CERN-TH.5877/90

COMPLETE CLASSIFICATION OF SIMPLE CURRENT AUTOMORPHISMS

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ABSTRACT

The complete classification of all fusion rule automorphisms within simple current orbits is presented.

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1. Introduction

Despite a lot of effort, relatively little progress has been made towards the classification of the modular invariant partition functions of conformal field theories. Indeed, even for the special case of Kac-Moody algebras a complete classification at arbitrary level exists only for $SU(2)$. On the other hand, in [1] a large subset of invariants based on simple currents has been identified. These invariants have a relatively simple structure, and it should be possible to classify them completely.

We define a simple current invariant as follows. Let M_{ab} be the matrix of multiplicities of operators $\phi_{ab}(z, \bar{z})$, where a is a representation of the left-moving chiral algebra and b a representation of the right-moving one. We will assume these algebras to be identical. Then M_{ab} defines a simple current invariant if the representations a and b can be mapped into each other by the action of a simple current under fusion, for each $M_{ab} \neq 0$. Here a simple current is a primary field whose fusion with any other field yields just one field. (Alternatively a field J is a simple current if and only if $J \times J^c = 1$ [2] or if and only if $S_{0J}/S_{00} = 1$ [3], in a unitary conformal field theory.)

In [1] examples of simple current invariants were constructed using an orbifold-like procedure. This procedure yields a non-diagonal invariant $M(J)$ for every simple current $J^\dagger$. If there is only one orbit of simple currents (i.e. all currents are of the form J^n , $n = 0, \dots, n-1$) this construction gives the complete answer, as will become clear below.

In general the simple currents of a conformal field theory generate an abelian group $\mathbf{Z}_{N_1} \times \dots \times \mathbf{Z}_{N_k}$ under fusion, which is called the center of the conformal field theory. If the center consists of more than one factor the orbifold procedure of [1] is not complete. First of all one has to include all possible products of matrices $M(J_i)$, where J_i are different simple currents. Although this obviously leads to a huge set of solutions, this set is, in general, still not complete. In [4] a class of modular invariants of fusion rule automorphism type (defined below) was found that could manifestly not be written as a product of $M(J_i)$'s. This class occurs if all currents have integer conformal dimension and are local with respect to each other. In this special case [4] does in fact give the complete set of solutions. In this paper we will present the complete set of solutions for arbitrary spin and relative monodromy of the currents.

† These invariants are not necessarily distinct and exist only if a mod-2 level matching condition is satisfied; see [1] for details.

In other words, we will enumerate for any (unitary*) conformal field theory all matrices M_{ab} which satisfy the following properties

1. $[M, S] = [M, T] = 0$
2. $M_{ab} \in \mathbf{Z}$ and $M_{ab} \geq 0$
3. $M_{00} = 1$
4. $M_{0b} = 0$ if $b \neq 0$
5. $M_{ab} = 0$ if a and b are not related to each other by simple currents.

Here S and T are the usual representation matrices of the modular group, satisfying $(ST)^3 = S^2 = C$. It is not difficult to show that any such matrix M defines a permutation of the fields [4] and that this permutation is in fact an automorphism of the fusion rules [3] [6].

This paper is organized as follows. In section 2 we briefly review some results of [1] and [4], and derive a composition formula for solutions. In section 3 we present some special solutions. In section 4 we consider general simple current automorphisms, and we show how any solution can be systematically decomposed into solutions belonging to smaller subgroups of the center. The most general solution can then be obtained by inverting this procedure. In section 5 we conclude with some comments on the problem of classifying all simple current invariants, i.e. including those that do not satisfy property 4. listed above and that therefore involve extensions of the left- and right chiral algebras.

2. Simple current properties

Consider a rational unitary conformal field theory with a set of simple currents $J_1, \dots, J_k$ generating a center $\mathbf{Z}_{N_1} \times \dots \times \mathbf{Z}_{N_k}$ (One may always choose a basis so that the N_i 's are all of the form $p_i^{n_i}$, $n_i \in \mathbf{Z}$, p_i prime, and we will assume from now on that this has been done). The set of distinct currents consists of combinations $J_1^{a_1} \times \dots \times J_k^{a_k}$, denoted as $[\vec{a}]$ in the following, where $\vec{a} \in \mathbf{Z}^k/\Lambda$. Here Λ is the lattice consisting of the points $(l_1 N_1, \dots, l_k N_k)$, $l_i \in \mathbf{Z}$.

* Non-unitary conformal field theories may, in principle, have simple currents satisfying $(S_{0J}/S_{00}) = -1$ [5]. Our results are valid for non-unitary theories provided that such currents are not used, i.e. M_{ab} is allowed to be non-zero only if a and b are related by a simple current with $(S_{0J}/S_{00}) = 1$.

The monodromies of all other fields in the conformal field theory with respect to the set of simple currents can be used to define a set of k charges, each defined modulo integers:

$$J_i(z)\phi(w) \approx (z-w)^{-Q_i(\phi)}\phi'(w) + \dots,$$

where ϕ' is the field $J \times \phi$ or one of its descendants. The allowed charges belong to the coset $\Lambda^*/\mathbb{Z}^k$, where Λ^* is the dual lattice of Λ . It is easy to show that the charges occurring in a conformal field theory must span all of $\Lambda^*/\mathbb{Z}^k$ in order to get a unitary matrix S [4].

The conformal weights of the currents and the current-current charges are parametrized by a monodromy matrix R_{ij} , which is of the form

$$R_{ij} = \frac{r_{ij}}{N_i} = \frac{r_{ji}}{N_j} \pmod{1},$$

where $r_{ij} \in \mathbb{Z}$. The matrix elements of r_{ij} are defined modulo N_i , except for the diagonal elements r_{ii} , which are defined modulo $2N_i$ if N_i is even [4].

