CERN-TH.5837/90

MULTIDIMENSIONAL EXACTLY SOLVABLE PROBLEMS IN QUANTUM MECHANICS
AND PULLBACKS OF AFFINE CO-ORDINATES ON THE GRASSMANNIAN

A. Losev

Institute for Theoretical and Experimental Physics
117259 Moscow, USSR

and

A. Turbiner ^{*)}

CERN - Geneva

ABSTRACT

Multidimensional exactly solvable problems related to compact hidden symmetry groups are discussed. Natural co-ordinates on homogeneous space are introduced. It is shown that a potential and scalar curvature of the problem considered have quite a simple form of quadratic polynomials in these co-ordinates. A mysterious relation between the potential and the curvature observed for $SU(2)$ in Refs. [2] and [3] is obtained in a simple way.

^{*)} Permanent address: Institute for Theoretical and Experimental Physics, B. Cheremushkinskaya ul. 25, 117259 Moscow, USSR.

1. - INTRODUCTION

Recently, quasi-exactly-solvable spectral problems for which a finite subset of eigenstates can be found algebraically were discovered [1]. They are related to a new realization of hidden symmetry based on the correspondence of a spectral problem for certain elements of the universal enveloping algebra of the hidden symmetry group and some Schrödinger equation. Consequent studies have led to the discovery of multidimensional quasi-exactly-solvable and exactly-solvable problems without separation of variables in general [2]. As a result, a definite construction has arisen: (1) hidden symmetry realized by compact groups gives rise to exactly-solvable spectral quantum-mechanical problems; (ii) non-compact groups can lead to quasi-exactly-solvable spectral problems^{*}, supergroups - to quasi-exactly-solvable matrix spectral problems, and affine groups - to conformal quasi-exactly-solvable matrix spectral problems, and affine groups - to conformal field theories [3,4]. Since there is a construction unifying special classes of quantum-mechanical and field-theoretical problems.

In this paper we will consider a case of multidimensional exactly-solvable problems related to a hidden symmetry and try to construct some economic representations for corresponding spectral problems. It was shown in Refs. [2,3] that these multidimensional exactly solvable problems are connected with the coset space $M = G/H$. The Lie algebra of the group G acts on the space of functions on M by first order differential operators. The bilinear form of such operators [see below (2.11)] is effectively diagonalizable in the space of functions on M . Moreover, this form was shown to be equivalent to the sum of the Laplacian in the metric (determined by this form) and the potential. In the case $SU(2)/U(1)$ and the bilinear form of rank two, this potential turned out to be a sum $3/16$ of the scalar curvature (in the above-mentioned metric) and some constant [2,3]. This result stemmed from straightforward, but quite complicated calculations. It has not given a possibility to analyze this and to understand underlying grounds.

This work was motivated by this mysterious connection between the potential and the scalar curvature. We found that in the case of an arbitrary coset and arbitrary quadratic form, both the potential and the scalar curvature can be simply expressed in terms of functions that are pullbacks of affine co-ordinates on the Grassmannian. We studied the properties of these functions and found that

^{*}) Well-known exactly-solvable quantum-mechanical problems are related to $SL(2, \mathbb{R})$ hidden symmetry [1]. They have arisen as a certain exceptional case and have a different structure from exactly-solvable problems originating from compact hidden symmetry groups.

- (a) they are constant on submanifolds of M which are isomorphic to a group manifold $N(H)/H$ where $N(H)$ is the normalizer of H in the group G ;
- (b) if H or G are semisimple and H contains the Cartan torus of G , then the only relations between these functions are polynomials of the third order.

In the case where the rank of the quadratic form [see below (2.11)] is equal to the dimension of the coset, both the potential and the scalar curvature are second-order polynomials (of the above-mentioned functions). In the case of $SU(2)/U(1)$ this leads to the mentioned connection between the curvature and the potential, but in general these polynomials are unfortunately different. This does not prove that there is no connection between the curvature and the potential due to relationships between our functions, but we cannot solve these relationships. We could not find other examples of the existence of a certain relationship between a curvature and a potential.

We hope that the functions studied and used in this article will be useful in further investigations of the multidimensional exactly solvable problems.

2. - REVIEW OF MULTIDIMENSIONAL EXACTLY SOLVABLE PROBLEMS

Let us consider the coset M of the compact Lie group G :

$$M = G/H \quad (2.1)$$

The action of the group G on M induces the action of this group on the space $C(M)$ of functions on M :

$$g: f \mapsto g(f); \quad g(f)(p) \equiv f(g(p)) \quad (2.2)$$

and $p \mapsto g(p)$ is an action of G on M . Here p is a point on M and f is a function on M .

Given a group action, one can form the action of the Lie algebra of this group by considering elements that are close to identity. Thus, the action of $\hat{G}$ (the Lie algebra of group G) on $C(M)$ is given by a first-order differential operator; for $v \in \hat{G}$

$$v(f) \equiv \lim_{t \rightarrow 0} \frac{f(e^{tv}(P)) - f(P)}{t} = \partial_{\vec{v}} f \quad (2.3)$$

where

$$\partial_{\vec{v}} \equiv v^{\mu} \partial_{\mu} \quad (2.4)$$

is an operator for taking derivatives along $\vec{v}$; the vector $\vec{v}$ corresponding to the element v of the Lie algebra is given by

$$\vec{v} = \frac{d}{dt} (e^{tv}(P)) \Big|_{t=0} \quad (2.5)$$

Thus, if $v_A, A = 1, \dots, n$ are generators of G , then the operators $\partial_{\vec{v}_A}$ [acting on elements of the vector space $C(M)$] form a representation of $\hat{G}$:

$$[\partial_{\vec{v}_A}, \partial_{\vec{v}_B}] = c_{AB}^C \partial_{\vec{v}_C} \quad (2.6)$$

where c_{AB}^C are structure constants of $\hat{G}$.

