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THEORY OF SOLAR OSCILLATIONS

J. Christensen-Dalsgaard and G. Berthomieu

Abstract

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The present chapter provides theoretical background for understanding the properties of solar oscillations and their relation to the structure of the Sun. Although accurate numerical computations are required to match the precision of the observed frequencies, a great deal of insight can be obtained from considering simple asymptotic approximations. In particular, the wide range of modes observed allows a separation of the effects of different parts of the Sun; as discussed in detail in Chapter 14, this allows *inversion* of the observed frequencies to determine localized properties of the solar interior. Also, changes in the assumed physics of solar models, within the range of uncertainty in the physics, lead to frequency changes that are substantial compared with the intrinsic precision of the observations. Thus as probes of the solar interior the frequencies are both sensitive and straightforward to interpret.

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CHAPTER 11:

Theory of solar oscillations

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Abstract. The last decade has witnessed a revolution in our knowledge about the interior of the Sun, through the development of helioseismology. By analyzing observed frequencies of solar oscillation it has been possible to obtain detailed information about many aspects of the properties of the Sun, and of the physics of matter in the solar interior. Major efforts are currently under way to improve the observations, and hence helioseismic investigations will probably remain a very active field of research for the foreseeable future.

The present chapter provides theoretical background for understanding the properties of solar oscillations and their relation to the structure of the Sun. Although accurate numerical computations are required to match the precision of the observed frequencies, a great deal of insight can be obtained from considering simple asymptotic approximations. In particular, the wide range of modes observed allows a separation of the effects of different parts of the Sun: as discussed in detail in Chapter 14, this allows *inversion* of the observed frequencies to determine localized properties of the solar interior. Also, changes in the assumed physics of solar models, within the range of uncertainty in the physics, lead to frequency changes that are substantial compared with the intrinsic precision of the observations. Thus as probes of the solar interior the frequencies are both sensitive and straightforward to interpret.

1. Introduction.

The observed frequencies of solar oscillation have opened a window on the solar interior. To use the information provided by the observations, however, it is necessary to consider the relationship between the behaviour of the oscillations and the underlying properties of the Sun. This requires the ability to compute sufficiently accurate oscillation frequencies, and if possible other aspects of the oscillations, for a given solar model; however, equally important is a well-developed understanding of how different modes of oscillation respond to particular features of the solar interior.

In principle all properties of the oscillations provide information about the structure and dynamics of the solar interior. In practice, however, only those aspects of the physics of the oscillations that determine their frequencies are sufficiently well understood to enable us to get reliable information about the interior of the Sun. The present chapter explores the properties of the modes of oscillation, and the relation between their frequencies and the structure of the solar interior. As a simplification, we assume that the oscillations are adiabatic, and hence neglect the processes that excite or damp the modes; these processes, which also have a significant effect on the oscillation frequencies, are treated in Chapter 16. Furthermore, we consider only the dependence of the frequencies on the *structure* of the solar interior, and neglect effects of rotation, as well as possible effects of a large-scale magnetic field. These effects form the subject of Chapters 12 and 13.

To provide a background for, and motivate, the development of the theory, it is useful to summarize the basic properties of the oscillations of the Sun. We assume that the oscillations can be described with sufficient accuracy in terms of linear perturbation theory. As discussed in Chapter 10, and justified in section 2.2 of the present chapter, for a single mode the dependence on colatitude θ and longitude ϕ is then given by a *spherical harmonic* $Y_l^m(\theta, \phi)$. Here the *degree* l determines the overall complexity of the perturbation over spherical surfaces. More precisely, the horizontal wavelength λ_h and the length k_h of the horizontal component of the wavenumber at a distance r from the centre of the Sun are given by

$$k_h = \frac{2\pi}{\lambda_h} = \frac{\sqrt{l(l+1)}}{r} \quad (1.1)$$

(*cf.* section 2.2). The *azimuthal order* m determines the number of nodes around the equator. For a spherically symmetric star, the orientation of the coordinate system used to describe the oscillation can have no effect on the properties of the modes, including their frequencies. Since the definition of m is related to the location of the equator, it follows that frequencies of a spherical star must be independent of m .

For each l and m there is a spectrum of modes, distinguished by the dependence of the perturbation on r , and characterized by the *radial order* n , which measures the number of zeros in the radial direction in, say, the radial component of the displacement associated with the perturbation.

Since we neglect the excitation and damping of the modes, as a function of time t they behave as purely harmonic functions $\cos \omega t$, where ω is the *angular frequency* of oscillation. For a theo-

retical description, the frequency is most simply characterized by ω ; on the other hand, observed frequencies are normally given in terms of the *cyclic frequency*

$$\nu = \frac{\omega}{2\pi} = \frac{1}{P}, \quad (1.2)$$

where P is the oscillation period.

To illustrate the basic properties of the spectrum of possible oscillations, Figure 1 shows computed frequencies for a typical solar model. It is immediately obvious that there are two distinct classes of modes, which are conventionally distinguished by the sign of n . Those with $n > 0$, which are normally called *p modes*, have frequencies that for large degree increase with l roughly as $l^{1/2}$, whereas for modes with $n < 0$, normally called *g modes*, the frequencies tend to a limit as l increases. The modes with $n = 0$ form an intermediate case, despite the similarity of the behaviour of their frequencies to that of the p modes; these modes are called *f modes*. The description of the physical nature of these modes forms an important part of this chapter.

2. Equations and boundary conditions of solar oscillations.

2.1 General properties of the motion.

Although the main purpose of this section is to present the equations and boundary conditions governing adiabatic stellar oscillations of small amplitude, we begin by considering briefly the more general equations of fluid motion. We neglect viscosity, and assume that gravity is the only body force; thus, effects of, *e.g.*, magnetic fields are neglected. Then the equations of motion can be written

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \rho \nabla \Phi \quad (2.1)$$

where $\mathbf{v}$ is the velocity, p is pressure, ρ is density, and Φ is the gravitational potential. As usual D/Dt denotes the material time derivative,

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f, \quad (2.2)$$

for any quantity f . The density satisfies the equation of continuity,

$$\frac{D\rho}{Dt} = -\rho \operatorname{div} \mathbf{v}, \quad (2.3)$$

and Φ is determined by Poisson's equation,

$$\nabla^2 \Phi = -4\pi G \rho, \quad (2.4)$$

where G is the gravitational constant.

The relation between p and ρ is determined by the energy equation, which may be written

$$\frac{1}{p} \frac{Dp}{Dt} - \Gamma_1 \frac{1}{\rho} \frac{D\rho}{Dt} = \frac{\Gamma_3 - 1}{p} (\rho \epsilon - \text{div} \mathbf{F}). \quad (2.5)$$

Here Γ_1 and Γ_3 are adiabatic indices, defined by

$$\Gamma_1 = \left(\frac{\partial \ln p}{\partial \ln \rho} \right)_s, \quad \Gamma_3 - 1 = \left(\frac{\partial \ln T}{\partial \ln \rho} \right)_s, \quad (2.6)$$

where T is temperature, and the derivatives are at constant specific entropy s . Also ϵ is the rate of energy generation ($\epsilon \cdot g$, from nuclear reactions) per unit mass, and $\mathbf{F}$ is the energy flux. Equation (2.5) is valid for radiative transfer in the equilibrium diffusion approximation, provided the thermodynamic quantities, including Γ_1 and Γ_3 , are based on an equation of state where the contribution from radiation, in the diffusion approximation, is included consistently in the pressure and internal energy (Mihalas 1984). This is an excellent approximation in the solar interior, but fails in the solar atmosphere; in this case the contributions from radiative pressure and energy density must be treated separately. However, radiation pressure makes a very small contribution to the dynamics of the motion throughout the Sun.

Equations (2.1) – (2.5) provide a description of the general behaviour of the fluid. However, in the case of solar oscillations the situation is vastly complicated by the instability of the outer layers to convection, which gives rise to convective motion. Thus, for example, the velocity field v in equation (2.1) contains a convective component, in addition to the oscillations which we wish to describe. To eliminate the convective fluctuations one has to carry out some suitable averaging procedure (e.g. Gough 1977a; Uno *et al.* 1979). This introduces second moments of the fluctuations in the equations describing the average motion. The most significant of these is the convective flux, which in most of the convection zone dominates the energy transport in the average energy equation corresponding to equation (2.5); however, the average momentum equation should also contain a term in the turbulent Reynolds stress tensor, often approximated by a turbulent pressure $p_t \sim \rho u_t^2$, where u_t is a typical convective velocity. Very near the solar surface, where the convective velocities are comparable with the sound speed, p_t may make a significant contribution to the pressure balance.

No satisfactory procedure exists for dealing with the convection in a static star, let alone in the presence of pulsations. As discussed in more detail in Chapter 16, this introduces severe uncertainty in the calculation of the stability of solar oscillations. Furthermore, the turbulent Reynolds stress is likely to have a significant effect on the oscillation frequencies. In this chapter, apart from the inclusion of the convective flux in the calculation of the equilibrium model, we shall generally ignore effects of convection; this introduces errors in the properties of the oscillations, in particular the frequencies, which must be kept in mind when computed frequencies are compared with the observations.

In most of the Sun the right hand side of equation (2.5) is negligible compared with the magnitude of the terms on the left hand side. If we approximate $\text{div} \mathbf{F}$ by F/d , where d is a typical

length scale, and ignore ϵ , we obtain to order of magnitude

$$\frac{\Gamma_3 - 1}{p} (\rho \epsilon - \text{div} \mathbf{F}) \sim \tau_{th}^{-1} = \frac{F}{pd} \approx \frac{1}{p} \frac{\text{dyn/cm}^2 R}{d} \text{ sec}^{-1}, \quad (2.7)$$

where R is the solar radius, if the solar surface flux and radius are assumed. When averaged over the entire Sun, and taking $d \approx R$, the thermal time scale τ_{th} is approximately the Kelvin-Helmholtz time of the Sun, which is of order 10^7 years. In the solar envelope we may take $d \sim H_p$, where

$$H_p = \frac{p}{g\rho} \quad (2.8)$$

is the pressure scale height, g being the gravitational acceleration. Very near the solar surface, τ_{th} is comparable with the typical oscillation periods, and the heating term in the energy equation has a substantial effect on the motion. However, due to the rapid increase of pressure with increasing depth (p increases by about 4 orders of magnitude in the outer one per cent of the solar radius), τ_{th} increases rapidly, and in most of the Sun the heating term is very small compared with the rate of change of p and ρ . Thus it is reasonable to treat the motion as being *adiabatic*, i.e. occurring without heat loss or gain, so that

$$\frac{1}{p} \frac{Dp}{Dt} = \Gamma_1 \frac{1}{\rho} \frac{D\rho}{Dt}. \quad (2.9)$$

The adiabatic approximation is evidently highly restrictive. In particular, the motion is conservative, and hence the total energy is constant. Thus, when considering a single mode of oscillation in the adiabatic approximation, the amplitude of the mode is constant, and therefore it is not possible to treat its excitation or damping. Also, close to the surface, and in the atmosphere, the thermal time scale is comparable with, or smaller than, the periods, and here there are large departures from adiabaticity. These effects are discussed in Chapter 16; they are crucial for the calculation of the stability of the modes, as well as for the interpretation of the phase and amplitude relations observed in the solar atmosphere. The nonadiabatic effects also modify the dynamical properties of the motion and hence affect the oscillation frequencies. However, by far the largest fraction of the mass involved in the oscillations is in regions where τ_{th} is large and the dynamics is barely affected by nonadiabaticity. Thus, although the nonadiabatic effects on the frequencies are not negligible, they are generally small.

The adiabatic approximation very considerably simplifies the discussion of the properties of the oscillations. Thus in most of this chapter we shall assume that the oscillations are adiabatic, keeping in mind that we thereby introduce errors in the computed frequencies.