The conformal weight of a current combination $[\vec{\alpha}]$ is

$$h([\vec{\alpha}]) = \frac{1}{2} \sum_i r_{ii} \alpha_i - \frac{1}{2} \sum_{ij} \alpha_i R_{ij} \alpha_j \pmod{1}. \quad (2.1)$$

and the relative charges of the currents are

$$Q_i(J_j) = R_{i,j} \pmod{1}.$$

All possible diagonal matrices R can be realized in a conformal field theory, for example by taking products of $SU(N)$ Kac-Moody algebras. Non-diagonal examples are harder to find, and many non-diagonal matrices R can be diagonalized by choosing a different set of basic currents. An example of a non-diagonal monodromy matrix that cannot be diagonalized occurs for the Kac-Moody algebra $SO(8)$, level 1, which has a center $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The three simple currents (v) , (s) and (c) have half-integer spin and are not local with respect to each other. In any basis, the matrix r is

$$r = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The presence of simple currents in a conformal field theory leads to relations among the matrix elements of S and T . These relations are respectively

$$S_{[\beta]b, [\vec{\alpha}]a} = e^{2\pi i(\vec{\alpha} \cdot \vec{Q}(b) + \vec{\beta} \cdot \vec{Q}(a) + \alpha_i R_{ij} \beta_j)} S_{ab} \quad (2.2)$$

and

$$h([\vec{\alpha}]a) = h(a) + h([\vec{\alpha}]) - \vec{\alpha} \cdot \vec{Q}(a) \pmod{1}, \quad (2.3)$$

where “ $[\vec{\alpha}]a$ ” denotes the field $J^{\alpha_1} \dots J^{\alpha_k} \phi_a$.

Not all currents in the center are really needed for the construction of modular invariant partition functions. One can distinguish two kinds of currents: those whose conformal weight h satisfies $Nh \in \mathbb{Z}$ and those whose conformal weight satisfies $Nh \in \mathbb{Z} + \frac{1}{2}$, where N is the order of the current. Since the charge $\vec{\alpha} \cdot \vec{Q}$ with respect to a current $[\vec{\alpha}]$ of order N is a multiple of $\frac{1}{N}$ for any field, it follows from (2.3) that currents of the second kind do not preserve the T -eigenvalues of any field. Therefore the modular invariant matrix M_{ab} cannot have any non-vanishing matrix elements between fields a and b mapped to each other by currents of the second kind. Hence we might as well leave such currents out of the discussion. Indeed, it is easy to show that the set of currents of the first kind closes under fusion, so that we can define an “effective” center, consisting only of such currents. When one restricts the monodromy matrix to the effective center, its matrix elements satisfy $r_{ii} = 0 \pmod{2}$ for N_i even; furthermore, for N_i odd we have the freedom of choosing r_{ii} even, since r_{ii} is defined modulo N_i in that case. Hence we may assume from now on that r_{ii} is even, and omit the first term in (2.1). A useful consequence of the restriction to the effective center is that all mod-2 level-matching conditions in the orbifold-inspired construction of [1] are automatically satisfied.

Since the matrices M we are considering define permutations of the fields, there is for each field a precisely one field a' and one field a'' with $M_{aa'} = M_{aa''} = 1$. Furthermore by assumption 5 listed above a' and a'' are related to a by the action of simple currents. For each field a we will denote those simple currents by $[\vec{\mu}(a)]$ and $[\vec{\nu}(a)]$ respectively, so that $a' = [\vec{\mu}(a)]a$ and $a'' = [\vec{\nu}(a)]a$. Using S -invariance we can now derive a relation between the functions $\vec{\mu}$ and $\vec{\nu}$ as well as an equation for these functions separately. These relations are

$$\vec{\mu}(a) \cdot \vec{Q}(b) = \vec{\nu}(b) \cdot \vec{Q}(a) \pmod{1} \quad (2.4)$$

and

$$\vec{\mu}(a) \cdot \vec{Q}(b) + \vec{\mu}(b) \cdot \vec{Q}(a) + \vec{\mu}(a) \cdot R\vec{\mu}(b) = 0 \pmod{1}, \quad (2.5)$$

and similarly for $\vec{\nu}$.

To derive these relations, we consider two different matrix elements of $MS = SM$, namely

$$M_{a,[\vec{\mu}(a)]a} S_{[\vec{\mu}(a)]a,b} = S_{a,[\vec{\nu}(b)]b} M_{[\vec{\nu}(b)]b,b},$$

and

$$M_{a,[\vec{\mu}(a)]a} S_{[\vec{\mu}(a)]a,[\vec{\nu}(b)]b} = S_{ab} M_{b,[\vec{\nu}(b)]b},$$

respectively, and use (2.2). In both cases one has to assume that $S_{ab} \neq 0$. We will discuss this assumption below.

It follows from (2.4) that $\vec{\mu}(a)$ and $\vec{\nu}(a)$ depend on the field a only via its charge $Q(a)$. Suppose two fields a and a' have the same charges $\vec{Q}(a) = \vec{Q}(a')$. Then from (2.4) we get $[\vec{\mu}(a) - \vec{\mu}(a')] \cdot \vec{Q}(b) = 0 \pmod{1}$. Since the charges $\vec{Q}(b)$ span the lattice of charges Λ^* (modulo integers), it follows that $\vec{\mu}(a) - \vec{\mu}(a')$ lies on the dual of Λ^* , so that $\vec{\mu}(a) = \vec{\mu}(a') \pmod{\Lambda}$. In exactly the same way one can show that the dependence on the charge is linear. Here the assumption of non-vanishing elements of S is obviously important. It follows from unitarity that every field a must have non-vanishing S -matrix elements with a set of fields whose charges span $\Lambda^*/\mathbb{Z}^k$, but it does not follow that this set of fields must be identical for a and a' [4]. On the other hand, it is clear that an 'accidental' zero of S does not matter, and that only a systematic pattern of vanishing matrix elements can invalidate the argument.

In general, if some matrix elements of S vanish this will lead to fewer constraints and hence potentially more modular invariants than those discussed here. In many cases, the matrix elements with the currents themselves (which never vanish) impose sufficient constraints. We do not know any examples with extra invariants due to unexpected zeroes of S , and we will not attempt to formulate the precise conditions for acceptable zeroes here.

The arguments given so far are valid for fields that are not fixed points of some current. It is easy to see from (2.2) that S cannot have matrix elements between fixed point fields and other fields with certain charges. When one considers the action of the currents on a fixed point f , the lattice Λ (which labels the current combinations whose action is trivial) is effectively enlarged to a lattice $\Lambda_f \supset \Lambda$. Hence its dual Λ_f^* (the lattice of allowed charges) contains fewer points than Λ^* . It is easy to show (see [4]) that a fixed point field must have non-vanishing S -elements with a set of charges that spans Λ^* . This is precisely sufficient to show that if $\vec{Q}(f) = \vec{Q}(f')$ mod 1, then $\vec{\mu}(f) = \vec{\mu}(f') \pmod{\Lambda_f}$, provided that the set of fields with which both f and f' have non-vanishing S -matrix elements spans $\Lambda_f^*/\mathbb{Z}^k$. Hence there is no essential difference in the argument for fixed points and normal fields.