If $\hat{G}$ is a compact Lie algebra, there is a G -invariant measure on M ; that is to say,

$$\int_M \Omega_0 \bar{f}_1 \partial_{\vec{v}_A} f_2 + \int_M \Omega_0 \bar{f}_2 \partial_{\vec{v}_A} f_1 = 0 \quad (2.7)$$

for all $f_1, f_2 \in C(M)$ and $v_A \in \hat{G}$.

Introducing the following scalar product on $C(M)$,

$$(f_1, f_2)_0 = \int_M \Omega_0 \bar{f}_1 \cdot f_2 \quad (2.8)$$

we form the Hilbert space H_0 . Equation (2.7) then means that $\partial_{\vec{v}_A}$ are anti-Hermitian operators in H_0 .

Due to the compactness of G , one can show that the representation H_0 of G can be decomposed into a direct sum of irreducible finite-dimensional representations

H_i :

$$H = \bigoplus_i H_i \quad (2.9)$$

and the operators $\partial_{\vec{v}_A}$ act in H_i :

$$\partial_{\vec{v}_A} H_i \subset H_i \quad (2.10)$$

The construction in [1,2] starts with the observation that the Hermitian operator in H_0

$$\hat{h} = c^{AB} \partial_{\vec{v}_A} \partial_{\vec{v}_B} \quad (2.11)$$

acts in the finite dimensional space H_i and, thus, $\hat{h}$ is effectively diagonalizable.

The main observation in [2,3] (described below) is that the operator $\hat{h}$ is isomorphic to the operator

$$\Delta_g + V \quad (2.12a)$$

where Δ_g is the Laplacian on the space M with the metric

$$g^{\mu\nu} = c^{AB} v_A^{\mu} v_B^{\nu} \quad (2.12b)$$

The explicit formula for the potential V will be derived below [see (2.19)].

To obtain (2.12), remember that canonically the Laplacian is defined by the following relationship:

$$\int_M \Omega_g g^{\mu\nu} \partial_{\mu} \bar{\varphi} \partial_{\nu} \varphi = - \int_M \Omega_g \bar{\varphi} \Delta_g \varphi \quad (2.13)$$

where Ω_g is a measure on M constructed with the help of the metric $g^{\mu\nu}$. Representing the left-hand side of (2.13) in the form

$$\int_M \Omega_0 \left(\frac{\Omega_g}{\Omega_0} \right) C^{AB} \partial_{\vec{V}_A} \partial_{\vec{V}_B} \psi \quad (2.14)$$

and using (2.7), we get

$$\Delta_g = C^{AB} \partial_{\vec{V}_A} \partial_{\vec{V}_B} + C^{AB} \partial_{\vec{V}_A} \ln \left(\frac{\Omega_g}{\Omega_0} \right) \partial_{\vec{V}_B} \quad (2.15)$$

Note that the Laplacian Δ_g is a Hermitian operator in the Hilbert space H_g , where H_g is a space $C(M)$ of functions (on M) equipped with the scalar product

$$(f_1, f_2)_g = \int \Omega_g f_1 \cdot f_2 \quad (2.16)$$

There is a natural isomorphism ψ between Hilbert spaces H and H_g that acts to functions on M through multiplication by $(\Omega_0/\Omega_g)^{1/2}$ (in Ref. [2] ψ was called the "quasi-gauge transformation" implying "gauge transformation with imaginary phase):

$$\begin{aligned} \psi : H_0 &\rightarrow H_g \\ f &\mapsto f \cdot \left(\frac{\Omega_0}{\Omega_g} \right)^{1/2} \end{aligned} \quad (2.17)$$

From representation (2.15) for the Laplacian, it follows that

$$\hat{h} = \psi^{-1} (\Delta_g + V) \psi \quad (2.18)$$

where the potential V is given by

$$\begin{aligned} V = \frac{1}{2} C^{AB} \left\{ \partial_{\vec{V}_A} \partial_{\vec{V}_B} \ln \left(\frac{\Omega_g}{\Omega_0} \right) + \right. \\ \left. \frac{1}{2} \partial_{\vec{V}_A} \ln \left(\frac{\Omega_g}{\Omega_0} \right) \partial_{\vec{V}_B} \ln \left(\frac{\Omega_g}{\Omega_0} \right) \right\} \end{aligned} \quad (2.19)$$

3. - PROPERTIES OF THE PULLBACKS OF AFFINE CO-ORDINATES ON THE GRASSMANNIAN

The functions presented in the introduction arise when we consider the map from the coset space $M = G/H$ to the Grassmannian $Gr(n, m)$ of m -dimensional hyperplanes in n -dimensional space; here and below $n = \dim G$ and $m = \dim H$.

One can treat the space M as a space of left conjugated classes with class C_P corresponding to the point P .

Choosing the representatives g_P in C_P , one obtains the following form for C_P :

$$C_P = g_P H \quad (3.1)$$

The group G acts on M treated as a set of classes via left multiplication:

$$C_{g(P)} = g C_P = g g_P H \quad (3.2)$$

It is evident that the little subgroup of point P (a group consisting of elements of G that leave point P fixed) is the subgroup H_P conjugated to H :

$$H_P = g_P H g_P^{-1} \quad (3.3)$$

Let us denote by $\hat{H}_P$ the subalgebra corresponding to the little group H_P . It is obvious that

$$\hat{H}_P = g_P \hat{H} g_P^{-1} \quad (3.4)$$

In other words, vector fields corresponding to elements of H are equal to zero at P .