2.2 Linearized equations.

The observed amplitudes of solar oscillations are very small. Thus in principle the modes can be treated in *linear perturbation theory*. We consider a static, spherically symmetric equilibrium model, with pressure $p_0(r)$, density $\rho_0(r)$, etc. These quantities satisfy the usual equations of stellar

structure. Thus the gravitational acceleration is $\nabla\Phi_0 = -g_0\mathbf{a}_r$, where $\mathbf{a}_r$ is a unit vector in the radial direction, and

$$g_0 = \frac{Gm_0}{r^2}, \quad (2.10)$$

here r is the distance to the centre, and m_0 is the mass of the sphere interior to r , which satisfies

$$\frac{dm_0}{dr} = 4\pi r^2 \rho_0. \quad (2.11)$$

The equation of hydrostatic support becomes

$$\frac{dp_0}{dr} = -g_0\rho_0. \quad (2.12)$$

The time-dependent quantities are written as, $e.g.$, $p(r, t) = p_0(r) + p'(r, t)$, where " $'$ " denotes the *Eulerian perturbation*, i.e. the perturbation at a given point. Also we introduce the displacement $\delta\mathbf{r}$ by $\mathbf{v} = \partial\delta\mathbf{r}/\partial t$. Dropping subscripts "0" on equilibrium quantities we obtain the following linearized equations:

$$\text{Momentum : } \rho \frac{\partial^2 \delta\mathbf{r}}{\partial t^2} = -\nabla p' + \frac{\rho'}{\rho} \nabla p + \rho \nabla \Phi' \quad (2.13)$$

$$\text{Continuity : } \rho' + \text{div}(\rho \delta\mathbf{r}) = 0 \quad (2.14)$$

$$\text{Adiabaticity : } \frac{\rho'}{\rho} + \frac{1}{\rho} \delta\mathbf{r} \cdot \nabla \rho = \frac{1}{\Gamma_1} \left(\frac{p'}{p} + \frac{1}{p} \delta\mathbf{r} \cdot \nabla p \right) \quad (2.15)$$

$$\text{Poisson's equation : } \nabla^2 \Phi' = -4\pi G \rho' \quad (2.16)$$

Because of the spherical symmetry and time independence of the equilibrium model, we can separate the perturbation quantities in spherical polar coordinates (r, θ, ϕ) and time. The equations can be combined in such a way that all derivatives with respect to θ and ϕ appear in the form of the tangential Laplace operator

$$\nabla_t^2 \equiv \frac{1}{r^2} \sin \theta \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}. \quad (2.17)$$

Consequently the variation of a scalar quantity with θ and ϕ can be written in terms of an eigenfunction of $r^2 \nabla_t^2$. This may be chosen as a spherical harmonic Y_l^m . Similarly the time dependence can be expressed in terms of a harmonic function. Thus, *e.g.*, the pressure perturbation is written as

$$p'(r, \theta, \phi, t) = \sqrt{4\pi} \text{Re} [p'(r) Y_l^m(\theta, \phi) e^{-i\omega t}], \quad (2.18)$$

where for simplicity we use p' to denote both the full perturbation and the amplitude function. Y_l^m is defined by

$$Y_l^m(\theta, \phi) = (-1)^m c_{lm} P_l^m(\cos \theta) e^{im\phi}, \quad (2.19)$$

where P_l^m is a Legendre function, and the normalization constant c_{lm} is determined by

$$c_{lm}^2 = \frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}, \quad (2.20)$$

such that the integral of $|Y_l^m|^2$ over the unit sphere is unity. It follows from the momentum equation (2.13) that the displacement vector can be written as

$$\delta\mathbf{r}(\tau, \theta, \phi, t) = \sqrt{4\pi} \text{Re} \left\{ \left[\xi_r(\tau) Y_l^m \mathbf{a}_r + \xi_h(\tau) \left(\frac{\partial Y_l^m}{\partial \theta} \mathbf{a}_\theta + \frac{1}{\sin \theta} \frac{\partial Y_l^m}{\partial \phi} \mathbf{a}_\phi \right) \right] e^{-i\omega t} \right\} \quad (2.21)$$

where $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$ are unit vectors in the θ and ϕ directions. It may be noted that with this separation of variables the effect of ∇_t^2 on any scalar variable corresponds to multiplication by $-k_h^2$, where

$$k_h = \frac{\sqrt{l(l+1)}}{r} \equiv \frac{L}{r}. \quad (2.22)$$

This justifies the identification, already made in equation (1.1), of k_h with the local horizontal wave number of the mode. The root mean squares, over a spherical surface at radius r and time, of the vertical and horizontal components of the displacement, are given by

$$< \delta r >_{rms} = \frac{1}{\sqrt{2}} \xi_r(\tau), \quad < \delta h >_{rms} = \frac{L}{\sqrt{2}} \xi_h(\tau). \quad (2.23)$$

The separation given in equations (2.18) and (2.21) describes the so-called *spheroidal modes*, which is the only class of modes considered in the following. It should be mentioned, however, that there exists a second class of modes, the *toroidal modes* (*e.g.* Aizenman & Smeyers 1977). In a non-rotating star these have purely tangential displacement, zero Eulerian perturbations of scalar quantities and zero frequency. Physically they correspond to infinitely slow rotation of spherical shells relative to each other. In a rotating star the toroidal modes give rise to oscillations whose frequencies are of order the rotation frequency (*e.g.* Papaloizou & Pringle 1978; Provost, Berthomieu & Rocca 1981).

By substituting the spherical harmonic representations into equations (2.13) – (2.16) the following relations between the amplitude functions are obtained:

$$-\omega^2 \rho \xi_r = -\frac{dp'}{dr} - \rho' g + \rho \frac{d\Phi'}{dr} \quad (2.24)$$

$$-\omega^2 \rho \xi_h = -\frac{1}{r} (p' - \rho \Phi') \quad (2.25)$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi'}{dr} \right) - \frac{L^2}{r^2} \Phi' = -4\pi G \rho' \quad (2.26)$$

$$\rho' = -\frac{1}{r^2} \frac{d}{dr} (r^2 \rho \xi_r) + \frac{L^2}{r} \rho \xi_h \quad (2.27)$$

$$p' = \rho \left[\Gamma_1 \frac{\rho'}{\rho} - \left(\frac{d \ln p}{dr} - \Gamma_1 \frac{d \ln \rho}{dr} \right) \xi_r \right]. \quad (2.28)$$

Equations (2.24) – (2.28) can be rearranged into the following fourth-order set of ordinary differential equations for the amplitude functions $(\xi_r(\tau), \xi_h(\tau), \Phi'(\tau))$:

$$\frac{d\xi_r}{d\tau} = -\left(\frac{2}{r} + \frac{1}{\Gamma_1} \frac{d \ln p}{dr} \right) \xi_r + \frac{r\omega^2}{c^2} \left(\frac{S_l^2}{\omega^2} - 1 \right) \xi_h - \frac{1}{c^2} \Phi' \quad (2.29)$$

$$\frac{d\xi_h}{dr} = \frac{1}{r} \left(1 - \frac{N^2}{\omega^2} \right) \xi_r + \left(\frac{1}{r} \frac{d \ln p}{dr} - \frac{d \ln \rho}{dr} - \frac{1}{r} \right) \xi_h + \frac{N^2}{r g \omega^2} \Phi' \quad (2.30)$$

$$\frac{d^2 \Phi'}{dr^2} = -\frac{2}{r} \frac{d \Phi'}{dr} - 4\pi G \rho \left(\frac{N^2}{g} \xi_r + \frac{r \omega^2}{c^2} \xi_h \right) + \left(\frac{L^2}{r^2} - \frac{4\pi G \rho}{c^2} \right) \Phi' \quad (2.31)$$

Here c is the adiabatic sound speed,

$$c^2 = \frac{\Gamma_1 p}{\rho} \approx \frac{\Gamma_1 k_B T}{\mu m_u}, \quad (2.32)$$

where the last approximation is valid for an ideal gas; k_B is Boltzmann's constant, m_u is the atomic mass unit and μ is the mean molecular weight. Also we have introduced the Lamb frequency

$$S_l = \frac{Lc}{r} \quad (2.33)$$

and the buoyancy frequency N , defined by

$$N^2 = g \left(\frac{1}{\Gamma_1} \frac{d \ln p}{dr} - \frac{d \ln \rho}{dr} \right). \quad (2.34)$$

For radial oscillations, with $l = 0$, it follows immediately from equations (2.26) and (2.27) that

$$\frac{d \Phi'}{dr} = 4\pi G \rho \xi_r, \quad (2.35)$$

if we require $d \Phi'/dr$ to be finite at the centre. If the differential equations are written in terms of ξ_r and p' , it is then possible to eliminate the terms in Φ' , to obtain a second-order system of differential equations. With a little further manipulation, this reduces to

$$\frac{1}{r^3} \frac{d}{dr} \left(r^4 \Gamma_1 p \frac{d \zeta}{dr} \right) + \frac{d}{dr} [(3 \Gamma_1 - 4) p] \zeta + \rho \omega^2 r \zeta = 0, \quad (2.36)$$

where $\zeta = \xi_r/r$. This equation is very convenient for discussing the properties of radial oscillations (e.g. Ledoux & Walraven 1958; Cox 1980). In the solar case, however, little is gained by distinguishing between the radial and the general nonradial case.

2.3 Boundary conditions.

The amplitude functions must satisfy boundary conditions at the centre and surface of the Sun. The centre is a regular singular point of equations (2.29) – (2.31). The behaviour of the solution can be analyzed by expansion in r around $r = 0$, using the expansion of the equilibrium quantities. For the regular solution, $\xi_r \sim r^{l-1}$ (except for $l = 0$, where $\xi_r \sim r$), and $p', \rho', \Phi' \sim r^l$. Furthermore one finds that

$$\xi_r \simeq l \xi_h, \quad \text{for } r \rightarrow 0. \quad (2.37)$$

From the expressions for the coefficients in the higher-order terms more precise relations can be established among $(\xi_r, \xi_h, \Phi', d \Phi'/dr)$ at small, but non-zero r (Christensen-Dalsgaard, Dike &

Gough 1974). When applied at the innermost meshpoint $r = r_1$ used in the computation, these relations determine two inner boundary conditions.

As discussed in the Section 3, p modes of high degree have substantial amplitudes only near the solar surface, and decrease exponentially with depth at greater depths. For such modes it may be sufficient to solve the oscillation equations in a model of the solar envelope, rather than in a complete solar model. Furthermore, as illustrated in section 4.4, the perturbation in the gravitational potential may be neglected. Thus one has to specify only a single boundary condition at the bottom of the envelope model. Given the small amplitude of the modes in the deep interior, it may be reasonable to take the displacement to be zero at this point (e.g. Ando & Osaki 1975). However, it is straightforward to derive a more accurate condition from the asymptotic behaviour of the eigenfunction, discussed in Section 4.

At the surface, Φ' and its first derivative must match continuously onto the decreasing solution to Laplace's equation outside the Sun. If the density in the solar atmosphere is neglected, this leads to the condition

$$\frac{d \Phi'}{dr} + \frac{l+1}{r} \Phi' = 0, \quad \text{at } r = R \quad (2.38)$$

(see also Christensen-Dalsgaard *et al.* 1974). The dynamical surface boundary condition should in principle take into account the detailed behaviour of the oscillation in the solar atmosphere. In the idealized case where the model is assumed to have a free outer surface, the absence of forces on the surface corresponds to demanding that the pressure perturbation vanish on the perturbed surface,

$$\delta p = p' + \frac{dp}{dr} \xi_r = 0, \quad \text{at } r = R, \quad (2.39)$$

where δp is the *Lagrangian* pressure perturbation, i.e. the perturbation following the motion. In practice the outer boundary condition is applied at a suitable point within the solar atmosphere. Since the equilibrium temperature varies slowly in the region around the temperature minimum, a reasonable condition can be obtained by matching the solution continuously to the analytically known solution to the adiabatic wave equation in an isothermal atmosphere (e.g. Uno *et al.* 1979). Also, as discussed in Section 3, at the frequencies where most of the oscillations are observed, the modes are evanescent in the atmosphere. Hence the solution is relatively insensitive to the details of the boundary condition, as long as it is applied sufficiently high in the atmosphere (see also Ulrich & Rhodes 1983; Noels, Scuflaire & Gabriel 1984).

By using equations (2.25), (2.12) and (2.10), equation (2.39) can also be written as

$$\xi_h = \frac{GM}{R^3 \omega^2} \xi_r - \frac{1}{R c^2} \Phi', \quad (2.40)$$

where M is the total mass of the Sun. As argued below, Φ' is often small. Hence equations (2.40) and (2.23) show that the ratio between the root-mean-squares of the horizontal and vertical components of the surface velocity is approximately given by

$$\frac{< V_h >_{rms}}{< V_r >_{rms}} \approx \frac{GM}{R^3 \omega^2} \frac{I_r}{I_h}, \quad (2.41)$$

and hence is a function of the frequency and the degree, but not of the detailed nature of the mode. Also it is evident that for high-frequency modes of low degree the surface velocity is predominantly in the radial direction.

2.4 The modes of oscillation.

A non-zero solution to equations (2.29) – (2.31), with the boundary conditions (2.37) – (2.39), is only possible for selected values of ω^2 , which is therefore an *eigenvalue* of the problem. For each value of l one obtains a discrete spectrum of eigenfrequencies ω_{nl} , labelled by the radial order n . It is convenient to let n take both positive and negative values, such that ω_{nl} increases with increasing n . The definition of n may be arranged in such a way that when large $|n|$ gives the number of zeros in ξ_r , both for positive and negative n , excluding (except for $l = 0$) a possible zero at $r = 0$. It can be shown that $\omega_{nl} \rightarrow 0$ for $n \rightarrow -\infty$. The behaviour at large positive n depends on the assumed surface properties of the model and the oscillations. For simplified models where the temperature tends to zero at the surface, or if the boundary condition (2.39) is used, $\omega_{nl} \rightarrow \infty$ for $n \rightarrow \infty$; as discussed in section 3.2, with more realistic models of the solar atmosphere the modes cease to be trapped at frequencies exceeding a critical frequency, and in this case at each value of l there is only a finite number of trapped modes with positive n .

The definition of n can be given a precise mathematical meaning, at least in the Cowling approximation (see below), where Φ' is neglected (Eckart 1960; Scuflaire 1974; Osaki 1975). It may be shown (e.g. Gabriel & Scuflaire 1979; Christensen-Dalsgaard 1980) that this definition is invariant under a continuous change of the equilibrium model. From a mathematical point of view it is customary to classify modes with $n < 0$, $n = 0$ and $n > 0$ as g, f and p modes, respectively. The relation of this classification to the physical nature of the modes is discussed in more detail in section 4.1.