From the foregoing we conclude that the automorphism can be specified completely by fixing it for a set of basic charges $\vec{e}_i = (0, \dots, 0, 1/N_i, 0, \dots, 0)$. Denote the functions $\vec{\mu}$ and $\vec{\nu}$ acting on a field with charge $\vec{e}_i$ as $\vec{\mu}_i$ and $\vec{\nu}_i$ respectively. Then for any other field a with charge $\vec{Q}(a) = \sum q_i \vec{e}_i$ one gets

$$\vec{\mu}(a) = \sum_i q_i \vec{\mu}_i, \quad (2.6)$$

and analogously for $\vec{\nu}$.

The vector components of $\vec{\mu}_i$ and $\vec{\nu}_i$ will be denoted as μ_{ij} and ν_{ij} , and are defined modulo N_j . In the following they will be treated as the elements of matrices, denoted respectively as μ and ν . The equations (2.4) and (2.5) imply the following relations for these components.

$$\frac{\mu_{ij}}{N_j} + \frac{\nu_{ji}}{N_i} \pmod{1}, \quad (2.7)$$

and

$$\frac{1}{N_j} + \mu_{ji} \frac{1}{N_i} + \sum_{l,n} \mu_{il} R_{ln} \mu_{jn} = 0 \pmod{1}, \quad (2.8)$$

and analogously for ν_{ij} .

Note that M is completely determined by the function $\vec{\mu}$ alone (or, alternatively, $\vec{\nu}$ alone). However, not every single-valued function $\vec{\mu}$ defines a permutation. To obtain a permutation it is essential that $[\vec{\mu}(a)]a \neq [\vec{\mu}(b)]b$ if $a \neq b$ (note that if this condition is not satisfied there does not exist a single-valued function $\vec{\nu}$ corresponding to M). We will now prove that any function $\vec{\mu}(a)$ that is a function only of the charges $\vec{Q}(a)$ of a , and that satisfies (2.8), defines in fact a permutation.

Suppose $[\vec{\mu}(a)]a = [\vec{\mu}(b)]b$. Then the respective charges are the same, so that

$$Q_i(a) + \frac{1}{N_i} \sum_j r_{ij} \mu_j(a) = Q_i(b) + \frac{1}{N_i} \sum_j r_{ij} \mu_j(b) \pmod{1},$$

where we used the additivity of charges, i.e. $Q(ab) = Q(a) + Q(b) \pmod{1}$. Using (2.6) we can write this as

$$\frac{q_i}{N_i} + \frac{1}{N_i} \sum_{j,l} r_{ij} q_l \mu_{lj} = \frac{\tilde{q}_i}{N_i} + \frac{1}{N_i} \sum_{j,l} r_{ij} \tilde{q}_l \mu_{lj} \pmod{1},$$

where $Q_i(a) \equiv q_i/N_i$ and $Q_i(b) \equiv \tilde{q}_i/N_i$. Now we multiply both sides with μ_{pi} and sum over i . Then, using (2.8) we get

$$\sum_i q_i \mu_{ip} / N_p = \sum_i \tilde{q}_i \mu_{ip} / N_p \pmod{1}.$$

Thus $\sum_i q_i \tilde{\mu}_i = \sum_i \tilde{q}_i \mu_i \pmod{\Lambda}$ (where Λ is the lattice defined in the beginning of this section), i.e. $\vec{\mu}(a) = \vec{\mu}(b)$. Then $[\vec{\mu}(a)]a = [\vec{\mu}(b)]b$ implies that $a = b$. Hence if two fields are mapped to the same field, they must be identical.

This implies that for every function $\vec{\mu}(a)$ defined by matrices μ_{ij} satisfying (2.8) a single-valued function $\vec{\nu}(a)$ exists. Note that (2.7) not only determines ν from μ , but also imposes quantization conditions on μ , since ν should be integer-valued.

This concludes the discussion of S -invariance. For T -invariance we get using (2.1), (2.3) and (2.6) the following condition

$$\frac{1}{2} \sum_{ij} q_i (\mu R \mu^T)_{ij} q_j + \sum_{ij} q_i \frac{\mu_{ij}}{N_j} q_j = 0 \pmod{1},$$

for a field a with $Q_i(a) = \frac{q_i}{N_i}$. It is easy to see that this yields exactly the same condition as (2.8) for $i \neq j$, but for the diagonal terms there is an extra factor $\frac{1}{2}$:

$$\frac{1}{2} (\mu R \mu^T)_{ii} + \frac{\mu_{ii}}{N_i} = 0 \pmod{1}. \quad (2.9)$$

(Note that this factor $\frac{1}{2}$ is relevant only for currents of even order.) Thus it appears that the conditions for T -invariance contain those of S -invariance, and are slightly more restrictive. Note however that we have used S -invariance to derive that $\vec{\mu}(a)$ is only a function of the charges of a , a fact that does not follow from T -invariance alone.

Finally, let us discuss the group properties of the solutions. Since the product of two matrices M_1 and M_2 satisfying conditions 1-5 listed above yields a third matrix M_3 satisfying the same conditions, it follows that there must exist a composition rule for the μ 's. Indeed, one can easily prove that two solutions μ_1 and μ_2 of (2.8) and (2.9) can be combined to a third solution μ_3 in the following way

$$(\mu_3)_{ij} = (\mu_1 + \mu_2 + \mu_1^T \mu_2)_{ij} \pmod{N_j}. \quad (2.10)$$

This formula can be derived from the matrix product of M_1 and M_2 , and one may verify that μ_3 is indeed a solution. We will denote this operation as $\mu_3 = \mu_1 * \mu_2$. Note that this operation is associative, but not commutative, just as the matrix product from which it was derived. The inverse of a solution is given by

$$(\mu^{-1})_{ij} = \frac{N_j}{N_i} \mu_{ji}. \quad (2.11)$$

To verify that $\mu * \mu^{-1} = 0 \pmod{N_j}$ one has to use equation (2.8). Note that $\mu^{-1} = \nu$. This is as it should be, since if μ defines an automorphism M , then ν defines $M^T = M^{-1}$.