Let us introduce a map φ of the coset M in the Grassmannian $Gr(n, m)$. The image of a point P is a subalgebra $\hat{H}_P$ considered as an m -dimensional hyperplane in the n -dimensional vector space G :

$$\varphi: P \rightarrow \hat{H}_P \quad (3.5)$$

It turns out that functions on M that are pullbacks of affine co-ordinates on the Grassmannian are useful for applications in exactly solvable problems. Below we will call these functions "Pullbacks of Affine co-ordinates" functions (PA functions).

Explicitly, PA functions can be described as follows. Let us choose the basis in G to consist of a basis $\{\vec{v}_a\}$ in H and a set of $(n-m)$ elements $\{\vec{v}_i\}$. Using this basis, we define affine co-ordinates on the Grassmannian as follows. Considering the hyperplane $\pi \in \hat{G}$ as a vector space, we find only one basis (if any at all) of the form:

$$\vec{v}_a = \tilde{z}_a^i(\pi) \vec{v}_i \quad (3.6)$$

The functions $\tilde{z}_a^i(\pi)$ are called affine co-ordinates. Note that for a given π the basis of the form (3.6) exists if, and only if, the vector spaces π and H are transversal, i.e., their intersection contains only a zero vector. Thus, affine co-ordinates are defined on the Grassmannian everywhere except in submanifolds of co-dimension 1.

The PA functions are constructed with the help of affine co-ordinates on the Grassmannian; the PA functions $Z_a^i(P)$ are just pullbacks

$$Z_a^i(P) = \tilde{z}_a^i(\hat{H}_P) = \tilde{z}_a^i(\varphi(P)) \quad (3.7)$$

where the subalgebra $\hat{H}_P$ (3.4) is treated as an m -dimensional vector space in the n -dimensional vector space G .

In other words, PA functions can be treated as coefficients in linear constraints on vector fields $\vec{V}_A(P)$ on M , i.e., the vector $\vec{V}_a(P)$ is a linear combination of vectors $\vec{V}_i(P)$ with the coefficients given by the PA functions

$$\vec{V}_a(P) = Z_a^i(P) \vec{V}_i(P) \quad (3.8)$$

From this point of view, PA functions are defined at point P if, and only if, the set $\{\vec{V}_i(P)\}$ is a basis in the tangent space to the manifold M at this point.

Before using PA functions, we will answer two (three) questions.

Question 1a

Do PA functions form a co-ordinate system on M , i.e., is it true that for any two points on M there exists a PA function that takes different values at these points?

Question 1b

If the answer to question 1a is "No", i.e., PA functions do not form a co-ordinate system on M , then what is the structure of the submanifolds of M , on which all PA functions are constant?

In other words, questions 1a and 1b are about the structure of the pre-image of the map φ of the points on $Gr(n,m)$.

Question 2

What are the relationships between the PA functions, i.e., what system of equations defines $\varphi(M)$ in $Gr(n,m)$?

Answer to Questions 1a and 1b

The values of all PA functions at point P are equal to the corresponding value at point P' [i.e., $\varphi(P) = \varphi(P')$] if, and only if, $\hat{H}_P = \hat{H}_{P'}$, i.e.,

$$g \hat{H} g^{-1} = \hat{H} \quad (3.9)$$

for

$$g = g_{P'}^{-1} g_P \quad (3.10)$$

From (3.9), it follows that

$$g H g^{-1} = H \quad (3.11)$$

The set of elements g that satisfy (3.11) form a subgroup $N(H)$ of the group G , called the normalizer of H in G . Thus $\varphi(P) = \varphi(P')$ if, and only if, $g_{P'}^{-1} g_P \in N(H)$. If $g_P = g_{P'} h$, $h \in H$, then $g_{P'}$ and g_P are different representatives of the same class, i.e., $P = P'$. Therefore, the manifold on which all PA functions are constant is isomorphic to the group manifold $N(H)/H$ [the coset $N(H)/H$ is a group because H is a normal subgroup in $N(H)$].

Answer to Question 2

It is evident that the image $\varphi(M)$ of the coset in the Grassmannian lies in the space of m -dimensional subalgebras in $\hat{G}$. We will denote this space by $\hat{Gr}(n,m)$. Moreover, since we consider the connected coset, the image $\varphi(M)$ lies in the connec-

ted component of $\text{Gr}(n, m)$. Below we will find conditions for $\cdot$ to be an isomorphism between M and one of the connected components of $\hat{G}(n, m)$; in other words, we will find when any subalgebra that lies "near" the subalgebra N is conjugate to N .

Since the set

$$\tilde{v}_a(p) = v_a - z_a^i(p) v_i \quad (3.12)$$

is a basis in H_p , PA functions $z_a^i(p)$ satisfy the following equations:

$$[v_a - z_a^i(p) v_i, v_b - z_b^j(p) v_j] = c_{AB}^c (v_c - z_c^k(p) v_k) \quad (3.13)$$

where c_{ab}^c are structure constants in H . Let us consider an m -dimensional algebra $\hat{A}$ that lies "near" $\hat{H}_p$. We can choose the following basis in $\hat{A}$:

$$\{\tilde{v}_a - w_a^i v^i\}, \quad a = 1, 2, \dots, m \quad (3.14)$$

Then the coefficients w_a^i satisfy the equation

$$[\tilde{v}_a - w_a^i v^i, \tilde{v}_b - w_b^j v^j] = f_{ab}^c (\tilde{v}_c - w_c^k v^k) \quad (3.15)$$

for some structure constants f_{ab}^c of subalgebra $\hat{A}$.