3. Asymptotic theory of the oscillations.

Numerical solution of the equations of adiabatic oscillations is straightforward. However, the understanding of the numerical results and the interpretation of the observed oscillations have been greatly assisted by asymptotic analyses of the oscillation equations. The usefulness of asymptotics is to a large extent due to the fact that the observed acoustic modes in the five minute region have high radial order or high degree; similarly gravity modes corresponding to the oscillations that may have been detected at long periods have high radial orders. For this reason asymptotics can provide relations of acceptable accuracy between the properties of the Sun and the properties of the oscillation frequencies.

3.1 Asymptotic properties of the oscillation equations.

The asymptotic analysis is considerably simplified by noting that at high degree or radial order the perturbation Φ' in the gravitational acceleration can be neglected. This approximation, known as

the *Cowling approximation*, was first suggested by Cowling (1941). It may be justified by noting that at high degree or order a mode gives rise to many regions of alternating sign in the density perturbation, the effects of which largely cancel in the perturbation of the gravitational potential. As a very rough approximation we may take

$$\frac{d^2}{dr^2} \sim -\frac{n^2}{r^2} \quad (3.1)$$

in Poisson's equation (2.31), where $|n|$ is the number of zeros in the radial direction. This results in the following estimate of the Φ' term in equation (2.29):

$$\frac{1}{c^2} \Phi' \sim \frac{4\pi G \rho r^2}{c^2} \frac{1}{n^2 + l^2} \frac{d \ln p}{dr} \xi_r. \quad (3.2)$$

Here the factor $4\pi G \rho r^2 / c^2$ is smaller than about 3 in a normal solar model, and hence for large $|n|$ or l the term is small compared with the other term in ξ_r . In equation (2.30) a similar estimate may be made. Thus when the order or the degree is large we may be able to neglect Φ' . The effects of the Cowling approximation is discussed in more detail in section 4.5 below.

When Φ' is neglected, the oscillation equations reduce to a second-order system consisting of equations (2.29) and (2.30) without the terms in Φ' , with one boundary condition (the first of equations (2.37)) at the centre and one (e.g. equation (2.39)) at the surface. It was shown by Gough (c.f. Deubner & Gough 1984) that a particularly convenient approximate representation of these equations can be obtained by generalizing an analysis by Lamb (1932) for the plane-parallel case. We introduce the quantity

$$\Psi(r) = c^2 \rho^{1/2} \text{div } \delta r; \quad (3.3)$$

neglecting derivatives of g and of r , one finds that Ψ satisfies

$$\frac{d^2 \Psi}{dr^2} + \frac{1}{c^2} \left[\omega^2 - \omega_{co}^2 - S_l^2 \left(1 - \frac{N^2}{\omega^2} \right) \right] \Psi = 0. \quad (3.4)$$

Here ω_{co} is a generalization of Lamb's (1909) acoustical cut-off frequency; it is defined by

$$\omega_{co}^2 = \frac{c^2}{4H^2} \left(1 - 2 \frac{dH}{dr} \right), \quad (3.5)$$

where $H = -(d \ln \rho / dr)^{-1}$ is the density scale height.

In addition to the modes described by equation (3.4), the approximate analysis shows the existence of modes whose frequencies are given by

$$\omega^2 = \frac{g_s L}{R}, \quad (3.6)$$

where g_s is the surface gravity. For these modes $\text{div } \delta r$ vanishes, and ξ_r decreases with increasing depth as $\exp[-k_\lambda(R - r)]$; they are entirely equivalent to surface gravity waves at a free surface. Corrections to equation (3.6) caused by curvature and the variation in g were discussed by Gough (1984a).

From equation (3.4) it follows that the behaviour of the mode is determined by the variation of the characteristic frequencies S_l , N and ω_{co} with r . They are illustrated in Figure 2, for a normal model of the present Sun. N has a maximum, where $N/2\pi \approx 450\mu\text{Hz}$, near the centre of the Sun; in the convection zone N is imaginary, and, except very near the surface, of small absolute value. S_l decreases monotonically from the centre towards the surface. Finally ω_{co} is large predominantly near the surface. It is illustrated in more detail in Figure 2b, together with the acoustical cut-off frequency according to Lamb's (1909) original definition,

$$\omega_{co,p} = \frac{c}{2H_p}, \quad (3.7)$$

where H_p is the pressure scale height (cf. equation (2.8)); ω_{co} reduces to $\omega_{co,p}$ for an isothermal layer. The rapid variation in ω_{co} just below $r/R = 1$ is associated with the region where convection is substantially superadiabatic. In the solar atmosphere $\omega_{co}/2\pi$ and $\omega_{co,p}/2\pi$ vary relatively little, around a value of about $5300\mu\text{Hz}$.

A mode is an oscillating function of r in the regions where

$$c^2 K \equiv \omega^2 - \omega_{co}^2 - S_l^2 \left(1 - \frac{N^2}{\omega^2}\right) > 0, \quad (3.8)$$

and behaves approximately exponentially outside them. A region where equation (3.8) is satisfied is known as a *trapping region* for the mode. From the behaviour of the characteristic frequencies shown in Figure 2 follows that equation (3.8) is satisfied, approximately, where

$$\omega^2 > S_l^2 \quad \text{and} \quad \omega^2 > \omega_{co}^2 \quad (3.9a)$$

or

$$\omega^2 < N^2. \quad (3.9b)$$

In general a given mode of oscillation has large amplitude in one of the trapping regions, and decreases exponentially outside it. Thus the frequency of the mode is mainly determined by the structure of the Sun within that region.

As shown in Figure 2, there are two distinct types of trapping regions:

- p) a region corresponding to equation (3.9a), extending from a point in the interior to just below the surface.
- g) a region in the interior corresponding to equation (3.9b), limited by N .

The p type of region typically occurs at high frequencies; modes corresponding to this type of trapping are called *p modes*. The g type typically occurs at low frequencies; the corresponding modes are called *g modes*.

Physically the terms in N^2 and S_l^2 in equation (3.4) correspond to the two dominant restoring forces for the oscillations, viz buoyancy and pressure fluctuations, respectively. Modes at low frequency where buoyancy dominates, i.e. g modes, correspond to standing gravity waves, whereas

modes at high frequency where pressure fluctuations dominate, i.e. p modes, correspond to standing acoustic waves.

By applying JWBK analysis to equation (3.4) one obtains the following approximate relation for the eigenfrequencies:

$$\omega \int_{r_1}^{r_2} \left[1 - \frac{\omega_{co}^2}{\omega^2} - \frac{S_l^2}{\omega^2} \left(1 - \frac{N^2}{\omega^2} \right) \right]^{1/2} dr \approx \pi(n' + \epsilon); \quad (3.10)$$

here r_1 and r_2 are consecutive *turning points* or zeros of K . The phase constant ϵ is $-\frac{1}{2}$, if K varies approximately linearly with r in the neighbourhood of both turning points. * The actual behaviour may be more complicated; as discussed in section 3.3 this is particularly true for g modes, due to the fact that $|N|$ is very nearly zero in the convection zone. (For p modes, on the other hand, the computed frequencies can be represented with reasonable accuracy by equation (3.10) with $\epsilon = -\frac{1}{2}$, provided ω_{co} is replaced by $\omega_{co,p}$; cf. section 3.2). Finally n' is here defined such that the number of zeros in the eigenfunction Ψ in the trapping region is $n' - 1$; it may be shown that for high order modes this definition corresponds to $n' = |n|$, where n is the formal labelling of computed modes introduced in section 2.4.

3.2 p-mode asymptotics.

Here we assume that $N^2/\omega^2 \ll 1$. This is certainly true for the solar five minute oscillations, whose frequencies are typically around $3000\mu\text{Hz}$. Furthermore, since in general ω is much larger than ω_{co} near the lower turning point, this is approximately at the point $r = r_1$ where $\omega = S_l$, or

$$\frac{r_1}{c(\tau_1)} = \frac{L}{\omega}. \quad (3.11)$$

Thus the location of the lower turning point is a function of ω/L alone. Figure 3 shows how r_1 varies with L and the frequency, for a normal solar model. Low-degree modes extend almost to the centre of the Sun, whereas modes at the highest degrees observed are confined to the outer fraction of a per cent of the solar radius. The upper turning point is approximately at the point where $\omega = \omega_{co}$. Within the present approximation waves whose frequencies exceed the maximum of ω_{co} in the solar atmosphere are not reflected at the photosphere, but propagate out into the atmosphere. Therefore the concept of trapped modes becomes questionable at high frequencies. In practice the modes may be partially reflected at the transition between the chromosphere and the corona, where the temperature increases steeply. Nonadiabatic calculations show strongly enhanced damping in the atmosphere at frequencies exceeding the adiabatic cut-off frequency (e.g. Christensen-Dalsgaard & Frandsen 1983). This, possibly together with damping by nonlinear effects, may account for the decrease in the observed amplitudes at higher frequencies.

* Note that we use the same symbol as for the rate of energy generation (cf. equation (2.6)). Since the latter quantity is not used below, this should cause little confusion.

We may obtain an approximate asymptotic expression for the p-mode oscillation frequencies from equation (3.10). We neglect the term in N^2/ω^2 , and note that the term in ω_{co} is only significant near the surface. By approximating the variation of ω_{co} and c near the surface by the appropriate expressions for a polytropic model, it is possible to expand the dependence of the integral on ω_{co} , to obtain

$$\omega \int_{r_i}^R \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{1/2} \frac{dr}{c} \approx \pi(n + \alpha). \quad (3.12)$$

Here α contains ϵ , as well as a contribution resulting from the expansion of ω_{co} and c ; hence, roughly speaking, α describes the phase change at the inner turning point and at the surface. Since $S_l \ll \omega$ near the surface, the l -dependent term in equation (3.10) can be neglected there; consequently α does not depend on l . In the polytropic case α is a constant, which is related to the polytropic index; for realistic models one finds that α is a function of frequency (cf. Figure 6).

From JWKB analysis of equation (3.4), neglecting the terms in N^2 and ω_{co}^2 , one obtains the following approximate solution for ξ_r , valid in the trapping region except near the turning point r_i :

$$\xi_r(r) \approx A_p(r) \cos \left[\omega \int_r^R \left(1 - \frac{L^2 c^2}{\omega^2 r'^2}\right)^{1/2} \frac{dr'}{c} - (1/4 + \alpha)\pi \right]. \quad (3.13)$$

Here $A_p(r)$ is, apart from a normalization constant, given by

$$A_p(r) \approx \tau^{-1}(\rho c)^{-1/2}, \quad (3.14)$$

at least when r is not close to r_i . Since ρ decreases by about 9 orders of magnitude from the centre to the surface of the Sun, equation (3.14) shows that the *displacement* is strongly concentrated towards the surface. The sensitivity of the frequencies to the structure of the model is approximately determined by the *energy density* in the mode, which is proportional to $\rho r^2 \xi_r^2 \sim c^{-1} \sim T^{-1/2}$, (cf. equation (2.32)). This varies by less than a factor 100 through the model. Thus energetically the p-modes have significant amplitude throughout the trapping region. In particular, low-degree modes are sensitive to the structure of the entire Sun.

The asymptotic behaviour of p-modes can be understood very simply in terms of the propagation of sound waves (Christensen-Dalsgaard *et al.* 1985). Since the wavelength of high-order modes is small compared with the typical scale over which equilibrium structure changes, the modes can be approximated locally by plane sound waves. These have the dispersion relation

$$k^2 \equiv k_r^2 + k_h^2 = \frac{\omega^2}{c^2}, \quad (3.15)$$

where k_r and k_h are the radial and horizontal components of the wave vector. For a wave corresponding to a mode of oscillation, k_h is given by equation (2.22). From equation (3.15) we then obtain

$$k_r^2 = \frac{\omega^2}{c^2} - \frac{L^2}{r^2}. \quad (3.16)$$

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Close to the surface, c is small and hence k_r is large. Here the wave propagates almost vertically. With increasing depth, c increases and k_r decreases (see Figure 4), until the point is reached where $k_r = 0$ and the wave propagates horizontally. It corresponds to the turning point $r = r_i$ defined previously; the condition in equation (3.11) is evidently equivalent to the vanishing of k_r . This is a point of total internal reflection; below r_i , $k_r^2 < 0$, and the mode decays exponentially. The reflection of the wave at the surface is not immediately contained in this simple description; however, it is at least plausible that it results from the steep density gradient. Thus the wave propagates in a series of "bounces" between the surface and the turning point (cf. Figure 4). A mode of oscillation is formed as the interference pattern between such bouncing waves.

A more direct derivation of this behaviour, which avoids explicit separation of the spherical harmonics, can be carried out on the basis of ray theory (Gough 1984b, 1986).

This description of the p-modes also yields the asymptotic dispersion relation for their frequencies. In fact it follows from equation (3.16) that the condition in equation (3.12) is equivalent to requiring that the change in radial phase between the lower turning point and the surface be an integral multiple of π , apart from α which describes the phase change at the turning points.

As r_i is a function of ω/L , equation (3.12) may be written as

$$\frac{\pi(n + \alpha)}{\omega} = F\left(\frac{\omega}{L}\right) \quad (3.17)$$

where

$$F(w) = \int_{r_i}^R \left[1 - \left(\frac{c}{r w}\right)^2\right]^{1/2} \frac{dr}{c}. \quad (3.18)$$

A relation like equation (3.17) was first found by Duvall (1982) for observed frequencies. In fact both observed and computed frequencies satisfy the Duvall relation quite accurately. Figure 5 shows a plot in this form of observed frequencies; α was assumed to be constant and was determined in such a way that the scatter in the relation was minimized.