3. Special solutions

In this section we present some special solutions to the equations derived above. The relevant equations are the off-diagonal part of (2.8), and (2.9). Furthermore we know from (2.7) that μ_{ij} must be an integer multiple of $\frac{N_i}{N_j}$ (in fact one can show that every solution to (2.8) and (2.9) automatically has that property). If N_j and N_i are relative prime this implies that $\mu_{ij} = 0 \bmod N_j$. Hence without loss of generality we may assume from now on that all N_i are of the form $N_i = p^{t_i}$, for some prime p , and with $t_i \in \mathbb{Z}$.

Define $M = \max(N_i)$, and consider the following *ansatz*

$$\mu_{ij} = x \frac{M}{N_i} n_i n_j, \quad (3.1)$$

where $n_i, i = 1, \dots, k$ and x are integer parameters. The fact that μ_{ij} is defined modulo N_j implies that we may regard n_j as defined modulo N_j and x as defined modulo M .

Note that the rôle of M is merely to ensure that all factors in (3.1) are manifestly integers. For generic $\vec{n}$, this requires M to be equal to the largest of the N_i 's. If some of the components of $\vec{n}$ vanish or contain factors of p one should choose

$$M = \max\left(\frac{N_i}{\text{GCD}(N_i, n_i)}\right), \quad (3.2)$$

where "GCD" denotes the greatest common divisor, with $\text{GCD}(N_i, 0) \equiv N_i$. This is the smallest value of M so that $M n_i / N_i \in \mathbb{Z}$ for all i . One may check that also in this case x is defined modulo M , i.e. if x is changed by a multiple of M , μ_{ij} changes by a multiple of N_j .

Substituting this *ansatz* into (2.8) and (2.9) yields the conditions

$$x \frac{M}{N_i} n_i n_j \frac{1}{N_j} [2 + xM \langle n, n \rangle] = 0 \bmod 1$$

and

$$x \frac{M}{N_i} n_i n_j \frac{1}{N_i} [1 + xM \frac{1}{2} \langle n, n \rangle] = 0 \bmod 1$$

Here we have introduced the inner product

$$\langle n, m \rangle = \sum_{k,l} n_l R_{kl} m_k.$$

(Note that $\langle n, m \rangle$ is defined modulo integers, and that the norm $\langle n, n \rangle$ is defined modulo even integers.) A sufficient condition for satisfying these two requirements is

$$1 + xM \frac{1}{2} \langle n, n \rangle = 0 \bmod M \quad (3.3)$$

We can find a parameter x satisfying this if $\frac{1}{2}M \langle n, n \rangle \neq 0 \bmod p$ in that case $x \frac{1}{2}M \langle n, n \rangle$ for x varying between 0 and $M-1$ takes precisely all M integer values that are different modulo M , and in particular the value -1 . Note the importance of choosing M as small as possible: if M contains redundant factors of p , $\frac{1}{2}M \langle n, n \rangle$ is also proportional to p , and then (3.3) has no solution.

This solution is the one obtained by means of the orbifold method of [1] for a simple current $J = [\vec{n}]$. We will refer to it as a solution of type I in the following. The orbifold method always yields a modular invariant partition function, without further conditions. The condition on the norm of $\vec{n}$ (or equivalently on the conformal weight of J) found above corresponds to the requirement that one gets a fusion rule automorphism, and not an extension of the chiral algebra. For example, if all N_i 's are equal to p the condition reads $\frac{1}{2} \langle n, n \rangle \neq 0 \bmod 1$ i.e. the current J must have fractional conformal spin.

A second kind of special solution, that will be referred to as type II in the following, can be obtained as follows. First rescale μ to remove the factors $1/N_i$ from the equations, by defining $\hat{\mu}_{ij} = \frac{\mu_{ij}}{N_j}$. Note that $\hat{\mu}$ is defined modulo integers. The equations for $\hat{\mu}$ are

$$\begin{aligned} (\hat{\mu} + \hat{\mu}^T + \hat{\mu} r N \hat{\mu}^T)_{ij} &= 0 \bmod 1 & (i \neq j) \\ (\hat{\mu} + \frac{1}{2} \hat{\mu} r N \hat{\mu}^T)_{ii} &= 0 \bmod 1 \end{aligned}$$

Here N denotes the diagonal matrix $N_{ij} = N_i \delta_{ij}$. Consider the *ansatz*

$$\hat{\mu}_{ij} = \frac{M}{N_i N_j} (n_i m_j - n_j m_i), \quad (3.4)$$

where n_i and m_i are integers defined modulo N_i .

Substituting this *ansatz* into the equations we observe that the linear terms vanish because of the anti-symmetry, while for the bilinear terms we find

$$\begin{aligned} \frac{M^2}{N_i N_j} (n_i m_j < n, m > + m_i n_j < n, m > - n_i m_j < m, m > - m_i m_j < n, n >) = 0 \pmod 1 \\ (i \neq j) \\ \frac{M^2}{N_i^2} (n_i m_i < n, m > - \frac{1}{2} n_i n_i < m, m > - \frac{1}{2} m_i m_i < n, n >) = 0 \pmod 1 \end{aligned}$$

Sufficient conditions for these terms to vanish are $< n, m > = 0 \pmod 1$, $\frac{1}{2} < n, n > = 0 \pmod 1$ and $\frac{1}{2} < m, m > = 0 \pmod 1$. Note that this implies that the currents $[\vec{n}]$ and $[\vec{m}]$ have integer and $\frac{1}{2}$ conformal spin and are local with respect to each other. This special solution is precisely the one described in [4], where systems of purely integer spin currents were considered. (Note that in this kind of system there exist additional modular invariants in which some of the integer spin currents extend the chiral algebra. Such invariants are not under consideration in this paper.)