Since in an algebra a linear subspace closed under commutation is a subalgebra, we can consider (3.15) to be a system of equations with variables w_a^i, f_{ab}^c . Solutions of this system would be in one-to-one correspondence with a subalgebra of $\hat{G}$.

Algebra $\hat{A}$ lies "near" $\hat{H}_p$, so we consider w_a^i and $f_{ab}^c = f_{ab}^c - \tilde{c}_{ab}^c$ as small parameters. When taking into account only the leading terms (due to small parameters) in Eq. (3.15), we have

$$[\tilde{v}_a, w_b] - [\tilde{v}_b, w_a] + \tilde{c}_{ab}^c w_c = -\delta f_{ab}^c \tilde{v}_c \quad (3.16)$$

where $w_a = w_a^i v^i$.

We will now look at (3.16) from the point of view of a linear algebra in the $\hat{H}_p$ -module $\hat{G}/\hat{H}_p$. In doing so let us consider a map

$$F: \hat{G} \rightarrow \hat{G}/\hat{H}_p \quad (3.17)$$

Here G and H_p are treated as vector spaces.

Let us define the following action ρ of $\hat{H}_p$ on a vector space $\hat{G}/\hat{H}_p$. If $F(x) = \tilde{x}$, then

$$\rho(v_a) \tilde{x} = F([v_a, x]) \quad (3.18)$$

So, the vector space $\hat{G}/\hat{H}_p$ becomes an $\hat{H}_p$ -module. Applying the map F to both sides of (3.16), we obtain an equation in $\hat{G}/\hat{H}_p$:

$$\rho(v_a) \tilde{w}_b - \rho(v_b) \tilde{w}_a + \tilde{c}_{ab}^c \tilde{w}_c = 0 \quad (3.19)$$

where $\tilde{w}_a = F(w_a)$.

It turns out that (3.19) has a rather nice interpretation in terms of Lie algebra cohomologies [5]. We will treat the set $\{\tilde{w}_a\}$ as a one-co-chain $\tilde{u}$ (a linear functional on $\hat{H}_p$) that takes a value in $\hat{G}/\hat{H}_p$; this one-co-chain $\tilde{u}$ acts on elements of $\hat{H}_p$ as follows: if $\tilde{v} = \tilde{v}_a^i v_a^i$, then

$$\tilde{u}(\tilde{v}) = \sum_a \tilde{v}_a^i \tilde{w}_i \quad (3.20)$$

From this point of view the left-hand side of (3.19) is just a two-co-chain $\tilde{d}\tilde{u}$ paired with the elements $\tilde{v}_a$ and $\tilde{v}_b$ of the Lie algebra $\hat{H}_p$:

$$d\tilde{u}(\tilde{v}_a, \tilde{v}_b) = \rho(\tilde{v}_a) \tilde{u}(\tilde{v}_b) - \rho(\tilde{v}_b) \tilde{u}(\tilde{v}_a) + \tilde{u}([\tilde{v}_a, \tilde{v}_b]) \quad (3.21)$$

Then (3.19) takes the form

$$d\tilde{u} = 0 \quad (3.22)$$

If $H^1(\hat{H}_p, \hat{G}/\hat{H}_p)$ (the first cohomology group of the algebra $\hat{H}_p$ with coefficients from the $\hat{H}_p$ -module of $\hat{G}/\hat{H}_p$) is equal to zero, then (3.22) has only trivial

solutions of the form

$$\tilde{u} = d\tilde{\mu} \quad (3.23)$$

where μ (an element of $\hat{G}/\hat{H}_p$) is a zero-co-chain and its differential (paired with $\tilde{u}$) is equal to

$$(d\tilde{\mu})(\tilde{v}_a) = \rho(\tilde{v}_a) \cdot \tilde{\mu} \quad (3.24)$$

In other words,

$$u_a - [\mu, v_a] \in \hat{H}_p \quad (3.25)$$

Equation (3.25) implies that

$$\hat{A} = e^\mu \hat{H}_p e^{-\mu} \quad (3.26)$$

when only accounting for the leading terms due to small parameters, or that the subalgebra $\hat{A}$ is conjugate to $\hat{H}_p$, i.e., $A = H_p$, for some point p' near p .

Putting everything together, we obtain that if $H(\hat{H}_p, \hat{G}/\hat{H}_p) = 0$, then the coset M coincides with some connected component of $Gr(n, m)$.

In other words, if this is the case, all algebraic relationships between PA co-ordinates are consequences of

$$[v_a - z_a^i v_i, v_b - z_b^j v_j] = 0 \pmod{v_c - z_c^k v_k} \quad (3.27)$$

i.e., PA functions satisfy $\binom{m(m-1)/2}{n-m}$ algebraic equations that are polynomials of the third degree:

$$c_{ab}^c z_c^k - (z_a^i (c_{ib}^k + c_{ib}^c z_c^k) + z_a^i z_b^j (c_{ij}^k + c_{ij}^c z_c^k)) - (a \leftrightarrow b) = 0 \quad (3.28)$$

In general, these equations are not algebraically independent.