It is of interest to use equations (3.17) and (3.18) to estimate the effects on the frequencies of changes in the equilibrium model. If δc denotes the difference in c between two solar models, at fixed r , and $\delta \alpha$ denotes the difference in α at fixed frequency, it follows by linearization, assuming δc and $\delta \alpha$ to be small, that (Christensen-Dalsgaard, Gough & Pérez Hernández 1988)

$$S \frac{\delta \omega}{\omega} \approx \int_{r_i}^R \left(1 - \frac{L^2 c^2}{r^2 \omega^2}\right)^{-1/2} \frac{\delta c}{c} \frac{dr}{c} + \pi \frac{\delta \alpha}{\omega} \quad (3.19)$$

where

$$S = \int_{r_i}^R \left(1 - \frac{L^2 c^2}{r^2 \omega^2}\right)^{-1/2} \frac{dr}{c} - \pi \frac{d\alpha}{d\omega}. \quad (3.20)$$

Since α is assumed to be a function of ω alone, and r_i is determined by ω/L , the scaled frequency difference $S\delta\omega/\omega$ is predicted to be of the form

$$S \frac{\delta \omega}{\omega} \approx H_1\left(\frac{\omega}{L}\right) + H_2(\omega), \quad (3.21)$$

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where the two functions H_1 and H_2 are determined by equation (3.19). Furthermore, the last term on the right hand side of equation (3.20) is in general relatively small compared with the first, and hence S is approximately a function of ω/L . Equation (3.21) evidently describes a very special dependence of $\delta\omega$ on ω and L .

Since c/τ decreases quite rapidly with increasing τ , $(Lc/\tau\omega)^2 \ll 1$ except near the turning point τ_i ; hence as a rough approximation $1 - L^2c^2/\tau^2\omega^2$ may be replaced by 1 in the integrals in equations (3.19) and (3.20). If furthermore the term in $\delta\omega$ can be neglected, the result is the very simple relation between the changes in sound speed and frequency:

$$\frac{\delta\omega}{\omega} \approx \frac{\int_{\tau_i}^R \frac{\delta c}{c} d\tau}{\int_{\tau_i}^R \frac{d\tau}{c}}. \quad (3.22)$$

This shows that the change in sound speed in a region of the Sun affects the frequency with a weight determined by the time the mode, regarded as a superposition of traveling waves, spends in that region. Thus changes near the surface, where the sound speed is low, have relatively large effects on the frequencies.

As discussed in Chapter 14, the asymptotic relations (3.12) and (3.19) permit an almost direct, and surprisingly precise, inversion for the solar sound speed or the sound-speed difference between the Sun and a model.

Due to the importance of the asymptotic relations for the frequency, it is of obvious interest to test their accuracy. One way of doing this is to compute effective phases by demanding that equations (3.10) and (3.12) be exactly satisfied for computed frequencies of oscillation. Thus we define ϵ_{nl} and α_{nl} by

$$\epsilon_{nl} \equiv \frac{\omega_{nl}}{\pi} \int_{\tau_i}^R \left[1 - \frac{\omega_{co}^2}{\omega_{nl}^2} - \frac{S_l^2}{\omega_{nl}^2} \left(1 - \frac{N^2}{\omega_{nl}^2} \right) \right]^{1/2} d\tau - n \quad (3.23)$$

and

$$\alpha_{nl} \equiv \frac{\omega_{nl}}{\pi} \int_{\tau_i}^R \left(1 - \frac{L^2c^2}{\omega_{nl}^2\tau^2} \right)^{1/2} d\tau - n. \quad (3.24)$$

Here ω_{nl} is the frequency of a mode of order n and degree l . Christensen-Dalsgaard (1984a) found that the rapid variation in ω_{co} near the surface (cf. Figure 2b) causes a jump in ϵ_{nl} at the frequency corresponding to the local maximum in ω_{co} where the upper turning point τ_i jumps discontinuously. This effect is evidently due to a failure of the JWKB analysis when the coefficients in the differential equation vary rapidly. It might be noted that if ω_{co} is replaced by $\omega_{co,p}$ in equation (3.23) this effect disappears, and ϵ_{nl} is close to $-\frac{1}{2}$, as predicted by simple JWKB theory.

Possibly more interesting is the behaviour of α_{nl} , which is more directly related to inversion by means of the asymptotic expressions. It has been found that if L as given in equation (2.22) is replaced by

$$L_0 = l + \frac{1}{2} \quad (3.25)$$

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in equation (3.24), α_{nl} for radial modes, with $l = 0$, shows a behaviour that is very similar to that obtained for nonradial modes; a corresponding improvement of the asymptotic description of the low-degree modes was noted by Brodsky & Vorontsov (1987, 1988). This property, which has a close analogy in the asymptotic theory of the Schrödinger equation (Kemble 1937), is related to the asymptotic behaviour of the solution near the singular point at $\tau = 0$ (see Vandakurov 1967; Tassoul 1980). Thus in the subsequent discussion of the asymptotic behaviour of the p modes we use the definition (3.25). Computed values of α_{nl} , based on frequencies of a normal solar model, are shown in Figure 6. At low degree α_{nl} shows some dependence on l . This, however, is almost entirely caused by effects of the perturbation in the gravitational potential; in a corresponding plot based on frequencies computed in the Cowling approximation the curves at all degrees essentially coincide with the upper envelope in Figure 6. Otherwise α_{nl} is largely a function of frequency alone. Hence the computed frequencies can be represented quite accurately by an expression of the form given in equation (3.12), α being a function of frequency. – Kosovichev (1988) made a careful analysis of the departures of computed frequencies from the asymptotic expression.

The dependence of α on frequency is determined by the structure of the outermost layers of the Sun. Thus it is possible to express the difference in α between two models in terms of integrals of appropriate kernels multiplying changes in the structure of the model (Christensen-Dalsgaard & Pérez Hernández 1988). This may eventually permit a study of the outer layers of the Sun from inversion of α as determined from the observed frequencies. It should be noted, however, that the computed α is undoubtedly affected by the uncertainties in our description of the physics of the outer layers of the Sun.

When l is small, $(L_0c/\omega\tau)^2 \ll 1$ except close to the centre of the Sun. Then it is possible to expand equation (3.12), with L replaced by L_0 , to take out the dependence on L_0 , in a manner similar to the derivation of equation (3.12) from equation (3.10). To lowest order in the expansion one obtains

$$\nu_{nl} = \frac{\omega_{nl}}{2\pi} \approx \left(n + \frac{l}{2} + \frac{1}{4} + \alpha \right) \Delta\nu \quad (3.26)$$

where

$$\Delta\nu = \left[2 \int_0^R \frac{d\tau}{c} \right]^{-1} \quad (3.27)$$

is the inverse of twice the sound travel time between the centre and the surface (see also Vandakurov 1967; Tassoul 1980). Thus to this asymptotic order there is a uniform spacing $\Delta\nu$ between modes of same degree but different order. In addition equation (3.26) predicts the approximate equality

$$\nu_{nl} \approx \nu_{n-1, l+2}. \quad (3.28)$$

This frequency pattern has been observed for the solar five minute modes of low degree (cf. Chapter 10), and may be used in the search for stellar oscillations of solar type.

The *deviations* from the simple relation (3.26) have considerable diagnostic potential. By taking the expansion of equation (3.10) to the next order (Gough 1986), or from a direct JWKB

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analysis of the oscillation equations (Tassoul 1980), one obtains

$$\delta\nu_{nl} \equiv \nu_{nl} - \nu_{n-1+l} \approx -(4l+6) \frac{\Delta\nu}{4\pi^2\nu_{nl}} \int_0^R \frac{dc}{dr} \frac{dr}{r}. \quad (3.29)$$

It is often convenient to represent observed or computed frequencies in terms of a limited set of parameters associated with the asymptotic description of the modes. This may be accomplished by fitting the asymptotic expression to the frequencies. By carrying out a polynomial fit in the quantity $x - x_0$, where $x = n + l/2$ and x_0 is a suitable reference value (Scherrer *et al.* 1983; Christensen-Dalsgaard 1988a) one obtains an average over n of $\delta\nu_{nl}$ as

$$< \delta\nu_{nl} >_n \approx (4l+6)D_0, \quad (3.30)$$

where

$$D_0 \approx -\frac{1}{4\pi^2x_0} \int_0^R \frac{dc}{dr} \frac{dr}{r}. \quad (3.31)$$

Thus $\delta\nu_{nl}$ is predominantly determined by conditions in the solar core. Physically this may be understood from the fact that only near the centre is k_r comparable with k_r . Elsewhere the wave vector is almost vertical, and the dynamics of the oscillations is largely independent of their horizontal structure, i.e. of l ; therefore at given frequency the contributions of these layers to the frequency are nearly the same, and hence almost cancel in the difference in equation (3.29).

It should be noted that the accuracy of expressions (3.30) and (3.31) is questionable; they appear to agree fortuitously with frequencies computed for models of the present Sun, whereas they are less successful for models of different ages or masses (Christensen-Dalsgaard 1988a). Indeed, it is shown in section 4.5 that the neglect of the perturbation of the gravitational potential causes frequency changes that are comparable with $\delta\nu_{nl}$ and depend strongly on l . Thus it is not surprising that the asymptotic description, assuming the Cowling approximation, cannot reproduce exactly the details of the frequencies computed from the full equations. However, the general form of the dependence of $< \delta\nu_{nl} >_n$ on l shown in equation (3.30), as well as the argument that this quantity is most sensitive to conditions in stellar cores, probably have a broader range of validity.

The solar convection zone is essentially adiabatically stratified, except for a thin boundary layer near the surface (e.g. Gough & Weiss 1976). Furthermore, except in the outer 2 per cent of the radius, Γ_1 is approximately constant. Thus we can approximate the structure of the convection zone by a simple adiabatically stratified polytropic layer. In this case equation (3.4) can be solved in terms of confluent hypergeometric functions (Lamb 1932; Spiegel & Unno 1962; Gough 1977b; Christensen-Dalsgaard 1980); the eigenfrequencies are given by

$$\omega_{nl}^2 = \frac{g_* L}{R} \left(1 + \frac{n}{\mu} \right), \quad (3.32)$$

where μ is an appropriately averaged polytropic index. Note that for $n = 0$ this expression fortuitously reduces to equation (3.6) for surface gravity waves.

3.3 g-mode asymptotics.

For g modes we assume that $S_l^2/\omega^2 \gg 1$. This is marginally true for a 160 minute oscillation with $l = 1$ (cf. Figure 2), but a good approximation otherwise. The g modes have frequencies such that $\omega < N$. They are evanescent in the convective regions ($N^2 < 0$); hence their trapping regions are confined to the radiative regions, the limits being approximately given by the zeros r_1 and r_2 of $N^2/\omega^2 - 1$ (cf. equation (3.9b)). For the model of the present Sun, the buoyancy frequency has a single maximum N_{max} in the inner radiative zone and increases to very high values in the outer radiative zone (see Figure 2); this defines two trapping regions. To each one corresponds a set of gravity modes. If we neglect the trapping surface layer, the internal gravity modes are confined in the inner radiative zone, with $\omega < N < N_{max}$, and strongly decaying in the convection zone except for very low degree. In a typical normal solar model, $N_{max}/2\pi$ is around 450 μHz . From equation (2.29), where Φ' is neglected and $S_l^2/\omega^2 \gg 1$, and assuming $d/dr \sim |n|/\tau$, we have

$$\xi_r \sim \frac{L^2}{|n|} \xi_h, \quad (3.33)$$

which indicates that low degree high radial order modes have a predominantly horizontal motion while for modes with $l \gg n$ the motion is mostly in the vertical direction.

When l becomes large, low order modes are split into modes trapped near the surface and modes trapped in the radiative zone, below the local maximum of the buoyancy frequency N_{max} . The frequencies of the latter modes tend to this maximum as l^{-1} . From equation (3.10), using a parabolic approximation to the buoyancy frequency around its maximum: $N^2 = N_{max}^2[1 - \beta^2(x/x_m - 1)^2]$ and assuming that S_l^2 is much greater than ω^2 , the periods of the modes are approximately given by (Christensen-Dalsgaard 1980):

$$P = \frac{2\pi}{N_{max}} \left[1 + \frac{\beta N_{max}^2 (2|n| + 1)}{L} \right]. \quad (3.34)$$

For low degree, high order modes, the theory developed up to the second order for the nonradial stellar pulsations by Tassoul (1980), taking into account the possible movable singularities of the coefficients, can be used. The solution is approximated by Bessel functions around the transition points which are the zeros of N^2 , the centre and the surface, and it depends on the behavior of the Väisälä frequency close to these points. However, in the major part of the convection zone, $|N|/2\pi$ is of the order of 1 μHz or less and the asymptotic approximation falls because $\omega > |N|$ in a large part of the low frequency range. Thus the behavior of the oscillation in the convection zone must be described either by the analytical solutions of the asymptotic equations with $N = 0$ (Zahn 1970), or by numerically integrated solutions (Provost & Berthomieu 1986; Gabriel 1986; Ellis 1986, 1987).

