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To Jens Vigen <Jens.Vigen@cern.ch>

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Carmen Vasini

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THEORETICAL STUDY DIVISION

Non-Perturbative Expansions

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(ricevuto il 16 Giugno 1955)

It has been shown previously that:

a) a given propagation kernel satisfies an integro differential relation ^(1,2), which contains the first and second derivatives of the kernel with respect to the interaction strength λ ;

b) all the λ derivatives of a kernel exist and are finite in the point $\lambda=0$, under the hypothesis that the kernel is continuous there ⁽³⁾

We propose here to show that, as a consequence of a) and b), it is possible to obtain, in several ways, expansions which are valid under the only assumptions that the kernels exist and are finite with their first and second λ derivatives (the latter hypothesis may be somehow relaxed, but it would be uninteresting to do it here) While a study of the convergence of perturbative expansions would be only partially conclusive, if the result be negative, a similar investigation on the expansions proposed here (which would present about the same degree of difficulty) would have a direct bearing on the problem of existence, and

eventually provide the analytical continuation of the perturbative expansions if these converge within some range of λ Renormalization presents no special difficulties and will be discussed at the end Since we only present here a general outline of the method, without any detailed discussion, we use as shortened a notation as possible

The rôle played by a) and b) is, intuitively, the following The result b), which gives all the desirable information on the behavior of a kernel in the origin, does not permit any deduction on its behavior in nearby points, since it is not known whether holomorphy holds An equation a), on the other hand, while clearly not sufficient to determine a kernel uniquely if only a finite amount of additional information (initial data) is given, yields however, because of its differential nature, a link between the point $\lambda=0$ and nearby points Combining a) with b), therefore, one has an infinite amount of information in $\lambda=0$ (all the λ derivatives) and a « bridge » connecting the origin with the points near to it, the « crossing » of which is quite independent of any assumption of holomorphy; sufficiency is secured trivially, since it is always possible to revert back, formally, to the perturbative expansion

It seems most convenient to work

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^(1,2,3) E R CAIANIELLO: *Nuovo Cimento* 11 492 (1954); 12, 561 (1954); 1 337 (1955)

with equations such as given in ref (2), and to express all quantities in terms of the new variable $\alpha = \lambda^2$. It will suffice, here, to write the fundamental equation *a*) in the symbolic form:

$$(1) \quad \mathcal{L}(K) = (L_0 \Theta_0 + L_1 \Theta_1 + L_2 \Theta_2)K = 0,$$

where $L_{0,1,2}$ are differential operators in α at most of the second order, and the $\Theta_{0,1,2}$ operate only on the spinor vector indices (suppressed) and on the space-time variables of K , which we denote, for short, with P : $K = K(\alpha, P)$. The most general expansion one can envisage for $K(\alpha, P)$ is:

$$(2) \quad K(\alpha, P) = \sum f_n(\alpha) k_n(P),$$

or

$$(3) \quad K(\alpha, P) = \sum a_{mn} g_m(\alpha) k_n(P),$$

where: $\{k_n(P)\}$ is some prescribed set of functions having the same tensorial character as $K(\alpha, P)$, which is known to be complete with respect to the problem of the linear approximation of functions having the nature of $K(\alpha, P)$; $\{g_m(\alpha)\}$ is a complete set of functions of α relative to some given interval $(0, A)$, conveniently chosen (α must be considered to be real and positive, or else λ would be complex and the interaction Hamiltonian non Hermitean: the whole formalism should have to be changed); the functions $f_n(\alpha)$, or alternatively the coefficients a_{mn} , are the quantities to be determined.

The usefulness of such an expansion depends upon our *a priori* knowledge of the set $\{k_n(P)\}$. The theorem *b*) permits to show that we know, indeed, all we need about it. For let $d^v K(\alpha, P)/d\alpha^v|_{\alpha=0} = K_v(P)$: then the functions $K_v(P)$ must be expandable, by hypothesis, in terms of the set $\{k_n(P)\}$, so that the latter is linearly related to the set $\{K_v(P)\}$. If the functions $K_v(P)$ are linearly independent (this we assume without proof, but there can be little doubt as to the validity of this hypothesis), then the set

$\{K_v(P)\}$ itself or any set

$$(4) \quad k_n(P) = \sum c_m K_m(P)$$

are complete and can be used in (2) or (3).