We will complete our answer to question 2 by pointing out cases where $H^1(\hat{H}_p, \hat{G}/\hat{H}_p) = 0$. According to Ref. [5], if $\hat{H} = \hat{S} \hat{\otimes} \hat{U}(1)^n$ and $\hat{S}$ is semi-simple, then

$$H^1(\hat{S} \hat{\otimes} U(1)^n, M) = R^n \hat{\otimes} I_{uv} M \quad (3.29)$$

where M is an $\hat{H}$ -module, $I_{uv} M$ is a linear subspace in M on which an action of $\hat{H}$ is equal to zero, and R^n is an n -dimensional real vector space.

Thus, $H^1(\hat{H}, \hat{G}/\hat{H}) = 0$ if either $\hat{H}$ or $\hat{G}$ are semi-simple and $\hat{H}$ contains a Cartan subalgebra of $\hat{G}$.

4. - CALCULATION OF THE POTENTIAL V AND THE RIEMANN CURVATURE IN TERMS OF PA FUNCTIONS

4a Calculation scheme for an arbitrary C^{AB}

Below we will intensively use the fact that the derivative of PA functions along vector fields $\hat{V}_i$ are polynomials of PA functions [see (4.4)].

To prove this, consider the commutator $[\hat{V}_i, \hat{V}_a]$. Because the operators $\hat{V}_a$ form a representation of an algebra $\hat{G}$, it is equal to

$$[\hat{V}_i, \hat{V}_a] = (c_{ia}^j + c_{ia}^b z_b^j) \hat{V}_j \quad (4.1)$$

Here we used

$$\partial_{\hat{V}_a} = z_a^k \partial_{\hat{V}_k} \quad (4.2)$$

If we first express $\hat{V}_a$ in terms of $\hat{V}_i$, and then calculate the commutator, we will obtain

$$[\hat{V}_i, \hat{V}_a] = \left\{ (\partial_{\hat{V}_i} z_a^j) + z_a^k c_{ik}^j + c_{ik}^b z_a^k z_b^j \right\} \hat{V}_j \quad (4.3)$$

From (4.1) and (4.3), we get

$$\partial_{\hat{V}_i} z_a^j = c_{ia}^j + c_{ia}^b z_b^j - c_{ik}^j z_a^k + c_{ik}^b z_a^k z_b^j \quad (4.4)$$

The calculation of the potential V in terms of PA functions amounts to computing $\partial \hat{v}_i \ln(Q_0/Q_0)$ in terms of them. The measure Q_0 can be constructed by using the invariant metric

$$g_0^{\mu\nu} = c_0^{AB} v_A^\mu v_B^\nu \quad (4.5)$$

where c_0^{AB} is the invariant (Killing) metric on the algebra $\hat{G}$.

In the basis $\hat{v}_i$ the metric g is as follows:

$$g^{ij} = c^{ij} + c^{ia} z_a^j + c^{aj} z_a^i + c^{ab} z_a^i z_b^j \quad (4.6)$$

The ratio of measures Q/Q_0 can be expressed as the ratio of the square roots of the determinants of the metrics:

$$\frac{\Omega_g}{\Omega_0} = \left(\frac{\det(g^{ij})}{\det(g_{0ij})} \right)^{1/2} = \left(\frac{\det(g^{ij}(z))}{\det(g_{0ij}(z))} \right)^{-1/2} \quad (4.7)$$

Then, from (4.3) and (2.15) the potential V can be obtained in terms of z_a^i ; thus the potential V is constant along manifolds where PA functions are constant (see Section 3, Question 2).

4b Calculation of the potential V for rank $c^{AB} = n-m$

It was found in Ref. [2] that in this case the potential is simpler than in the general case. We will prove that the potential here is a polynomial of the second order of PA co-ordinates.

If rank $c^{AB} = n-m$, we can choose a basis $\{\hat{v}_i\}$ in $\hat{G}$ such that the matrix c^{AB} in this basis is diagonal. Let us consider a case:

$$c^{ij} = \text{diag}(\underbrace{1, 1, \dots, 1}_{n-m}, \underbrace{0, \dots, 0}_m) \quad (4.8)$$

Let us take the first $(n-m)$ vectors $\hat{v}_i$ to define PA functions. Note that in the basis $\{\hat{v}_i\}$ on M (in the tangent space to M) the metric g will be a unity matrix.

We will see that $\partial \hat{v}_i \ln(Q_0/Q_0)$ is a first-order polynomial of PA functions. This result may be obtained in two different ways. The first is straightforward, but here the simple form of V looks a little like a miracle. The second way explains the simplicity of the answer using the concept of the Lie derivative.

First way

In this derivation we will use the following method. Invariance of the measure Q_0 [see (2.7)] leads to

$$\begin{aligned} 0 &= \int_M \sqrt{g_0} \bar{f}_2 v_A^\mu \partial_\mu \bar{f}_1 + \int_M \sqrt{g_0} \bar{f}_1 v_A^\mu \partial_\mu \bar{f}_2 = \\ &= - \int_M \bar{f}_1 \bar{f}_2 \partial_\mu (v_A^\mu \sqrt{g_0}) \end{aligned} \quad (4.9)$$

Therefore

$$\partial_\mu v_A^\mu = - v_A^\mu \partial_\mu \ln(\sqrt{g_0}) \quad (4.10)$$

We will calculate

$$\partial_{\hat{v}_i} \ln \frac{\Omega_g}{\Omega_0} = v_i^\mu \partial_\mu \ln \left\{ \left(\frac{\det(g_{\lambda\nu})}{\det(g_{0\lambda\nu})} \right)^{1/2} \right\} =$$

$$= - \frac{1}{2} v_i^\mu \partial_\mu \ln |\det(g^{\lambda\nu})| + \partial_\mu v_i^\mu \quad (4.11)$$