The matching of the solutions, subject to the boundary conditions, gives the asymptotic relation for the periods:

$$P = \frac{P_0}{L} \left(|n| + \frac{l}{2} - \frac{1}{4} - \theta \right) + \frac{P_0^2 L^2 V_1 + V_2}{P L^2}, \quad (3.35)$$

with

$$P_0 = \frac{2\pi^2}{\int_0^{x_1} \frac{N}{x} dx}, \quad (3.36)$$

and

$$V_1 = \frac{1}{P_0} \lim_{\delta \rightarrow 0} \left(\int_{\delta}^{x_1} \frac{dx}{N} - \frac{1}{N(\delta)} \right). \quad (3.37)$$

The coefficient V_2 is given by a more intricate formula (Tassoul 1980). The phase factor θ is determined by

$$\theta = \frac{v}{2} - \frac{1}{\pi} \arctan \left(\frac{\sin v\pi}{K + \cos v\pi} \right); \quad (3.38)$$

here N^2 is assumed to vanish as $(x_1 - x)^\zeta$ at the base of the convection zone, and $v = 1/(\zeta + 2)$. For a standard solar model, N^2 behaves almost linearly below the convection zone; hence $\zeta = 1$ and $v = \frac{1}{3}$.

The value of K is obtained by matching the eigenfunctions on both sides of the lower boundary of the convection zone, assuming continuity of the displacement and of the Lagrangian pressure perturbation (or the horizontal displacement). It is related to the ratio of these two quantities at that point and depends on the derivatives of the Väisälä frequency just below the convection zone. For $v = \frac{1}{3}$

$$K = (3N_0)^{\frac{1}{3}} \frac{\Gamma(2/3)}{\Gamma(1/3)} r_c^{-1} \left[L^2 \left(\frac{\xi_0}{r_c} \right) - \frac{1}{2\Gamma_1} \frac{d \ln P}{d \ln r} - \frac{N_1}{5} \right]^{-1} (LP)^{2/3}, \quad (3.39)$$

with

$$N_0 = \lim_{r \rightarrow r_c^-} \frac{N^2}{r_c - r}, \quad N_1 = -\frac{1}{2} \frac{d}{dr} \ln \left(\frac{d N^2}{dr} \frac{r^2}{r^2} \right) \Big|_{r_c^-}.$$

N_1 is related to the curvature of N^2 . Thus θ is a function of the frequency and of the degree of the modes. In the limit of very low frequency,

$$K \sim K_0 (LP)^{2/3}$$

and

$$\theta \sim \frac{1}{6} - \frac{\sqrt{3}}{(2\pi)^{\frac{1}{3}} K_0} (LP)^{-2/3}, \quad (3.40)$$

where K_0 is independent of l and P .

For large l , θ tends to $\frac{1}{6}$ and the observational determination of this asymptotic limit would provide a constraint on the stratification below the convection zone. However, for a standard model, θ varies significantly as a function of ω and l and approaches its asymptotic behavior only for very large values of the radial order.

The asymptotic solution in terms of Bessel functions does not exist for all values of ζ (Oliver 1956). Some restrictive conditions on the coefficients of the initial equations are required for $\zeta \geq 1$. In the case of g -mode oscillations, these conditions are not satisfied for $\zeta \geq 2$, leading to divergent integrals in the second order terms. For $\zeta = 2$, i.e. $v = 1/4$ a development in Weber functions just below the convection zone leads to an expression for P which has the same form as in equation (3.35) with modified expressions for V_1 and V_2 which include a dependence on the logarithm of the period; a similar expression is also obtained for θ , giving the asymptotic limit $\theta = 1/8$ (Ellis 1987). An estimate from observed periods of the limiting behaviour of θ could therefore provide information about the behavior of N^2 just below the convection zone.

In the case $v = 1/3$, Gabriel (1986) noted that θ depends on both the slope of N^2 and the depth of the convection zone through the term between brackets in equation (3.39). This last dependence is dominant so that if the slope of N^2 is not too poorly known, it is possible to obtain an estimate of the depth of the convection zone from the determination of θ as a function of P . From theoretical periods of modes of radial order n from 30 to 100, for which the second order terms can be neglected in the asymptotic formula, Gabriel gave a fitting procedure for the variation of θ with frequency in terms of two parameters which depend on the depth of the convection zone.

The main feature which results from the asymptotic analysis is that, to first order, the periods of the modes of a given degree are equally spaced, with a period spacing that depends on the degree. Given a set of observed periods, this property can be used to identify sets of equally spaced periods, each corresponding to a given degree, and hence ideally can lead to an identification of the degree of the modes as well as a determination of P_0 (Delache & Scherrer 1983; van der Raay 1988; see also Kawaler 1988, who used this technique on white dwarf oscillations).

The way in which the periods P are approximated by the expression (3.35) is illustrated in Figure 7, where the quantities LP are plotted for $l = 1$ and 4 as functions of $(|n| + l/2 - 1/4 - \theta)$, using the asymptotic limit $\theta = 1/6$. The first and second order asymptotic approximation are given respectively by the straight line with slope P_0 and the two hyperbolas. The computed periods are very close to the asymptotic predictions from equation (3.35). A more careful analysis of computed periods in terms of the asymptotic expression, for a number of different models, is given in section 6.3.

Except in the vicinity of the turning points, an approximate expression for the radial variation of the displacement in the trapping region may be derived from the asymptotic expansion of the Bessel functions:

$$\begin{aligned}\xi_r(\tau) &\sim A_g(\tau) \cos\left(\frac{L}{\omega} \int_{\tau}^{\tau} \frac{N}{r'} dr'\right) \\ \xi_h(\tau) &\sim \frac{N}{L\omega} A_g(\tau) \sin\left(\frac{L}{\omega} \int_{\tau}^{\tau} \frac{N}{r'} dr'\right)\end{aligned}\quad (3.41)$$

with

$$A_g(\tau) \sim r^{-3/2}(\rho N)^{-1/2}. \quad (3.42)$$

In a normal solar model, the buoyancy frequency N is proportional to τ at the centre; the ratio N/r is maximum there (the dimensionless quantity $\tilde{N}/x \equiv N/(r\sqrt{R/GM}) \sim 90$) and decreases rapidly to zero at the bottom of the convection zone. As a consequence the amplitude of the g modes is very large in the core of the Sun with a small secondary maximum near the boundary of the convection zone. The energy density of the modes is proportional to $\rho^2 \xi_h^2 \sim N/r$. Equation (3.41) also shows that the radial wavelength around the point τ , which can be approximated by $\Lambda \sim 2\pi\omega r/LN$, gets smaller toward the centre so that the radial nodes are concentrated in the inner layers.

In the convection zone, the g modes are evanescent and their amplitude decreases rapidly toward the surface following the law (Christensen-Dalsgaard, Dziembowski & Gough 1980):

$$\log\left(\frac{\xi_r}{\xi_h}\right) \sim \log\left(\frac{x_c}{x_s}\right) \sqrt{L} + \frac{1}{2} \log\left(\frac{x_c^3 \rho_c}{x_s^3 \rho_s}\right) \quad (3.43)$$

This means that when their degree increases g modes are strongly attenuated by the effect of the convection zone.

4. Properties of adiabatic oscillations.

The dependence of the oscillation frequencies on the equilibrium structure is determined by the coefficients in equations (2.29) – (2.31). They can be calculated from $p(r)$, $\rho(r)$, $g(r)$ and $\Gamma_1(r)$. However, assuming that the Sun is in hydrostatic equilibrium, these quantities are related by equations (2.10) – (2.12). Also, the mass $m(r)$ must satisfy the boundary condition $m(0) = 0$. Thus, for example, if $\rho(r)$ is known, $m(r)$ may be calculated from equation (2.11); given $m(r)$, $g(r)$ is determined by equation (2.10), and finally $p(r)$ can be found by integrating equation (2.12), with the boundary condition $p(R) = 0$. Hence if we know $\rho(r)$ we can obtain $m(r)$, $g(r)$ and $p(r)$ by integration. Thus all coefficients in equation (2.29) – (2.31), and hence ω , are determined completely by $\rho(r)$ and $\Gamma_1(r)$. Conversely, if the oscillations can be adequately represented as adiabatic modes of a spherical star in hydrostatic equilibrium, observations of their frequencies *only* give information about $\rho(r)$ and $\Gamma_1(r)$, or equivalently about any other pair of independent quantities from which the coefficients in equations (2.29) – (2.31) can be derived. In fact, it follows from the asymptotic properties of p modes that their frequencies predominantly provide information about the adiabatic sound speed $c(r)$. As discussed in more detail in Chapter 14, a consequence of this dependence of the frequencies on solar structure is that inversion for the structure can be organized such as

to determine a pair amongst the variables $(\rho(r), \Gamma_1(r), p(r), c(r))$, or any other variable, such as the buoyancy frequency N , which may be derived from them. – To get information about other properties of the Sun, such as the temperature or the composition, one must use the equations of stellar structure.

The equations and boundary conditions do not contain the azimuthal order m . Thus the solution, in particular the eigenfrequencies, are independent of m . This is a consequence of the assumption of spherical symmetry. Physically it follows from the fact that in this case there is no preferred axis in the Sun; since m depends on the choice of coordinate axis, the physics of the oscillations, and hence their frequencies, must be independent of m . For slow rotation a modal description as in equations (2.18) and (2.21) is still possible, provided that the rotation axis is chosen as coordinate axis. As discussed in Chapter 12, the dependence of ω on m can then be found from a perturbation analysis: rotation introduces a splitting in ω of order $m\bar{\Omega}$, where $\bar{\Omega}$ is an average of the rotation frequency.

4.1 Oscillation frequencies.

It is instructive to consider various properties of oscillations computed for a normal solar model. These can to a large extent be interpreted by means of the asymptotic results presented in Section 3.

Computed frequencies for a model of the present Sun were shown in Figure 1. Here the modes were labelled according to the mathematical classification discussed in section 2.4. It is clear, however, that the physical nature of the modes, and the behaviour of the frequencies as functions of l , in general correspond to the asymptotic properties discussed in Section 3. The p -mode frequencies increase roughly as $l^{1/2}$ for high l , in accordance with the polytropic approximation in equation (3.32). The g -mode frequencies tend towards a constant limit at large l , given by the maximum in the solar interior of the buoyancy frequency N . Finally the f modes are intermediate in frequency between the p and g modes, and, for degrees exceeding 16, behave approximately as the surface gravity wave described by equation (3.6).

At degrees below about 16 the f -mode frequency apparently intersects the g -mode frequencies. To understand this behaviour, it is convenient to follow the frequencies as continuous functions of the degree, which from a mathematical point of view may take non-integral values; this shows that the interaction between the modes takes place in a sequence of *avoided crossings* (Christensen-Dalsgaard 1980) where the frequencies approach very closely without actually crossing. This creates a discrepancy between the mathematical classification, which is unchanged as one follows an eigenvalue through the avoided crossing, and the physical nature. Thus for $l < 16$ the modes labelled as f modes in the figure behave physically as the lowest-order trapped g mode, as described by equation (3.34); on the other hand with decreasing l a sequence of “mathematical” g modes of increasing order take on the physical nature of the surface gravity wave, with most of their energy concentrated near the surface. Similar avoided crossings, and consequent conflicts between mathematical classification and physical nature, are found among modes of evolving massive stars (e.g. Osaki 1975; Shibahashi & Osaki 1976; Aizenman, Smeyers & Weigert 1977; Roth & Weigert 1979; Shiba-

hashi 1979). The study of analogous phenomena has a long history in quantum mechanics; in fact the analysis of von Neuman & Wigner (1929) has been very useful in understanding the properties of the avoided crossings found among stellar eigenfrequencies (Gabriel 1980; Christensen-Dalsgaard 1981).

4.2 Oscillation eigenfunctions.

Figure 8 shows eigenfunctions for a selection of p modes with approximately the same frequency, but varying degree. To show the distribution of kinetic energy in the modes, $r\sqrt{\rho}\xi_r$ is plotted. As also found from the asymptotic analysis the low-degree modes extend to the solar core, whereas modes with higher l are increasingly confined close to the solar surface. Notice also that except near and below the turning points, the eigenfunctions are very similar. Physically this may be understood from the fact that only near r_1 is the term in L in equation (3.13) significant. Elsewhere the wave vector is almost vertical, and the dynamics of the oscillations is largely independent of their horizontal structure, i.e. of l ; the same is therefore true for the eigenfunctions. The location of the asymptotic turning point, shown in Figure 3, is indicated with arrows in the figure; it is clearly in good agreement with the actual behaviour of the eigenfunctions.

To illustrate the reflection of the modes at the surface, Figure 9 shows the eigenfunctions in the outer few per cent of the solar radius of modes of degree 1 with different frequencies. From the behaviour of ω_∞ (cf. Figure 2b) follows that for $\nu \gtrsim 2500 \mu\text{Hz}$, the modes are propagating essentially to the photosphere; hence energetically, the amplitude of the mode is as large at the photosphere as in the interior. However, even at $2286 \mu\text{Hz}$ the effect of ω_∞ already causes a decrease in the photospheric amplitude relative to its interior value, and at $1604 \mu\text{Hz}$ there is a substantial evanescent region between the interior and the photosphere, causing the photospheric amplitude to be considerably smaller than the amplitude in the interior. Notice that except when ν is close to the aconustical cut-off frequency in the atmosphere, the pulsation energy decreases rapidly above the photosphere.