This class of solutions can be generalized further. Consider the following *ansatz*

$$\vec{\mu} = \epsilon + \Delta, \quad (3.5)$$

where ϵ is anti-symmetric and Δ symmetric. Furthermore we require that ϵN and ΔN are integer matrices, to ensure that μ is integer as well. The solution obtained above was of this type, but with $\Delta = 0$. Now consider for a given ϵ the following *ansatz* for Δ

$$\Delta = F(X)\epsilon, \quad \text{with } X = \epsilon r N, \quad (3.6)$$

where F is some polynomial to be determined from the equations. Substituting this *ansatz* into the two equations one finds

$$\begin{aligned} (2F + (F^2 - 1)X)\epsilon)_{ij} &= 0 \pmod 1 \quad (i \neq j) \\ (F + \frac{1}{2}(F^2 - 1)X)\epsilon)_{ii} &= 0 \pmod 1. \end{aligned} \quad (3.7)$$

A formal solution to these equations is

$$F(X) = \frac{1}{X}(\sqrt{X^2 + 1} - 1)$$

One can verify that Δ is indeed symmetric, using $F(-X) = -F(X)$ and $rN = Nr^T$. As it stands, this solution does not make much sense, since X is an integer-valued matrix, and F should be an integer-valued matrix as well. We can define F by means of the power series expansion of the right-hand side:

$$F(X) = \sum_{n=1} (-1)^{n+1} \frac{1}{2n-1} \binom{2n-1}{n} \left(\frac{1}{2}X\right)^{2n-1}. \quad (3.8)$$

This expansion makes sense if the coefficients are integers and if it truncates at some finite order. It is not hard to show that $\frac{1}{2n-1} \binom{2n-1}{n}$ is an integer. Furthermore, if p (the common prime factor of all N_i 's) is odd, one may always define ϵ so that X is even (since X_{ij} is defined modulo N_j). Then all expansion coefficients are integers, and the factor $\frac{1}{2}$ in (3.7) is harmless. A sufficient condition for truncation of the series is that for some finite power l , $X^l = 0 \pmod p$. In that case there exists a finite power l' so that $X^{l'} = 0 \pmod M$. Since F_{ij} is defined modulo N_j , any contribution to F from the terms of order l' and higher are irrelevant.

For $p = 2$ the discussion is essentially the same, except that one has to worry also about the factors $\frac{1}{2}$ in (3.7) and (3.8). These expressions make sense if, for example, X is even and some power of $\frac{1}{2}X$ is even, so that the sum truncates. There are other possibilities, but since we are making no claims of generality in this section we will not investigate this systematically.

We conclude this section with an example. Consider a theory with a center $\mathbf{Z}_9 \times \mathbf{Z}_9$ and monodromy parameters

$$r = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}.$$

Such a set of currents occurs for example in $SU(9) \times SU(9)$, with levels that are equal to 6 modulo 9. None of the currents in this theory yields a solution of type I. However, there

are integer spin currents J_1^z and J_2^z that are mutually local and can be used to build an invariant of type II. There are in fact two such invariants, given by

$$\mu = \begin{pmatrix} 0 & 3 \\ 6 & 0 \end{pmatrix} \quad \text{and} \quad \mu = \begin{pmatrix} 0 & 6 \\ 3 & 0 \end{pmatrix}.$$

These are the solutions described in [4]. Using the generalization discussed above, we can find still more solutions. Consider the ansatz

$$\epsilon N = \begin{pmatrix} 0 & 8 \\ 1 & 0 \end{pmatrix} \quad (N = 9),$$

which is anti-symmetric modulo 9. One finds

$$X = \epsilon N r^T = \begin{pmatrix} 0 & -6 \\ 6 & 0 \end{pmatrix},$$

which is a multiple of 3, so that only the first term in the expansion for F has to be kept. Using (3.5) and (3.6) we find the new solution

$$\mu = \begin{pmatrix} 6 & 8 \\ 1 & 6 \end{pmatrix}.$$

Starting with other ansätze for ϵ one obtains in total six similar solutions of the form

$$\mu = \begin{pmatrix} 6 & 9-n \\ n & 6 \end{pmatrix},$$

with $n = 1, 2, 4, 5, 7, 8$. The results of the next section will enable us to conclude that this completes the set of solutions for this theory.

4. The general solution

To find the complete set of simple current automorphisms we will use the following strategy. First we will show that any matrix μ describing an automorphism that is non-trivial in K basic Z_N -factors of the center can be reduced to a matrix μ' describing an automorphism that is non-trivial in $K-1$ factors (an automorphism is trivial in a certain Z_N factor of the center if the corresponding row and column vanish completely). The reduction is achieved by combining μ with a certain special solution using the composition formula (2.10). By repeating this any solution can be completely reduced to a single Z_N factor. By inverting this procedure we can then construct the complete set of solutions.

4.1 The reduction procedure

Observe first of all that if $\mu_{i_0 j} = 0 \bmod N_j$ for all j and some fixed value i_0 , then, according to (2.8) $\mu_{j i_0} = 0 \bmod N_{i_0}$. Hence we can make a solution trivial in the i_0^{th} Z_N factor by cancelling the i_0^{th} row. For convenience of notation we will henceforth assume that the row to be cancelled is the first one.

For every solution μ we would like to construct another solution ρ so that $\mu' = \mu * \rho$ satisfies $\mu'_{1j} = 0$. In many cases this can be achieved by choosing for ρ a solution of one of the two types discussed in the previous section.

Since μ has no matrix elements between Z_{N_1} and Z_{N_2} if N_1 and N_2 are relative prime, we can restrict ourselves without loss of generality to a center of the form $Z_{N_1} \times \dots \times Z_{N_k}$, where all N_i are integer powers of the same basic prime number p .

Suppose that $\mu_{11} \neq 0 \bmod p$. Then there exists an integer $I(\mu_{11}, N_1)$, defined modulo N_1 , with the property

$$I(\mu_{11}, N_1)\mu_{11} = 1 \bmod N_1 \quad (4.1)$$

Using this integer we can write down the following solution

$$\begin{aligned} \rho_{ij} &= I(\mu_{11}, N_1) \frac{N_1}{N_i} \mu_{1i} \mu_{1j} \\ &= I(\mu_{11}, N_1) \nu_{i1} \mu_{1j}. \end{aligned} \quad (4.2)$$

In the second line we have expressed ρ in terms of ν using (2.7) to show that it is integer-valued. To verify that ρ is indeed a solution note first of all that (2.9) applied to μ_{11}

gives

$$\mu_{11} \frac{1}{N_1} + \frac{1}{2} \mu_{11} \frac{1}{N_i} r_{ik} \mu_{1k} = 0 \pmod{1}. \quad (4.3)$$

Substituting ρ into (2.8) yields the following condition

$$I(\mu_{11}, N_1) \frac{N_1}{N_i} \mu_{1i} \mu_{1j} \frac{2}{N_j} \left[1 + I(\mu_{11}, N_1) N_1 \left(\frac{1}{2} \mu_{1i} \frac{1}{N_i} r_{ik} \mu_{1k} \right) \right] = 0 \pmod{1}.$$

Because of (4.3) and the definition of I this condition is indeed fulfilled (again we are using here that $N_1 \mu_{11}/N_i = \nu_{11} \in \mathbb{Z}$). From (2.9) one gets the same condition without the factor two, which does not alter the argument. Note that ρ is a solution of type I, described in the previous section, with μ_{1j} playing the rôle of n_j . Now one can compute $(\mu * \rho)_{1j}$ to verify that the result is zero modulo N_j^* .