In practice, one shall consider the N^{th} approximation to $K(\alpha, P)$, obtained by neglecting in (2) or (3) all the terms with $m, n > N$. For any finite N :

$$(5) \quad \frac{d^N K^{(N)}(\alpha, P)}{d\alpha^N} \Big|_{\alpha=0} = K^{(N)}_1(P) = \sum_0^N f_n^{(N)}(0) k_n(P),$$

or

$$(6) \quad K^{(N)}_1(P) = \sum_0^N a_{mn} g_m^{(N)}(0) k_n(P);$$

we need not assume, though, that term by term differentiation to all orders is legitimate in the complete series.

The way to follow for the determination of the functions $f_n(\alpha)$ or of the coefficients a_{mn} is quite clear in principle, and will be hinted to only very briefly. As in all such problems, one can ask either that the equation (1) be satisfied exactly by $k^{(N)}(\alpha, P)$ and the initial conditions $K^{(N)}_v(P) = K_v(P)$ be satisfied only approximately, or viceversa, or that both be satisfied approximately. (In any case, of course, the initial conditions can be satisfied exactly, at most, only for $v \leq N$. The first instance is not possible if one starts from (3)). A variational procedure consisting in minimizing expressions:

$$(7) \quad \int_0^A d\alpha \int dP \varrho(P) \mathcal{L}(K^{(N)}) \mathcal{L}(K^{(N)})^*,$$

$$(8) \quad \int dP \varrho(P) |K^{(N)}_1 - K_1|^2,$$

where $\varrho(P)$ is some suitable positive tensor weight (to smooth out singularities) may then provide, according to the criterion chosen, the best fit for the $f_n(\alpha)$

or the a_{nn} . The coefficients c_{nn} in (4) are related to the initial values of the $f_n(\alpha)$ and their derivatives.

This outline may be concluded with two remarks. First, the ordinary differential equations one obtains for the $f_n(\alpha)$ are singular in the points $\alpha=0$ and $\alpha=\infty$; this is a feature of general interest, which may throw some light on the nature of the kernels. Second, the equation (1) is satisfied *identically* by the formal power series expansion; that is, on equating the coefficient of α^n to zero, one obtains a linear identity connecting $\theta_0(K_n(P))$, $\theta_1(K_n(P))$, $\theta_2(K_n(P))$, which is of great help in practical calculations and can be used, conversely, to establish equation (1). The latter point shows also that no troubles arise because *a)* of renormalization, *b)* of normalization to unity vacuum vacuum transition amplitude (division by K_0). For if renormalization is performed with the standard subtractive technique (as we assume here), then this identity remains true also for quantities calculated in this manner: the renormalized perturbative expansion satisfies, formally, the same equation (1), where now everything is calculated with the subtractive prescriptions. Moreover, FER-

RETTI⁽⁴⁾ has proved, quite generally, that divergencies arising in the calculation of λ derivatives of kernels are compensated by the same counter terms of the usual theory: this shows, independently of perturbative methods, that the standard subtractive procedure can be legitimately applied in our case. Exactly the same argument shows that the normalized kernels K/K_0 are obtained by simply dropping, throughout, the terms which correspond, in the usual graph terminology, to vacuum vacuum fluctuations.

A method such as the one proposed here yields, of course, only a formal expansion; however, the conditions for its convergence are much less restrictive than those for the corresponding perturbative expansion. One should prove first that, in the limit $N \rightarrow \infty$, both the equation (1) and the initial conditions are satisfied exactly; then investigate the convergence of the expansion: even if only a weighed mean convergence should hold, rather than point by point convergence, definite conclusions might then be drawn as to the existence of a solution.

(4) B. FERRETTI: *Nuovo Cimento* **10** 3 (1953).