Figure 10 shows g modes with different degrees and periods of approximately 160 minute; the energy of such modes is clearly large near the centre. Thus observation and identification of g modes would provide data with high sensitivity to conditions in the solar core.

The envelopes of the eigenfunctions shown in Figure 8 and 10 closely follow the asymptotic equations (3.14) and (3.42) in the trapping regions. Thus for p modes $r\sqrt{\rho}\xi_r$, and for g modes $r^{3/2}\sqrt{\rho}N\xi_r$, are very close to being harmonic functions with a nonuniform phase.

An important global property of a mode of oscillation, besides the frequency, is its integrated

kinetic energy or, equivalently, its inertia. Here we consider the dimensionless inertia*

$$E_{nl} = \frac{4\pi \int_0^R [\xi_r^2 + l(l+1)\xi_\theta^2] \rho r^2 dr}{M[\xi_r^2(R_{phot}) + (l(l+1)\xi_\theta^2(R_{phot}))]}, \quad (4.1)$$

where R_{phot} is the photospheric radius. This is related to the modal mass M_{nl} (cf. Goldreich & Keeley 1977) by $M_{nl} = E_{nl}M$, and defined such that the time-averaged kinetic energy in the oscillation is

$$\frac{1}{2} M_{nl} V_{rms}^2 = \frac{1}{2} E_{nl} M V_{rms}^2, \quad (4.2)$$

where V_{rms}^2 is the mean, over the solar surface and time, of the squared total photospheric velocity of the mode. Figure 11 shows E_{nl} for selected p modes in the present Sun. The dominant feature is the strong decrease with increasing frequency up to about $3000 \mu\text{Hz}$. This is caused by the behaviour of the eigenfunctions shown in Figure 9, as determined by the variation of ω_∞ . At low frequencies the interior amplitude is substantially larger than the surface value, and hence E_{nl} is large. At high frequencies, on the other hand, the mode is oscillatory almost to the surface, and the surface and interior amplitudes are comparable. It is also noticeable that E_{nl} decreases substantially with increasing l at fixed frequency; the increase in l causes the mode to be confined closer to the surface, and hence to involve a smaller fraction of the mass of the Sun.

The normalized inertia of g modes in the Sun is illustrated in Figure 12. It shows a much more dramatic variation than the inertia for p modes, spanning about 30 orders of magnitude for $l \leq 25$. The reason for this is the trapping of the modes at high degrees and frequencies near the maximum in the buoyancy frequency N . It is particularly pronounced for modes whose frequencies exceed the secondary maximum in N at $400 \mu\text{Hz}$. The very low values of E_{nl} found at selected frequencies for higher values of l occur where the g modes pass through avoided crossings and take on the character of surface gravity waves; this happens when their frequencies are near the extensions of the f-mode frequencies shown in Figure 1. It is interesting that this phenomenon can be traced in inertia to a degree as low as 3.

Whatever the excitation mechanism that may give rise to g modes in the Sun, it is unlikely that g modes of high degree can be excited to observable surface amplitudes. In particular, if all modes are arbitrarily assumed to be excited to the same kinetic energy, the predicted velocity amplitudes at high degrees are very small (e.g. Dziembowski, Paterno & Ventura 1985; Provost & Berthomieu 1988; Berthomieu & Provost 1990). On the other hand it is interesting that the increase in E_{nl} with l is weak for $l \leq 10$. Although all g modes are evanescent in at least the convection zone, the decrease in amplitude predicted by equation (3.43) is relatively modest at low l .

Predictions of the corresponding amplitudes in intensity require consideration of the non-adiabatic behaviour of the modes near the solar surface, and hence may be subject to the uncertainty

* Note that the quantity considered here is greater by a factor 4π than the quantity plotted in Figures 7 – 9 of Christensen-Dalsgaard (1986) and in Figure 6 of Christensen-Dalsgaard (1988b). The “ 4π ” in the denominator of equation (4.1) of the former paper, and equation (3.5) of the latter, should be removed.

in the description of convection (*cf.* Chapter 16). Cox, Guzik & Kidman (1989) estimated the temperature perturbation at the photosphere, neglecting the perturbation of the convective flux. The observed amplitude of the intensity results from a mean over the surface of the Sun of the intensity perturbation, integrated along the line of sight in the atmospheric layers. With simplifying assumptions concerning the transfer of radiation, and neglecting the perturbation of the convective flux, Provost & Berthomieu (1988) and Berthomieu & Provost (1990) obtained results for the intensity fluctuations, which indicated a zero in the intensity amplitude for dipole modes at a frequency of about 60 μHz . However, a full account of the transfer, including the effects of opacity fluctuations, could modify these features (Toutain & Gouttebroze 1988).

4.3 The functional analysis of adiabatic oscillations.

A great deal of insight into the properties of adiabatic oscillations can be obtained by regarding the equations as an eigenvalue problem in a Hilbert space (Eisenfeld 1969; Dyson & Schutz 1979; Christensen-Dalsgaard 1981). We write equations (2.24) and (2.25) as

$$\frac{1}{\rho} \frac{dp'}{dr} + \frac{\rho'}{\rho} g - \frac{d\Phi'}{dr} = \omega^2 \xi_r \quad (4.3)$$

$$\frac{1}{r} \left(\frac{p'}{\rho} - \Phi' \right) = \omega^2 \xi_h. \quad (4.4)$$

Here ρ' is determined in terms of ξ_r and ξ_h by equation (2.27), and, given ρ' , p' is determined by equation (2.28). Also Φ' can be found in terms of ρ' from the usual integral solution to Poisson's equation:

$$\Phi' = \frac{4\pi G}{2I+1} \left[r^{-(I+1)} \int_0^r \rho' r'^{I+2} dr' + r^I \int_r^\infty \rho' r'^{I-1} dr' \right]. \quad (4.5)$$

We introduce the vector

$$\xi \equiv (\xi_r, \xi_h) \in \mathcal{H}, \quad (4.6)$$

where $\mathcal{H}$ is a subspace of the space of pairs of quadratically integrable functions on the interval $[0, R]$. Then ρ' , p' and Φ' are obtained by linear operations on ξ . The same is therefore true of the left hand sides of equations (4.3) and (4.4), and thus these equations can be written as

$$\mathcal{L}\xi = \omega^2 \xi \quad (4.7)$$

which defines the linear operator $\mathcal{L}$ in $\mathcal{H}$. The boundary conditions on ξ_r and ξ_h can be imposed by restricting the part of $\mathcal{H}$ where $\mathcal{L}$ is defined. For reasons which become clear below we choose to use the surface condition (2.39), and define the *domain* of $\mathcal{L}$ by

$$\mathcal{D}(\mathcal{L}) = \{(\xi_r, \xi_h) | \xi_r - l\xi_h \rightarrow 0 \text{ for } r \rightarrow 0 \wedge \delta p(R) = 0\}; \quad (4.8)$$

the boundary conditions on Φ' are satisfied automatically by the integral expression (4.5).

We introduce a scalar product on $\mathcal{H}$. Let $\xi = (\xi_r, \xi_h)$ and $\eta = (\eta_r, \eta_h)$ be two vectors in $\mathcal{H}$. Then we define

$$\langle \xi, \eta \rangle \equiv 4\pi \int_0^R [\xi_r^*(r)\eta_r(r) + L^2 \xi_h^*(r)\eta_h(r)] r^2 \rho dr, \quad (4.9)$$

where the star denotes the complex conjugate. It is straightforward, using the explicit expression, to show that the operator $\mathcal{L}$ is symmetric, *i.e.*

$$\langle \mathcal{L}\xi, \eta \rangle = \langle \xi, \mathcal{L}\eta \rangle \quad \text{for all } \xi, \eta \in \mathcal{D}(\mathcal{L}). \quad (4.10)$$

However, for unbounded operators such as $\mathcal{L}$ symmetry is not enough for the completeness of the (possibly generalized) eigenfunctions: we have to show self-adjointness, *i.e.* that the domain of $\mathcal{L}$ is equal to that of its adjoint. Certain symmetric operators can be successfully extended to self-adjoint operators, by enlarging their domain (which entails a similar reduction of the domain of its adjoint) precisely to the point that both domains become identical. There are sufficient conditions for such extensions: a frequent criterion is given when the quadratic form associated with the operator is bounded from below. This permits the so-called Friedrich extension (Reed & Simon 1975). Dyson & Schutz (1979) have based their completeness proof (valid even for differentially rotating stars) on this idea.

From equation (4.10) follows immediately a number of useful properties of $\mathcal{L}$. The simplest is that the squared eigenfrequencies are real. We introduce the functional Σ on $\mathcal{D}(\mathcal{L})$ by

$$\Sigma(\xi) = \frac{\langle \xi, \mathcal{L}\xi \rangle}{\langle \xi, \xi \rangle}; \quad (4.11)$$

it follows from equation (4.10), that $\Sigma(\xi)$ is real. If ω_0^2 is an eigenvalue of the problem with eigenvector ξ_0 , *i.e.*

$$\mathcal{L}\xi_0 = \omega_0^2 \xi_0, \quad (4.12)$$

then

$$\Sigma(\xi_0) = \omega_0^2 \quad (4.13)$$

and hence ω_0^2 is real. Since the coefficients in equations (2.29) – (2.31) are then real, it follows that we may also choose the eigenfunctions to be real at all r .

As is well known, a second property of a Hermitian operator is that eigenvectors corresponding to different eigenvalues are orthogonal. Thus if

$$\mathcal{L}\xi_1 = \omega_1^2 \xi_1; \quad \mathcal{L}\xi_2 = \omega_2^2 \xi_2; \quad \omega_1^2 \neq \omega_2^2, \quad (4.14)$$

then

$$\langle \xi_1, \xi_2 \rangle = 0. \quad (4.15)$$

In addition ω^2 satisfies a variational principle (*cf.* Chandrasekhar 1964). From equation (4.10) it is easy to show that if $\delta\xi \in \mathcal{H}$ is a small change to the eigenvector, then

$$\Sigma(\xi_0 + \delta\xi) = \omega_0^2 + O(\|\delta\xi\|^2) \quad (4.16)$$

where $\|\dots\|$ is a suitable norm. Thus Σ is stationary at the eigenfrequencies. From a physical point of view this reflects Hamilton's principle for the system consisting of the pulsating star. It is conservative, because of the assumption of adiabaticity, and isolated because of the boundary condition (2.39).

It is evident that the variational property, and equivalently the Hermiticity of the operator $\mathcal{L}$, are not valid for non-adiabatic oscillations, where energy is not conserved. Also it depends on the choice of boundary conditions that isolate the system, such as the vanishing of the Lagrangian pressure perturbation. If a different condition is used, such as the match to a solution in an isothermal atmosphere discussed in section 2.3, the variational property is no longer guaranteed. However, regardless of the boundary condition it is possible to write down expressions analogous to equation (4.13), but possibly containing surface terms; if the effects of the surface properties on the oscillations are small, these expressions may be expected to be approximately variational. In fact it follows from the equations of linear adiabatic oscillations that for any radius r^*

$$\begin{aligned} \omega^2 \int_0^{r^*} \rho (\xi_r^2 + L^2 \xi_h^2) r^2 dr &= \int_0^{r^*} \left(\Gamma_1 p D_1^2 + \frac{2}{dr} \xi_r D_1 + \frac{1}{\rho} \frac{dp}{dr} \frac{d\xi_r}{dr} \right) r^2 dr \\ &\quad - \frac{8\pi G}{2l+1} \int_0^{r^*} r^{-(l-1)} D_2(r) \int_0^r r'^{l+2} D_2(r') dr' dr \\ &\quad + p'(r^*) \xi_r(r^*) r^{*2} \\ &\quad + \frac{4\pi G}{2l+1} \left\{ 2\rho(r^*) \xi_r(r^*) r^{*-(l-1)} \int_0^{r^*} D_2(r') r'^{l+2} dr - [\rho(r^*) \xi_r(r^*)]^2 r^{*3} \right\}; \end{aligned} \quad (4.17)$$

here

$$D_1 = \frac{1}{r^2} \frac{d}{dr} (r^2 \xi_r) - \frac{l(l+1)}{r} \xi_h, \quad D_2 = \frac{1}{r^2} \frac{d}{dr} (r^2 \rho \xi_r) - \frac{l(l+1)}{r} \rho \xi_h \quad (4.18)$$

are the amplitudes of $\text{div}(\delta \mathbf{r})$ and $\text{div}(\rho \delta \mathbf{r})$ respectively. For radial oscillations equation (4.17) may be considerably simplified, to give

$$\begin{aligned} \omega^2 \int_0^{r^*} \xi_r^2 \rho r^2 dr &= \int_0^{r^*} \left\{ \Gamma_1 p r^4 \left[\frac{d}{dr} \left(\frac{\xi_r}{r} \right) \right]^2 - r \xi_r^2 \frac{d}{dr} [(3\Gamma_1 - 4)p] \right\} dr \\ &\quad + r^* [3\Gamma_1 p(r^*) \xi_r(r^*) + r^* \delta p(r^*)] \xi_r(r^*). \end{aligned} \quad (4.19)$$

These equations are valid at any r^* . If the surface radius is chosen for r^* , and the surface terms are neglected, they reduce to the expression defined symbolically in equation (4.13). In fact, the surface terms are in general relatively small, and even the complete expressions are approximately variational. As discussed by Christensen-Dalsgaard, Gough & Morgan (1979) and J. Christensen-Dalsgaard (1982), they may be utilized in the computation of accurate oscillation frequencies (see also section 5.6 below).