If $\mu_{11} = 0 \pmod{p}$ one may proceed in one of the following ways. Either one can use a solution of type II to cancel the first row, or one may try to modify μ by means of an auxiliary solution ρ^{aux} of type I so that $\mu^{\text{aux}} = \mu * \rho^{\text{aux}}$ satisfies $\mu_{11}^{\text{aux}} \not\equiv 0 \pmod{p}$. In this case one can cancel the first row of μ^{aux} by means of a second type I solution, as described above. We have found that in the special case $N_i = p$ for all i , at least one of these alternatives works. This is presumably also true in general, and it means that every automorphism can be written as a product of basic automorphisms of type I and type II.

There is however a more direct way of dealing with the general case. Consider the second alternative described above. After modifying μ by ρ^{aux} and using a second type I solution ρ to cancel the first row of μ^{aux} , the final result of the reduction procedure is

$$\mu' = (\mu * \rho^{\text{aux}}) * \rho = \mu * \rho',$$

where, because of the associativity of the product, $\rho' = \rho^{\text{aux}} * \rho$. It is now instructive to compute ρ' directly rather than perform the reduction in two steps. For convenience we will

* If one keeps track of the arbitrary integers in the definition of I and in (4.3) one finds that $(\mu * \rho)_{1j}$ is an integer multiple of $N_i \mu_{1j} = N_i \nu_{1j}$. Thus it is zero modulo N_j as well as modulo N_i .

order the Z_{N_i} factors in the center in such a way that $N_1 \geq N_2 \geq \dots \geq N_k$. Now choose

$$\rho_{ij}^{\text{aux}} = x(N_1/N_i) n_i n_j.$$

Note that here we have chosen the parameter M used in (3.1) equal to N_1 . Because of the ordering of the N_i 's this ensures that ρ^{aux} is integer-valued. The parameter x is as before determined by solving (3.3). The choice of N_1 may not be the optimal one for the existence of x , (a better choice would be (3.2)), but we will see in a moment that this is irrelevant.

If x (and ρ^{aux}) exists, we can compute $\mu^{\text{aux}} = \mu * \rho^{\text{aux}}$ and substitute its components into (4.2) to obtain ρ . Finally we compute the product $\rho^{\text{aux}} * \rho$, and find

$$\rho'_{ij} = \frac{N_1}{N_i} I \left[-\mu_{11} n_i n_j + n_1 n_i \mu_{1j} - (n_1 + N_1 < n, m >) n_j \mu_{1i} + \mu_{1i} \mu_{1j} N_1^{\frac{1}{2}} < n, n > \right], \quad (4.4)$$

where $m_j = \mu_{1j}$ and

$$I = I[\mu_{11} N_1^{\frac{1}{2}} < n, n > - n_1(n_1 + N_1 < n, m >), N_1].$$

(Note that all quantities, such as $N_1^{\frac{1}{2}} < n, n >$ and N_1/N_i are integers!) This result has a surprising pleasant feature: its existence does not depend on that of ρ^{aux} (i.e. of x) or on the existence of ρ , but only on the existence of the new quantity I (see (4.1)). Thus a necessary and sufficient condition for the existence of ρ' is

$$\mu_{11} N_1^{\frac{1}{2}} < n, n > - n_1(n_1 + N_1 < n, m >) \not\equiv 0 \pmod{p}. \quad (4.5)$$

By construction, ρ' is a solution to (2.8) and (2.9) and cancels the first row of μ , at least if ρ^{aux} and ρ exist. One might expect that it still has these properties even if ρ^{aux} and/or ρ do not exist, and indeed this can be verified explicitly. Hence we can cancel the first row of any solution μ provided (4.5) can be satisfied.

If $\mu_{11} \neq 0 \pmod p$ we know already that we can cancel the first row by means of a solution of type I. Suppose thus that $\mu_{11} = 0 \pmod p$. Then the first term in (4.5) is irrelevant, and the second term imposes the conditions

$$n_1 \neq 0 \pmod p \quad \text{and} \quad n_1 + N_1 < n, m > \neq 0 \pmod p.$$

Suppose there exists no vector $\vec{n}$ satisfying these conditions. Then for every vector $\vec{n}$ with $n_1 \neq 0 \pmod p$ one has

$$n_1(1 + \sum_l \mu_{1l} \frac{N_l}{N_l} r_{1l}) + \sum_{i=2}^k n_i (\sum_l \mu_{il} \frac{N_l}{N_l} r_{il}) = 0 \pmod p. \quad (4.6)$$

Since this condition is linear in $\vec{n}$, it must in fact hold for any vector $\vec{n}$, also for those with $n_1 = 0 \pmod p$. Now consider the $1j$ -component of equation (2.8), satisfied by μ . From this equation we derive

$$\mu_{1j} \frac{N_j}{N_j} + \mu_{j1} (1 + \sum_l \mu_{1l} \frac{N_l}{N_l} r_{1l}) + \sum_{i=2}^k \mu_{ji} (\sum_l \mu_{il} \frac{N_l}{N_l} r_{il}) = 0 \pmod{N_1}. \quad (4.7)$$

Since (4.6) holds for every vector $\vec{n}$ it holds in particular for $n_i = \mu_{ji}$. Hence from (4.7) we get

$$\mu_{1j} \frac{N_j}{N_j} = 0 \pmod p.$$

This, however, contradicts (4.6) if one chooses the vector $\vec{n} = (1, 0, \dots, 0)$. Therefore a vector $\vec{n}$ must exist for any solution μ with $\mu_{11} = 0 \pmod p$.

