dK_0}{d(\lambda^2)} = \frac{1}{2} \int \int \sum_{1,2} d\xi_1 d\xi_2 \gamma^1 \gamma^2 [\xi_1 \xi_2] G \begin{pmatrix} \xi_1 \xi_2 \\ \xi_1 \xi_2 \end{pmatrix};$$

this yields

$$(9) \quad \lg K_0 = \sum_{p=1}^{\infty} \frac{\lambda^{2p}}{2^p} \int \sum \gamma^1 \gamma^{2p} [\xi_1 \xi_2] [\xi_{2p-1} \xi_{2p}] \begin{pmatrix} \xi_1 \xi_2 & \xi_3 \xi_4 \\ \xi_1 \xi_2 & \xi_3 \xi_4 \end{pmatrix} \begin{pmatrix} \xi_{2p-1} \xi_{2p} \\ \xi_{2p-1} \xi_{2p} \end{pmatrix}$$

Comparison of this expression with the familiar one of the Fredholm theory shows most clearly the structural difference between a one field and a two field theory

2 - We touch here upon another point of more general interest. It is not known, at present, whether the expansions here represented have some region of convergence, or should rather be considered as asymptotic, or, worse, as completely devoid of meaning. It is therefore useful, in the hope of future advances of the theory, to obtain compact, though formal, expressions for their remainders, which may be eventually of help in deciding on these questions and in testing the validity of approximative methods. This aim can be achieved in a variety of ways, by using the techniques of II; we shall treat a few typical cases. By iterating (II 55) one finds the exact relation:

$$(10) \quad \frac{d^n}{d(\lambda^2)^n} K \begin{pmatrix} x_1 & x_{N_0} \\ y_1 & y_{N_0} \end{pmatrix} = \frac{1}{2^n} \int \sum \gamma^1 \gamma^{2n} [\xi_1 \xi_2] [\xi_{2n-1} \xi_{2n}] K \begin{pmatrix} x_1 & x_{N_0} \xi_1 & \xi_{2n} \\ y_1 & y_{N_0} \xi_1 & \xi_{2n} \end{pmatrix};$$

an equivalent expression is:

$$(11) \quad \frac{d^n}{d\lambda^n} K \begin{pmatrix} x_1 & x_{N_0} \\ y_1 & y_{N_0} \end{pmatrix} = \int \sum \gamma^1 \gamma^n K \begin{pmatrix} x_1 & x_{N_0} \xi_1 & \xi_n \\ y_1 & y_{N_0} \xi_1 & \xi_n \end{pmatrix} \begin{pmatrix} \xi_1 & \xi_n \\ \xi_1 & \xi_n \end{pmatrix}$$

These relations can be considered as equations for K_{N_0} , which are readily solved by using, again, the results of II. One finds, writing $K_{(\lambda)}$ to express the dependence of K on λ :

$$(12) \quad K_{(\lambda^2)} \begin{pmatrix} x_1 & x_{N_0} \\ y_1 & y_{N_0} \end{pmatrix} = \sum_{p=0}^{n-1} \left(\frac{\lambda^2}{2} \right)^p \frac{1}{p!} \int \sum \gamma^1 \gamma^{2p} [\xi_1 \xi_2] [\xi_{2p-1} \xi_{2p}] \begin{pmatrix} x_1 & x_{N_0} \xi_1 & \xi_{2p} \\ y_1 & y_{N_0} \xi_1 & \xi_{2p} \end{pmatrix} + \\ + \int_0^{\lambda^2} \frac{(\lambda^2 - \mu^2)^{n-1}}{(n-1)!} d(\mu^2) \frac{1}{2^n} \int \sum \gamma^1 \gamma^{2n} [\xi_1 \xi_2] [\xi_{2n-1} \xi_{2n}] K_{(\mu^2)} \begin{pmatrix} x_1 & x_{N_0} \xi_1 & \xi_{2n} \\ y_1 & y_{N_0} \xi_1 & \xi_{2n} \end{pmatrix};$$

$$(13) \quad K_{(\lambda)} \begin{pmatrix} x_1 & x_{N_0} \\ y_1 & y_{N_0} \end{pmatrix} = \sum_{p=0}^{n-1} \left(\frac{\lambda^2}{2} \right)^p \frac{1}{p!} \int \sum \gamma^1 \gamma^{2p} [\xi_1 \xi_2] [\xi_{2p-1} \xi_{2p}] \begin{pmatrix} x_1 & x_{N_0} \xi_1 & \xi_{2p} \\ y_1 & y_{N_0} \xi_1 & \xi_{2p} \end{pmatrix} + \\ + \int_0^{\lambda} \frac{(\lambda - \mu)^{2n-1}}{(2n-1)!} d\mu \int \sum \gamma^1 \gamma^{2n} K_{(\mu)} \begin{pmatrix} x_1 & x_{N_0} \xi_1 & \xi_{2n} \\ y_1 & y_{N_0} \xi_1 & \xi_{2n} \end{pmatrix} \begin{pmatrix} \xi_1 & \xi_{2n} \\ \xi_1 & \xi_{2n} \end{pmatrix}$$

One can infer then without difficulty from (7) that similar, although less simple, expressions hold also for the normalized kernels G_{N_0} .

Finally, we record one more among the host of possible formal expansions, which is non perturbative and may become of interest if the kernels $K(t_1 \dots t_{p_0})$ turn out to have a simpler structure than the purely fermionic kernels (the proof, which requires some lengthy algebra, is omitted in view of the intuitive character of the result):

$$(15) \quad K \begin{pmatrix} x \\ y \end{pmatrix} = \sum_{p=0}^{\infty} \lambda^p \left(\sum \{ (\xi_1 \xi_2) \gamma^1 (\xi_1 \xi_2) \gamma^2 \dots (\xi_{p-1} \xi_p) \gamma^p (\xi_p y) \} K(\xi_1 \xi_2 \dots \xi_p) \right)$$

3 — In conclusion, and as a justification for our laying so much stress on formal matters, we should like to mention a result, the proof of which will appear in a subsequent paper together with a fuller discussion: if convergence factors are introduced in the momentum expansions of the functions D^F and S^F (or, equivalently and perhaps more significantly, if one works in a lattice space depending upon some fundamental length), then all the series given here certainly converge within a finite range of values of λ (which depends, of course, in a non renormalized theory, upon the parameters of the convergence factors). Such a thing is not trivial, because finiteness which one would expect in this case—does by no means imply holomorphy: indeed, this result can only be proved to hold in the case of fermion boson interactions, but not in the case of a four fermion or of a quadratic boson boson interaction.