[Here we used (4.10)]. Since

$$\det(g^{\lambda\nu}) = (\det(v_i^\mu))^2 \quad (4.12)$$

we can calculate the first term on the right-hand side of (4.11) with the help of the identity

$$\ln \det A = \text{Sp} \ln A \quad (4.13)$$

We thus obtain

$$\begin{aligned} \partial_{\vec{v}_i} \ln \frac{\Omega_g}{\Omega_o} &= -(\nu^{-1})^j_{\mu} \nu_i^{\nu} \partial_{\nu} \nu_j^{\mu} + \partial_{\nu} \nu_i^{\nu} = \\ &= -(\nu^{-1})^j_{\mu} ([\vec{v}_i, \vec{v}_j])^{\mu} \end{aligned} \quad (4.14)$$

Here, $(\nu^{-1})^j_{\mu}$ is a matrix inverse to ν^{μ}_j , and the commutator of vector fields is a vector field with co-ordinates

$$([\vec{v}_i, \vec{v}_j])^{\mu} = \nu_i^{\nu} \partial_{\nu} \nu_j^{\mu} - \nu_j^{\nu} \partial_{\nu} \nu_i^{\mu} \quad (4.15)$$

One can check that

$$[\partial_{\vec{v}_i}, \partial_{\vec{v}_j}] = \partial_{[\vec{v}_i, \vec{v}_j]} \quad (4.16)$$

Thus

$$[\vec{v}_i, \vec{v}_j]^{\mu} = (c^k_{ij} + c^g_{ij} z^k_a) \nu_k^{\mu} \quad (4.17)$$

and, finally, we get

$$\partial_{\vec{v}_i} \ln \left(\frac{\Omega_g}{\Omega_o} \right) = -c^j_{ij} - c^a_{ij} z^j_a \quad (4.18)$$

Second way

The simplicity [6] of (4.18) can be understood as follows. Let us introduce the Lie derivative [7] which generalizes $\partial_{\vec{v}}$ on tensors in a co-ordinate-invariant way. The Lie derivative $\mathcal{L}_{\vec{v}}$ acts on vectors as a commutator

$$\mathcal{L}_{\vec{v}} \vec{u} = [\vec{v}, \vec{u}] \quad (4.19)$$

Its action on tensors obeys two rules:

$$\mathcal{L}_{\vec{v}} (T_1 \otimes T_2) = \mathcal{L}_{\vec{v}} T_1 \otimes T_2 + T_1 \otimes \mathcal{L}_{\vec{v}} T_2 \quad (4.20a)$$

$$\frac{\partial}{\partial \vec{v}} (w(\vec{u})) = \mathcal{L}_{\vec{v}} (w(\vec{u})) = (\mathcal{L}_{\vec{v}} w)(\vec{u}) + w(\mathcal{L}_{\vec{v}} \vec{u}) \quad (4.20b)$$

Here T_1, T_2 are tensors and w is a one-form. Rule (4.20a) means that the Lie derivative of a tensor product obeys the Leibnitz rule, and (4.20b) means that the Lie derivative of a pairing between vectors and forms also obeys this rule.

One can check that the Lie derivative commutes with $d = dx^{\mu} \partial_{\mu}$. The concept of the Lie derivative is used in an alternative definition of a G-invariant measure; the measure Q_0 on a coset is G-invariant if

$$\mathcal{L}_{\vec{v}} (\Omega_o) = 0 \quad (4.21)$$

In order to check the equivalence between (4.21) and (4.10), we will calculate

$$\begin{aligned} \mathcal{L}_{\vec{v}} (\Omega_o) &= \nu_A^{\mu} \partial_{\mu} (\sqrt{g_o}) dx^1 \wedge \dots \wedge dx^{n-m} + \\ &+ \sum_{\nu=1}^{n-m} \sqrt{g_o} dx^1 \wedge \dots \wedge dx^{\nu-1} \wedge d(\nu^{\lambda}_{\nu} x^{\nu}) \wedge \dots \wedge dx^{n-m} \end{aligned} \quad (4.22)$$

The second term in (4.22) is equal to $\Sigma_{\lambda} (\partial_{\lambda} \nu^{\lambda}_{\nu}) Q_0$, so (4.21) is really equal to (4.10).

Now we can proceed to the calculations:

$$\partial_{\vec{v}_i} \ln \left(\frac{\Omega_g}{\Omega_o} \right) = \mathcal{L}_{\vec{v}_i} \ln \left(\frac{\Omega_g}{\Omega_o} \right)$$

From the G-invariance of Q_0 it follows that

$$\mathcal{L}_{\vec{v}_i} \ln \left(\frac{\Omega_g}{\Omega_o} \right) = \frac{\mathcal{L}_{\vec{v}_i} (\Omega_g)}{\Omega_g} \quad (4.23)$$

The measure Ω_g can be written in the following form:

$$\Omega_g = \omega_1 \wedge \dots \wedge \omega_{n-m} \quad (4.24)$$

where ω_i is the basis of one-forms, dual to the basis $\vec{v}_i$:

$$\omega_i(\vec{v}_j) = \delta_{ij} \quad (4.25)$$

Since

$$\Omega_g(\vec{v}_1 \wedge \dots \wedge \vec{v}_{n-m}) = 1,$$

$$\mathcal{L}_{\vec{v}_i}(\Omega_g)(\vec{v}_1 \wedge \dots \wedge \vec{v}_{n-m}) + \Omega_g(\mathcal{L}_{\vec{v}_i}(\vec{v}_1 \wedge \dots \wedge \vec{v}_{n-m})) = 0 \quad (4.26)$$