Thus we have shown that any solution μ can be reduced to a solution that is trivial in Z_{N_1} , by means of either a solution of type I, or by means of a solution of type (4.4). One can repeat this procedure in an obvious way to cancel the second row and column, etc., until one is left with a solution that is non-trivial only in Z_{N_k} .

4.2 Construction of all solutions

By applying the inverse of the reduction procedure, the complete set of solutions can now be constructed systematically. Given all solutions within the last $k - q$ factors $Z_{N_{q+1}} \times \dots \times Z_{N_k}$ of the centers one can construct all solutions in the last $k - q + 1$ factors $Z_{N_q} \times \dots \times Z_{N_k}$ by means of the following algorithm

a. Construct all $N_{q+1} \times \dots \times N_k$ currents $\{\vec{m}\}$ in the last $k - q$ factors (including the identity). These vectors are candidates for the components $\mu_q, l > q$ of the new solution we are trying to construct.

b. For each current $[\vec{m}]$, solve the following equation for m_q

$$m_q \left[1 + \sum_{l=q+1}^k r_q m_l \right] + \frac{1}{2} r_{qq} m_q^2 + \frac{1}{2} \sum_{j=q+1}^k m_j \frac{N_q}{N_j} m_j = 0 \pmod{N_q}. \quad (4.8)$$

Here we are using equation (2.9) to determine $m_q \equiv \mu_{qq}$. If there is no solution, one has to generate the next vector $\vec{m}$. If there is more than one solution, they should all be considered in the following.

c. If $m_q = 0 \pmod p$, find a vector $\vec{n} = (n_q, n_l), l = q + 1, \dots, k$ so that $n_q \neq 0 \pmod p$ and $n_q + N_q < m, n > \neq 0 \pmod p$. Here $<, >$ denotes the same inner product as before, restricted to the last $k - q + 1$ factors of the center. This vector plays the rôle of the auxiliary vector $\vec{n}$ of the previous sub-section. If there are several such vectors $\vec{n}$, choose one of them. If there are no such vectors (which in practice happens very rarely) the vector $\vec{m}$ cannot appear as the q^{th} row of a solution, and one should go back to point b. and generate the next solution for m_q .

d. If $m_q \neq 0 \pmod p$ define

$$\sigma_{ij}(\vec{m}, 0) \equiv I(m_q, N_q) \frac{N_q}{N_i} m_i m_j. \quad (4.9)$$

Otherwise, define

$$\sigma_{ij}(\vec{m}, \vec{n}) \equiv \frac{N_q}{N_i} I(\frac{1}{2} m_q N_q < n, n > -n_q(n_q + N_q < m, n >), N_q) \quad (4.10)$$

$$\times [-m_q n_i n_j + n_q n_j m_i - (n_q + N_q < n, m >) n_i m_j + m_i m_j N_q \frac{1}{2} < n, n >] .$$

This matrix σ is the inverse of ρ' (eqn. (4.4)), restricted to the last $k - q + 1$ factors of the center. Note that we set the second argument of σ equal to 0 to indicate the first case. (This is not intended as a suggestion that this matrix can be viewed as a formal limit of $\sigma(\vec{m}, \vec{n})$.)

e. The complete set of solutions in the last $k - q + 1$ factors of the centers is obtained by computing all products $\mu_{k-q} * \sigma$, with μ_{k-q} equal to each of the solutions in the last $k - 1$ factors, and σ equal to each of the matrices obtained above.

All solutions for the complete center are then obtained by applying this procedure iteratively, starting with the full set of solutions for $\mathbf{Z}_{N_k}$. The latter are obtained by solving the equation $\mu_{kk} + \frac{1}{2} \gamma_{kk} \mu_{kk}^2 = 0 \bmod N_k$.

A possibly somewhat subtle point in this procedure is the choice of $\vec{n}$. Suppose for a given solution μ_{k-q} and a given vector $\vec{m}$ one can choose two different vectors $\vec{n}_1$ and $\vec{n}_2$, and corresponding matrices $\sigma(\vec{m}, \vec{n}_1)$ and $\sigma(\vec{m}, \vec{n}_2)$. Obviously, this yields two *different* solutions $\mu_1 = \mu_{k-q} * \sigma(\vec{m}, \vec{n}_1)$ and $\mu_2 = \mu_{k-q} * \sigma(\vec{m}, \vec{n}_2)$. Nevertheless, nothing is lost if we choose only one of the two vectors $\vec{n}_1$ and $\vec{n}_2$, because the second solution can be written as

$$\mu_2 = [\mu_{k-q} * \sigma(\vec{m}, \vec{n}_2) * \sigma^{-1}(\vec{m}, \vec{n}_1)] * \sigma(\vec{m}, \vec{n}_1) .$$

Since $\sigma^{-1}(\vec{m}, \vec{n})$ cancels the q^{th} row $\vec{m}$ for any valid choice of $\vec{n}$, it follows that the expression within the square brackets is a different solution μ'_{k-q} that lives entirely within the last $k - q$ factors. Since in point e. above we consider all such matrices, the solution μ_2 is already included. Thus although σ depends on $\vec{n}$, the set of solutions that is finally obtained does not. Hence we can fix the ambiguity by choosing any convenient rule* that uniquely assigns

* For example: if $m_q \neq 0 \bmod p$ choose $\sigma(\vec{m}, 0)$. If $m_q = 0 \bmod p$, use $\sigma(\vec{m}, \vec{n})$, with $\vec{n}$ chosen as follows. We may always set $n_q = 1$. If $\sum_i m_i(N_q/N_i)r_{iq} \neq -1$ choose all other components of $\vec{n}$ equal to zero. If $\sum_i m_i(N_q/N_i)r_{iq} = -1$ choose $n_q = 1$, and $n_i = 1$, where n_i is the smallest i for which $\sum_j m_j(N_q/N_j)r_{ji} \neq 0 \bmod p$. If no such i exists, then $\vec{n}$ and σ do not exist, which implies the $\vec{m}$ cannot appear as the q^{th} row of a solution.

a matrix σ (either of the form (4.9) or (4.10)) to any $\vec{m}$ for which such a matrix exists. We may then omit the second argument of σ in the following, since it is completely determined by the first argument.