Finally we consider the effects of a perturbation $\delta \mathcal{L}$ in the operator $\mathcal{L}$. This perturbation could be due to changes in the equilibrium model, or to changes in the physics of the oscillations.* Thus

* *e.g.*, the inclusion of nonadiabatic effects; see Christensen-Dalsgaard (1981).

we consider the eigenvalue problem

$$(\mathcal{L} + \delta \mathcal{L}) \xi = \omega^2 \xi, \quad (4.20)$$

where $\omega^2 = \omega_0^2 + \delta \omega^2$ and, as in equation (4.12), ω_0^2 is an eigenvalue of the unperturbed problem with eigenvector ξ_0 . From first-order perturbation theory (*e.g.* Schiff 1949) then follows that

$$\delta \omega^2 \approx \frac{\langle \xi_0, \delta \mathcal{L} \xi_0 \rangle}{\langle \xi_0, \xi_0 \rangle}. \quad (4.21)$$

Thus the frequency change can be computed from the unperturbed eigenvector.

4.4 Effects on frequencies of a change in the model.

As an example of the use of equation (4.21), we consider in more detail the interpretation of changes in the frequencies caused by changes in the equilibrium model. We consider a mode (n, l) , with eigenvector $\xi_{nl} = (\xi_{r,nl}, \xi_{h,nl}) \in \mathcal{H}$; without loss of generality we can assume that $\xi_{r,nl}(r)$ and $\xi_{h,nl}(r)$ are real. The relative frequency change caused by $\delta \mathcal{L}$ is then, according to equation (4.21)

$$\frac{\delta \omega_{nl}^2}{\omega_{nl}^2} = \frac{1}{2} \frac{\delta \omega_{nl}^2}{\omega_{nl}^2} = \frac{\langle \xi_{nl}, \delta \mathcal{L} \xi_{nl} \rangle}{2 \omega_{nl}^2 \langle \xi_{nl}, \xi_{nl} \rangle}. \quad (4.22)$$

Here the denominator is proportional $\omega_{nl}^2 E_{nl}$, where E_{nl} was defined in equation (4.1). Also, we represent $\delta \mathcal{L}$ on component form as

$$\delta \mathcal{L} \xi_{nl} = (\phi_r[\xi_{nl}], \phi_h[\xi_{nl}]) \quad (4.23)$$

where $\phi_r[\xi_{nl}](r)$ and $\phi_h[\xi_{nl}](r)$ are functions of r . Then we can write equation (4.22) as

$$\frac{\delta \omega_{nl}^2}{\omega_{nl}^2} = I_{nl} \quad (4.24)$$

where

$$I_{nl} = \frac{2\pi \int_0^R [\xi_{r,nl}(r) \phi_r[\xi_{nl}](r) + L^2 \xi_{h,nl}(r) \phi_h[\xi_{nl}](r)] \rho r^2 dr}{M \omega_{nl}^2 [\xi_{nl}^2(R_{pho}) + l(l+1) \xi_h^2(R_{pho})]}. \quad (4.25)$$

Thus I_{nl} gives the integrated effect of the perturbation, normalized to the total photospheric displacement.

There is a close analogy between the exact equations (4.24) and (4.25) and the asymptotic expressions (3.19) and (3.20). In both cases the factor multiplying $\delta \omega$ represents the fact that modes with larger inertia are more difficult to perturb. The effect on the frequencies of the change in the model is described by the right hand sides of equations (3.19) and (4.25), and depends on the overlap between the change in the model and the eigenfunction.

We now assume that $\delta \mathcal{L}$ is localized near the solar surface, in the sense that

$$\phi_r[\xi](r) \simeq 0, \quad \phi_h[\xi](r) \simeq 0 \quad \text{for } R - r > \delta, \quad (4.26)$$

for some small δ . For modes extending substantially more deeply than the region of the perturbation, i.e. with $R - r_l \gg \delta$, the eigenfunctions are nearly independent of l at fixed frequency in that region (see also Figure 8 and the associated discussion). Hence I_{nl} depends little on l at fixed ω . To get a more convenient representation of this property, we introduce

$$Q_{nl} = \frac{E_{nl}}{\bar{E}_0(\omega_{nl})} \quad (4.27)$$

where $\bar{E}_l(\omega)$ is obtained by interpolating to ω in E_{nl} at fixed l . Then

$$Q_{nl}\delta\omega_{nl} \quad (4.28)$$

is independent of l , at fixed ω , for modes such that $R - r_l \gg \delta$. Conversely, if $Q_{nl}\delta\omega_{nl}$ is independent of l at fixed ω for a given set of modes, then $\delta\mathcal{L}$ is probably largely localized outside $r = \max(r_l)$ over the set of modes considered.

Q_{nl} has been plotted in Figure 13, for selected values of l . Its variation with l is largely determined by the change in the penetration depth. Modes with higher degree penetrate less deeply and hence have a smaller inertia at given photospheric displacement. As a consequence of this their frequencies are more susceptible to changes in the model.

It should be noticed that equations (4.24) and (4.25) provide a linear relation between the change in the model and the change in the frequency. As discussed in Chapter 14 this may be expressed in terms of kernels relating, for example, changes in ρ and Γ_1 to the changes in the frequencies.

4.5 The Cowling approximation.

Given the importance of the Cowling approximation for the asymptotic analysis, it is of some interest to consider its implications in more detail. Results for polytropic models were obtained by Robe (1968). The effects in a model of the present Sun may be judged from Figure 14, which shows differences between frequencies computed in the Cowling approximation and from the full equations. As discussed in Section 3 (cf. equation (3.2)) the effect of the perturbation in the gravitational potential decreases rapidly with increasing mode order and degree.

It was pointed out by Cowling (1941) that the effects of Φ' can be estimated from perturbation analysis. This in fact follows from the analysis in section 4.3, if we regard the terms in Φ' in equations (4.3) and (4.4) as small perturbations, to be evaluated from the unperturbed eigenfunctions by means of equation (4.5). Christensen-Dalsgaard (1984a) found that the resulting corrected frequencies differed by less than 0.05 per cent from the frequencies computed without making the Cowling approximation, for p modes of degree greater than 0 and frequency greater than 1700 μHz .

It should be mentioned that in principle the Cowling approximation involves mathematical subtleties which have not been fully resolved. These are related to the fact that the approximation reduces the order of the system by two, and hence eliminates two independent solutions of the

differential equations. A striking manifestation is the fact that in the Cowling approximation there is an $l = 1$ f mode with no nodes in the radial displacement, intermediate in frequency between the p and the g modes; in the full case there is also a solution to the equations with no nodes in the radial displacement, but this has zero frequency, and corresponds to an infinitely slow uniform translation of the model (see also Christensen-Dalsgaard 1976). If the effects of the gravitational potential are gradually introduced, the transition from the dipolar f mode in the Cowling approximation to the zero-frequency "mode" in the full case takes place through a sequence of avoided crossings with the g modes (Christensen-Dalsgaard 1978). A related problem concerns the classification of the modes. As discussed in section 2.4, a precise mathematical classification is possible in the Cowling approximation. In contrast, a naive application of this classification procedure to the lowest-order dipolar p modes, computed from the full equations, results in a misidentification of the modes with $n = 1$ and 2 in typical models of the present Sun; for more evolved models the problem apparently extends to modes of higher order. However, for those modes that have been observed on the Sun the neglect of Φ' does not change the overall structure of the spectrum.

In the light of the utility of the asymptotic treatment, there is evidently a great need for an asymptotic description that takes the effects of Φ' into account. The leading-order correction to equation (3.12) was presented by Vorontsov (1988). A more complete description has recently been developed by Dziembowski & Gough (in preparation).

5. Numerical techniques.

The differential equations (2.29) – (2.31), in combination with boundary conditions such as equations (2.37) – (2.39), constitute a two point boundary value problem. Non-trivial solutions to the problem can be obtained only at selected values of the frequency ω , which is therefore an eigenvalue of the problem. Problems of this nature are extremely common in theoretical physics, and hence there exists a variety of techniques for solving them. Nevertheless, the computation of solar adiabatic oscillations possesses special features, which merit discussion. In particular, we typically need to determine a large number of frequencies very accurately, to match the volume and precision of the observed data.

Specific numerical techniques are discussed in considerable detail by, for example, Unno *et al.* (1979) and Cox (1980). Here we concentrate more on general properties of the solution method, and give examples illustrating typical results on the numerical precision of the solution. The choice of techniques and examples is unavoidably biased by our personal experience, but should at least give an impression of what can be achieved, and how to achieve it.

5.1 Difference equations.

The numerical problem can be formulated generally as that of solving

$$\frac{dy_i}{dx} = \sum_{j=1}^I a_{ij}(x)y_j(x), \quad \text{for } i = 1, \dots, I, \quad (5.1)$$

with suitable boundary conditions at $x = x_1$ and x_2 , say. Here the order I of the system is four for the full nonradial case, and two for radial oscillations or nonradial oscillations in the Cowling approximation.

To handle these equations numerically, we introduce a mesh $x_1 = x^{(1)} < x^{(2)} < \dots < x^{(N_m)} = x_2$ in x , where N_m is the total number of mesh points. Similarly we introduce $y_i^{(n)} \equiv y_i(x^{(n)})$, and $a_{ij}^{(n)} \equiv a_{ij}(x^{(n)})$. A commonly used, very simple representation of the differential equations is in terms of *second-order centred differences*, where the differential equations are replaced by the difference equations

$$\frac{y_i^{(n+1)} - y_i^{(n)}}{x^{(n+1)} - x^{(n)}} = \frac{1}{2} \sum_{j=1}^I \left[a_{ij}^{(n)} y_j^{(n)} + a_{ij}^{(n+1)} y_j^{(n+1)} \right], \quad i = 1, \dots, I. \quad (5.2)$$

These equations allow the solution at $x = x^{(n+1)}$ to be determined from the solution at $x = x^{(n)}$.

More elaborate and accurate difference schemes (e.g. Press *et al.* 1986; Cash & Moore 1980), can be set up which allow the rapid variation in high-order eigenfunctions to be represented with adequate accuracy on a relatively modest number of meshpoints. Alternatively one may approximate the differential equations on each mesh interval $(x^{(n)}, x^{(n+1)})$ by a set of equations with constant coefficients, given by

$$\frac{dy_i^{(n)}}{dx} = \sum_{j=1}^I a_{ij}^{(n)} y_j^{(n)}(x), \quad \text{for } i = 1, \dots, I, \quad (5.3)$$

where $a_{ij}^{(n)} \equiv \frac{1}{2}(a_{ij}^{(n)} + a_{ij}^{(n+1)})$ (Gabriel & Noels 1976). These equations may be solved analytically on the mesh intervals, and the complete solution is obtained by continuous matching at the mesh points. This technique clearly permits the computation of modes of arbitrarily high order. We have considered its use only for systems of order two, i.e. for radial oscillations or non-radial oscillations in the Cowling approximation.

5.2 Shooting techniques.

Perhaps the conceptually simplest technique for handling a boundary value problem is the shooting technique. For simplicity we consider first the case of a second-order system, such as results from making the Cowling approximation. Then there is one boundary condition, namely the first of equations (2.37), at the centre, and one condition, equation (2.39), at the surface. For any value of ω the equations may be integrated numerically, imposing the central boundary condition on ξ_r and ξ_h , and the quantity

$$\Delta(\omega) \equiv \left(p' + \frac{dp}{dr} \xi_r' \right) \Big|_{r=R} \quad (5.4)$$

may be evaluated. The eigenfrequencies are obviously the zeros of $\Delta(\omega)$. A convenient method of locating them is to evaluate $\Delta(\omega)$ at a sequence of points $\omega_1, \omega_2, \dots$; once an interval has been found where Δ changes sign the zero can be found, for instance, by applying the secant method.

An attractive feature of the method is precisely this ability to search automatically for all modes in a given frequency range, particularly when it is combined with a method for determining the order of a given mode, so that a check can be made that no modes have been skipped.

A slight elaboration of this basic technique is required to make it computationally efficient. Due to the rapid decrease of temperature near the solar surface, the equations are almost singular here. Far from the eigenfrequencies the solution therefore generally increases rapidly towards the surface; this translates into a dramatic variation of Δ with ω , which complicates the determination of the zeros. To avoid this problem, one may compute solutions $(\xi_r^{(i)}, \xi_h^{(i)})$ and $(\xi_r^{(s)}, \xi_h^{(s)})$ satisfying the inner and the surface boundary conditions, respectively. A continuous match of the interior and exterior solutions requires the existence of non-zero constants $C^{(i)}$ and $C^{(s)}$ such that

$$\begin{aligned} C^{(i)} \xi_r^{(i)}(r_j) &= C^{(s)} \xi_r^{(s)}(r_j) \\ C^{(i)} \xi_h^{(i)}(r_j) &= C^{(s)} \xi_h^{(s)}(r_j), \end{aligned} \quad (5.5)$$

where r_j is an appropriately chosen fitting point. This set of equations has a solution only if the determinant

$$\Delta_f(\omega) = \xi_r^{(i)}(r_j) \xi_h^{(s)}(r_j) - \xi_h^{(i)}(r_j) \xi_r^{(s)}(r_j) \quad (5.6)$$

vanishes. Hence the eigenfrequencies are determined as the zeros of Δ_f , as before.