From (4.26) we have

$$\begin{aligned} \frac{\mathcal{L}_{\vec{v}_i}(\Omega_g)}{\Omega_g} &= - \frac{\mathcal{L}_{\vec{v}_i}(\vec{v}_1 \wedge \dots \wedge \vec{v}_{n-m})}{\vec{v}_1 \wedge \dots \wedge \vec{v}_{n-m}} = \\ &= - \sum_j \frac{(\vec{v}_1 \wedge \dots \wedge \mathcal{L}_{\vec{v}_i} \vec{v}_j \wedge \dots \wedge \vec{v}_{n-m})}{(\vec{v}_1 \wedge \dots \wedge \vec{v}_{n-m})} = -c_{ij}^j - c_{ij}^a z_a^j \end{aligned} \quad (4.27)$$

Finally, we obtain that the potential is a second-order polynomial of PA functions:

$$\begin{aligned} V &= \frac{1}{2} \left[c_{ij}^a (c_{ia}^j + c_{ib}^b z_b^j - c_{ik}^k z_k^j + c_{ik}^b z_a^k z_b^j) + \right. \\ &\quad \left. + \frac{1}{2} (c_{ij}^j + c_{ij}^a z_a^j)^2 \right]. \end{aligned} \quad (4.28)$$

4c Calculation of the Riemann curvature

We will calculate the curvature using the basis $\vec{v}_i$ on M . In this basis, the metric g is represented by an identity matrix.

To determine the Riemann curvature, we will find a connection that preserves the metric g and has a torsion equal to zero. Then the Riemann curvature is the curvature of such a connection. If a connection preserves the metric g , then its action on the vector fields $\vec{v}_i$ is an orthogonal rotation. In other words, the action of the covariant derivative (built using such a connection) on the vectors $\vec{v}_i$ is equal to

$$\mathcal{D}_\mu \vec{v}_i = A_{\mu,ij} \vec{v}_j \quad (4.29)$$

where the matrix $A_{\mu,ij}$ is antisymmetric on the indices i and j .

It is useful to work with covariant derivatives along the vectors $\vec{v}_i$ rather than along the co-ordinates [as in (4.29)]. So, we define

$$A_{i,jk} = v_i^\mu A_{\mu,jk} \quad (4.30)$$

and the covariant derivative

$$\mathcal{D}_i = v_i^\mu \mathcal{D}_\mu \quad (4.31)$$

We can find $A_{i,jk}$ from the condition of zero torsion [7]

$$\mathcal{D}_i \vec{v}_j - \mathcal{D}_j \vec{v}_i - [\vec{v}_i, \vec{v}_j] = 0 \quad (4.32)$$

that is, from

$$A_{i,jk} - A_{j,ik} - \tilde{c}_{ij}^k = 0 \quad (4.33)$$

where

$$\tilde{c}_{ij}^k = (c_{ij}^k + c_{ij}^a z_a^k) \quad (4.34)$$

are structure constants in a vector field algebra on homogeneous space $[\vec{v}_i, \vec{v}_j] = \tilde{c}_{ij}^p \vec{v}_p$. Using the antisymmetry of $\tilde{c}_{ij}^k$ in the first two indices, we can solve (4.33) (according to Ref. [7]) and get

*) Both C_{ij}^a and $\tilde{c}_{ij}^a$ are structure constants of a Lie algebra $\hat{G}$; index a belongs to a subalgebra $\hat{H}$ and index p belongs to $\hat{G}/\hat{H}$. Since the bases " p " and " a " are non-orthonormal with respect to the Killing metric, then, generally speaking, $C_{ab}^c \neq -C_{ac}^b$. Hence $C_{ij}^j \neq 0$.

$$A_{i,jk} = \frac{1}{2} (\tilde{c}_{i,j}^k - \tilde{c}_{i,k}^j - \tilde{c}_{j,k}^i) \quad (4.35)$$

Then the Riemann curvature can be calculated from the representation

$$-R(\vec{v}_i, \vec{v}_j) = [D_i, D_j] - ([\vec{v}_i, \vec{v}_j]^k D_k) \quad (4.36)$$

This formula may be rewritten in another representation:

$$\begin{aligned} R_{ij,kl} &= -\left\{ [D_i, D_j]_{kl} - (\tilde{c}_{ij}^p D_p)_{kl} \right\} = \\ &= -\left\{ (\partial_i A_{j,kl} - \partial_j A_{i,kl}) - \right. \\ &\quad \left. - (A_{i,kl} A_{j,pl} - A_{j,kl} A_{i,pl}) - \right. \\ &\quad \left. - (\tilde{c}_{ij}^p A_{p,kl}) \right\} \end{aligned} \quad (4.37)$$

Thus, after tedious calculations, we get for the scalar curvature the following expression:

$$\begin{aligned} R &= 2 \left(c_{jk}^a c_{ji}^b z_a^i z_b^k - c_{kj}^a c_{ka}^b z_b^j + \right. \\ &\quad \left. + c_{jk}^a c_{je}^k z_a^e - c_{kj}^a c_{ka}^j \right) + \end{aligned}$$

$$\begin{aligned} &+ (\tilde{c}_{kj}^j \tilde{c}_{ki}^i - \frac{1}{2} \tilde{c}_{ki}^j \tilde{c}_{kj}^i - \frac{1}{4} \tilde{c}_{ij}^k \tilde{c}_{ij}^k) + \\ &+ \left(\frac{1}{2} \tilde{c}_{ik}^l \tilde{c}_{ik}^l + \tilde{c}_{ij}^k \tilde{c}_{ik}^j \right) \end{aligned} \quad (4.38)$$