This procedure yields all solutions, since in the previous subsection we have shown that any solution can be reduced by means of the inverse procedure. Furthermore, all solutions that are obtained are different. Suppose $\mu_{k-q} * \sigma(\vec{m}_1) = \mu'_{k-q} * \sigma(\vec{m}_2)$. Since the q^{th} row of the result is equal to respectively $\vec{m}_1$ and $\vec{m}_2$, it follows that $\vec{m}_1 = \vec{m}_2$. But then one can use $\sigma^{-1}(\vec{m}_1)$ to show that $\mu_{k-q} = \mu'_{k-q}$.

The precise number of solutions depends non-trivially on the orders $N_1 \dots N_k$ of the basic currents and on the matrix R . An obvious brute force method for generating all of them is to check (2.8) and (2.9) for all possible matrices μ . The number of possibilities is finite but large, namely $(N_1 N_2 \dots N_k)^k$. The method described here requires a substantially smaller set of numbers to be generated, namely $N_5 = (N_k)^{k-1} N_{k-1}^{k-2} \dots N_2$ (this is the number of vectors $\vec{m}$ generated in point (a)). Since the conditions to be satisfied in point (b) and (c) are not very transparent, it is hard to appreciate whether this is still in a sense a "brute force" method, i.e. whether the actual number of solutions is much smaller than N_5 . In practice one finds that the number of solutions is of the same order of magnitude as N_5 (condition (b) typically yields, with similar frequency, either 0, 1 or 2 solutions, while condition (c) can almost always be satisfied).

In the special case $R = 0$ the number of solutions is equal to N_5 , since condition (b) has always just one solution, $m_q = 0 \bmod N_q$, and for $\vec{n}$ one can always choose $n_q = 1, n_j = 0, j \neq q$. (This solution for $R = 0$ agrees with the one presented in [4].) Hence to obtain all solutions it is not only sufficient, but also necessary to generate all N_5 integers parametrizing the vectors $\vec{m}$.

For example, consider a center $(\mathbf{Z}_3)^k$. If $r = \text{diag}(0, \dots, 0)$ the number of solutions is equal to $N_5 = 3^{k(k-1)/2}$. If we choose $r = \text{diag}(2, \dots, 2)$ we get 2, 8, 48, 1152, 103680, 26127360, 18341406720, ... solutions for $k = 1, 2, \dots, 7, \dots$ respectively. This number is roughly equal to $2N_5$. This example shows that for a conformal field theory whose center has several factors the number of fusion rule automorphisms is huge. If such a theory is obtained by tensoring many identical copies of some other theory the number of genuinely distinct modular invariants is reduced, because many of them are related to each other by the interchange of identical factors.

One may simplify the computations somewhat by choosing a basis of currents in which R is diagonal, which is possible in many (though not all) cases, as pointed out in section 2. The main advantage of a diagonal form is that there always exists a solution for the auxiliary vector $\vec{n}$ (for $\mu_{qq} = 0 \bmod p$), namely $n_q = 1, n_i = 0, i \neq q$. For example, consider a center $\mathbb{Z}_3 \times \mathbb{Z}_3$ and monodromy matrix

$$r = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

The complete set of solutions μ is

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that there is no solution with $\mu_{12} = 2$, even though the vector $\vec{m} = (0, 2)$ satisfies (4.8). The reason is of course that no auxiliary vector $\vec{n}$ exists for this $\vec{m}$. One can solve the same problem by choosing new basis currents J_1 and J_2 , which generate the same set of currents. In this new basis,

$$r = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix},$$

and the complete set of solutions is

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now all vectors $\vec{m}$ satisfying (4.8) do appear in the first row. One may check that the solutions for the second choice of basis are obtained (in the same order) from the set of solutions for the first choice upon changing the basis.

5. Concluding Remarks

We have developed a method to enumerate all fusion rule automorphisms on simple current orbits. Our results are valid for any unitary conformal field theory with simple currents, under the assumption that the matrix elements of S are non-zero (except for certain matrix elements involving fixed points that are required to vanish by (2.4)). This technical assumption is a sufficient but not a strictly necessary condition, and the results still hold if certain elements of S are zero. In fact we do not know any conformal field theory where it is violated severely enough to invalidate the conclusion.

Fixed points do not affect the result, except that in some cases modular invariant partition functions that are in principle different become identical. For example theories with a single simple current J of order 2 and half-integer spin have two modular invariants, the diagonal one and an automorphism invariant in which all Ramond fields a are paired with Ja . In the special case of the Ising model these two invariants are identical, since the only Ramond field is a fixed point of the current.

We would like to extend this work to cover *all* simple current invariants, including those that involve an extension of the chiral algebra. In other words, we would like to omit condition 4. from the ones listed in the introduction. It is not difficult to show that M_{00} can only be zero or one. The possible extensions of the chiral algebra are simply all possible subgroups of the center consisting entirely of integer spin currents. Furthermore, it is easy to show that the left and right extensions must involve the same number of currents. The main remaining difficulty is to determine which left algebras can be combined with which right algebras.

Given certain left- and right algebras, the fields that appear are precisely those that have vanishing charges with respect to the currents in these algebras, i.e. the representations of the chiral algebras. To specify the modular invariant partition function one has to choose a pairing of the left- and right representations. If the left- and right algebras are the same this problem is precisely the one solved in the present paper (if the two algebras are different, the discussion will probably be very similar).

If the center is just a single $\mathbb{Z}_N$ factor, each subgroup is uniquely determined by its size, so that the left and right algebras must be the same. Hence in this case the problem is now completely solved. The complete set of modular invariants is in one-to-one correspondence

with the subgroups of the center, and each distinct invariant is given by a matrix $M(J)$, with J equal to the generator of a subgroup. This was shown to be the complete result for $SU(N)$ level 1 in [7], and argued to be the complete result for any conformal field theory with such a center in [1]. Note that by “center” we mean here the effective center introduced in section 2. For example, the effective center of $SU(2)$ at odd levels is trivial.

Our present results for automorphisms suggest that the characterization of the complete result for more complicated centers is far less straightforward. Nevertheless, it should be possible to find a procedure that systematically enumerates all solutions, including extensions of the chiral algebra. We hope to present such a procedure in the near future.

ACKNOWLEDGEMENTS

One of us (B. G.-R.) acknowledges the “Fundación Banco Exterior” (Spain) for financial support. The algebraic program FORM by Jos Vermaseren has been used to check some of the algebra.

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