There is a correspondence between the terms in brackets of Eqs. (4.37) and (4.38). In turn, the expression for the potential V (4.28) can be written as

$$\begin{aligned} V &= \left(\frac{1}{4} c_{ij}^j c_{ik}^k - \frac{1}{2} c_{ij}^a c_{ia}^j \right) + \\ &+ \frac{1}{2} \left[c_{ip}^p c_{ij}^a z_a^j + c_{ij}^a (c_{ik}^j z_a^k - c_{ia}^b z_b^j) \right] + \\ &+ \left(\frac{1}{4} c_{ij}^a c_{ik}^b z_a^j z_b^k + \frac{1}{2} c_{ij}^a c_{ik}^b z_a^k z_b^j \right) \end{aligned} \quad (4.39)$$

It is worth noting that Eqs. (4.38) and (4.39) have a quite simple form, convenient for real calculations. Besides that, comparing these two expressions, we can see a certain similarity: both of them are polynomials of the second degree in terms of the variables "z". Let us try to search for some situation where they can be reduced each other.

5. - THE SU(2) CASE

As an instructive example, let us study the case of the SU(2) group. There is one subgroup U(1) and correspondingly $M = \text{SU}(2)/\text{U}(1)$. We can choose the basis $\{\vec{V}_i\}$, $i = 1, 2$ in such a way that orthogonality takes place in the Killing metric. Furthermore, in this case the generator $\vec{V}_a$ [belonging to U(1)] can be made

orthogonal in $\hat{G}$ to a plane formed by $\hat{V}_1$ and $\hat{V}_2^*$. This choice of basis means that the structure constants will be vanishing in the same way as for the antisymmetric symbol ϵ_{ABC} . The non-vanishing elements of $\hat{C}_{AB}$ have the form

$$C_{12}^0 = -C_{21}^0, \quad C_{01}^2 = -C_{10}^2, \quad C_{02}^1 = -C_{20}^1 \quad (5.1)$$

[the index "0" belongs to $U(1)$]. Besides that, denoting $\hat{z}_0^k \equiv \hat{z}^k$, we can calculate the structure constants $\tilde{C}$ and their bilinears explicitly:

$$\tilde{C}_{ij}^k = C_{ij}^0 \hat{z}^k; \quad \tilde{C}_{ij}^k \tilde{C}_{ij}^k = 2 (C_{12}^0)^2 \{(\hat{z}^1)^2 + (\hat{z}^2)^2\};$$

$$\tilde{C}_{ij}^k \tilde{C}_{ik}^j = (C_{12}^0)^2 \{(\hat{z}^1)^2 + (\hat{z}^2)^2\}; \quad \tilde{C}_{ij}^j \tilde{C}_{ik}^k = (C_{12}^0)^2 \{(\hat{z}^1)^2 + (\hat{z}^2)^2\} \quad (5.2)$$

Substituting (5.2) for (4.38) and (4.39), and taking into account (5.1), we can obtain explicit formulas for the potential and scalar curvature:

$$V = -\frac{1}{2} C_{12}^0 (C_{10}^2 + C_{20}^1) + \frac{3}{4} (C_{12}^0)^2 \{(\hat{z}^1)^2 + (\hat{z}^2)^2\} \quad (5.3)$$

$$R = -2 C_{12}^0 (C_{10}^2 + C_{20}^1) + 4 (C_{12}^0)^2 \{(\hat{z}^1)^2 + (\hat{z}^2)^2\} \quad (5.4)$$

Comparing (5.3) and (5.4), we arrive at

$$V = \frac{3}{16} R - \frac{1}{8} C_{12}^0 (C_{10}^2 + C_{20}^1) \quad (5.5)$$

which reproduces a mysterious relation in Refs. [2,3].

*) Since G/H is isomorphic to $G/G^{-1}Hg$, then by doing conjugations $g^{-1} \hat{z}_a g$ one generator can be put in orthogonal direction with respect to $\hat{V}_1, \hat{V}_2$.

6. - CONCLUSION

We investigated exactly-solvable problems related to the compact hidden symmetry group. It was shown that expressions for the potential and the scalar curvature have a simple form in PA co-ordinates (as certain quadratic polynomials in these co-ordinates), which appeared as natural co-ordinates in a homogeneous space.

An obvious problem arises when searching for other homogeneous spaces for which a certain relation between the potential and the scalar curvature will emerge.

ACKNOWLEDGEMENTS

We would like to thank A. Rosly for enlightening to us the possible way for getting an answer to question 1, and V. Fock for drawing our attention to the concept of the Lie derivative.

REFERENCES

- [1] A.V. Turbiner, Comm. Math. Phys. 118 (1988) 467.
- [2] M.A. Shifman and A.V. Turbiner, Comm. Math. Phys. 126 (1989) 347.
- [3] A.Yu. Morozov, A.M. Perelomov, A.A. Rosly, M.A. Shifman and A.V. Turbiner, Int. J. Mod. Phys. A5 (1990) 803.
- [4] A.Yu. Morozov, M.A. Shifman and A.V. Turbiner, Int. J. Mod. Phys. A5 (1990) 2953.
- [5] D.B. Fuks, "Cohomology of Infinite-Dimensional Lie Algebras", Nauka, 1984.
- [6] S. Kobayashi and K. Nomizu, "Foundations of Differential Geometry", Interscience Publishers (New York/London), 1963.
- [7] B.A. Dubrovinn, S.P. Novikov and A.T. Pomenko, "Modern Geometry", Nauka, 1979.