The choice of fitting point should be guided by the expected behaviour of the eigenfunction, in such a way that the integration of the differential equations proceeds in a stable fashion. Thus, for instance, when solving for a strongly trapped g mode of high degree, r_j should be near the maximum in the buoyancy frequency where the mode is trapped; in this way the integration from both the centre and the surface is in the direction where the solution increases. Similarly, when integrating for a p mode of high degree, r_j should be in the oscillatory region near the surface.

The solution of the full fourth-order problem proceeds in a very similar fashion. Here there are two linearly independent solutions that satisfy the boundary conditions at the centre, and two linearly independent solutions that satisfy the conditions at the surface. The condition that these two sets of solutions match continuously at a point r_j leads to a set of equations whose solution requires the vanishing of a 4×4 determinant. It should be noted, however, that problems arise when the effect of the perturbation in the gravitational potential is small. In this case, although the two separate solutions from e.g. the centre are formally linearly independent, they are in practice very close to being linearly dependent, and the zeros of Δ_f are therefore ill-determined. This is no major concern in practice, since under these circumstances the Cowling approximation is in general adequate. However, as discussed below the problem may be avoided through the use of some variant of the relaxation technique.

5.3 Relaxation techniques.

The relaxation technique considers the set of difference equations, such as equations (5.2), together with the homogeneous boundary conditions and a normalization condition, as a set of equations

for the unknown quantities $\{y_i^{(n)}; i = 1, \dots, I; n = 1, \dots, N_{me}; \omega\}$. Due to the appearance of the eigenfrequency, the equations are non-linear in the unknowns. They are solved by linearizing around an assumed initial trial solution, and the solution is obtained by iteration. This technique is equivalent to what is commonly known as the Henyey technique in computations of stellar evolution (Henyey, Forbes & Gould 1964; see also Baker, Moore & Spiegel 1974).

A disadvantage of this technique is that it requires a reasonably accurate trial solution, both for the eigenfrequency and the eigenfunction, if the iteration is to converge to the desired mode. Also it is not immediately possible to search a given part of the spectrum. These problems may be avoided by dropping one of the boundary conditions, and regarding ω as given (e.g. Castor 1971; Osaki & Hansen 1973). The difference equations are then a linear set of equations for the $\{y_i^{(n)}\}$ which may be solved directly. Given the solution, the remaining boundary condition, now regarded as a function of ω , is solved to obtain the eigenfrequencies. Thus in this form the relaxation technique retains the advantages of the shooting method, in that a region of the spectrum can be scanned. Once a sufficiently close approximation to the solution has been found, the rate of convergence can be increased by switching to simultaneous iteration for the eigenfrequency and eigenfunction.

As for the shooting technique, the straight determination of the eigenfrequency through root-seeking on one of the boundary conditions is rather ill-behaved. This problem may be avoided by imposing all boundary conditions, but permitting for general ω a discontinuity in one component of the eigenfunction at a suitable interior fitting point r_f . The eigenfrequencies are then determined by requiring that the discontinuity vanish. We have found that this technique allows stable solution of the full set of equations for all relevant degrees and frequencies.

5.4 Formulation as a matrix eigenvalue problem.

As discussed in section 4.3 the equations of adiabatic oscillation, written as in equations (4.3) and (4.4), constitute a linear eigenvalue problem in function space. If the operator on the right hand side is discretized, the result is a linear discrete eigenvalue problem. By solving this, one obtains (approximations to) the eigenvalues and eigenfunctions of the continuous problem.

A method of this nature (but generalized to the non-adiabatic case) was used by Keeley (1977) for radial oscillations. Knölker & Stix (1983) used it for adiabatic non-radial oscillations in the Cowling approximation. In these cases, the operator describing the left hand side of the oscillation equations is a pure differential operator; hence its discrete representation only couples the solution at a few neighbouring meshpoints and results in an eigenvalue problem where the matrix is banded, with only a few off-diagonal elements. Consequently, efficient techniques exist for the determination of the eigenvalues. In contrast, in the full non-radial problem the terms in Φ' couple all parts of the model (see also equation (4.5)); then the corresponding matrix is full, although for large l it is diagonally dominated, due to the factors $(r'/r)^{l+1}$ and $(r/r')^l$ occurring respectively in the first and second term on the right hand side of equation (4.5). In this case it is not evident that sufficiently fast algorithms exist for the determination of the matrix eigenvalues to make the

technique competitive with the shooting or relaxation techniques. No attempt has apparently been made to apply it to this problem.

The matrix eigenvalue problem can also be derived directly from the variational principle as expressed in equation (4.17), by means of the so-called Rayleigh-Ritz method (e.g. Strang & Fix 1973). To do so the eigenfunction is expanded on a set of suitable basis functions, and the expansion coefficients are determined by imposing the condition that the expression (4.17) be stationary. Although this method has proven useful in atomic physics (e.g. B. L. Christensen-Dalsgaard 1982), the effects of the gravitational potential once again lead to a full matrix in the resulting eigenvalue problem.

We finally note that Pesnell (1990) has developed an efficient algorithm, based on the method of Castor (1971), for computing nonradial oscillations both in the adiabatic and the nonadiabatic case. This involves formulating the oscillation equations as a generalized linear algebraic eigenvalue problem; in contrast to the techniques discussed above, Poisson's equation is left on differential form, and hence the resulting matrices are sparse. Cox *et al.* (1989) applied this method to the computation of solar oscillations.

5.5 Richardson extrapolation.

The difference scheme (5.2), which is used by at least some versions of the shooting, relaxation and matrix eigenvalue techniques, is of second order. Consequently the truncation errors in the eigenfrequency and eigenfunction scale as N_{me}^{-2} . If $\omega(\frac{1}{2}N_{me})$ and $\omega(N_{me})$ are the eigenfrequencies obtained from solutions with $\frac{1}{2}N_{me}$ and N_{me} meshpoints, the leading order error term therefore cancels in

$$\omega(Ri) \equiv \frac{1}{3}[4\omega(N_{me}) - \omega(\frac{1}{2}N_{me})]. \quad (5.7)$$

The evaluation of $\omega(Ri)$, known as *Richardson extrapolation*, was used by Shibahashi & Osaki (1981) to compute frequencies of solar oscillation. As shown in section 5.8 it provides an estimate of the eigenfrequency that is substantially more accurate than $\omega(N_{me})$, although of course at some added computational expense.

5.6 Variational frequencies.

The variational property discussed in section 4.3 can be used to obtain an estimate of the oscillation frequency which is at least formally more accurate than the frequency obtained as eigenvalue of the solution of the oscillation equations (Christensen-Dalsgaard, Gough & Morgan 1979; J. Christensen-Dalsgaard 1982). It follows from equation (4.16) that if the computed eigenfunction is substituted into the functional $\Sigma(\xi)$, the result agrees with the squared eigenfrequency to within an error that is quadratic in the error in the eigenfunction. If the latter error goes as N_{me}^{-2} , the error in $\Sigma(\xi)$ would be expected to vary as N_{me}^{-4} ; this assumes that the evaluation of $\Sigma(\xi)$, for given ξ , is sufficiently accurate.

For realistic solar models the complete expressions (4.17) or (4.19) must be used, including the surface terms. However, since these terms are in general relatively small, the variational property is still approximately satisfied. Hence the expressions may be used to provide estimates of the frequency which are less sensitive to numerical error than the eigenfrequency. On the other hand, it should be noted that the variational property, and the analysis leading to equations (4.17) and (4.19), assume that the solar model satisfies the equations of hydrostatic equilibrium and the mass equation exactly. When the model is itself the result of a numerical solution of the equations of stellar structure this is evidently not the case; then, even if they were to be evaluated with infinitely high precision for the given model, the variational frequency and the eigenfrequency would not agree. The discrepancy provides an estimate of the effect on the frequencies of the inconsistencies in the model. Examples of this are discussed below.

5.7 The determination of the mesh.

Computational efficiency demands that the distribution of mesh points be chosen appropriately. It is immediately obvious from the eigenfunctions (*cf.* Figures 8 and 10) that a mesh uniform in r is far from optimal; also the distribution of points should clearly be different for p and for g modes.

Procedures exist that determine the optimal mesh as part of the numerical solution of a set of differential equations (Gough, Spiegel & Toomre 1975); the use of such techniques in calculations of stellar evolution was discussed, for example, by Eggleton (1971) and J. Christensen-Dalsgaard (1982). In the present case, however, the requirements on the mesh are essentially driven by the behaviour of the modes of high radial order, whose eigenfunctions are given, with considerable precision, by the asymptotic expressions (3.13) and (3.41). Thus to have a roughly constant number of meshpoints between the nodes in the eigenfunction, the mesh should be approximately uniformly spaced in terms of the integrals in these equations.

To define a flexible method for setting up the mesh we have adopted a simplified version of the procedure developed by Gough *et al.* (1975). Thus we introduce a variable z , with a range from 0 to 1, such that the mesh is uniform in z , and determined by

$$\frac{dz}{dr} = \lambda H(r); \quad (5.8)$$

here λ is a normalization constant, relating the ranges of z and r , and the function H determines the properties of the mesh. Given H , z is obtained as

$$z(r) = \lambda \int_0^r H(r') dr', \quad (5.9)$$

with

$$\lambda = \left(\int_0^R H(r) dr \right)^{-1}. \quad (5.10)$$

The mesh $\{r^{(n)}, n = 1, \dots, N_{me}\}$ is finally determined by solving the equations

$$z(r^{(n)}) = \frac{n-1}{N_{me}-1}, \quad (5.11)$$

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by interpolating in the computed $z(r)$.

The choice of the function H must be guided by the asymptotic behaviour of the modes. We have used

$$H(r)^2 = R^{-2} + c_1 \frac{\omega_p^2}{c^2} + c_2 \frac{|N|^2}{\omega_p^2 r^2} + c_3 \left(\frac{d \ln p}{dr} \right)^2, \quad (5.12)$$

where $\omega_p^2 \equiv GM/R^3$ is a characteristic squared frequency. Here the terms in c_1 and c_2 are clearly aimed at reproducing the asymptotic behaviour of the p - and the g -mode eigenfunctions, respectively. The term in c_3 provides extra meshpoints near the surface, where the reflection of the p modes takes place. Finally the constant term ensures a reasonable resolution of regions where the other terms are small.

The parameters in this expression can be determined by testing the numerical accuracy of the computed frequencies, as discussed in the following section. For normal solar models reasonable choices, based on a fairly extensive (but far from exhaustive) set of calculations, are given in Table 1. In the p -mode case the mesh is predominantly determined by the variation of sound speed, with the term in N giving a significant contribution near the centre and the term in $d \ln p/dr$ contributing very near the surface. The g -mode mesh is dominated by the term in N in most of the radiative interior, whereas in the convection zone, where $|N|$ is generally small, the constant term dominates; the term in $d \ln p/dr$ is again important in the surface layers.

5.8 Numerical accuracy.

Estimates of the numerical accuracy of computed frequencies should in principle regard the physics and assumptions of the underlying model as given, and should therefore take into account the numerical errors introduced in the computation of the models. However, here we restrict the discussion to the calculation of frequencies for a given model. As an indication of the desired accuracy, it might be recalled that the typical errors in the observed frequencies are around, or less than, $0.1 \mu\text{Hz}$ over a wide range of modes (*cf.* Chapter 10). It is clearly desirable that the errors in the computed frequencies should be no greater.

The only practical way of estimating the numerical error in the computed frequencies is to compare results obtained with different numbers of meshpoints. This technique was applied to adiabatic oscillation frequencies of solar models by, *e.g.*, Shibahashi & Osaki (1981), J. Christensen-Dalsgaard (1982), Noels, Scuflaire & Gabriel (1984) and Guenther & Sarajedini (1988). Mullan & Ulrich (1988) and Mullan (1989) made a careful analysis of the accuracy of frequencies of polytropic models, although restricted to calculations in the Cowling approximation. Here we give examples of such estimates of the errors in calculations of p -mode frequencies by various techniques. In each case the error is obtained as the difference between the results of a computation with $N_{me} = 600$ points, and a computation with $N_{me} = 2400$ points. Except where otherwise noted the computations used the centred difference approximation (5.2) to the differential equations. For $l \leq 20$ the full set of equations were solved, by means of the relaxation technique with interior fitting discussed in section 5.3. For higher degree the Cowling approximation was employed, the

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frequency being corrected for the effect of the perturbation in the gravitational potential by means of the perturbation analysis discussed in section 4.5; for modes of degree 20 and frequency less than 5000 μHz the error resulting from this approach is less than 0.01 μHz for the Richardson extrapolated frequencies, and less than 0.05 μHz for the variational frequencies.

Figure 15 shows the errors in the straight eigenfrequency, the result of applying Richardson extrapolation according to equation (5.7), and in the variational frequency. In each case the error generally grows with increasing frequency, and hence mode order, as would be expected. The eigenfrequencies computed with 600 points, using the centred difference equations (5.2), are evidently totally inadequate, the maximum error within the observed frequency range being as high as about 15 μHz . In contrast the error in the frequencies obtained from Richardson extrapolation, and using the variational method, are modest, although still somewhat higher than our goal of 0.1 μHz .

The variation in error with number of meshpoints was estimated by also computing frequencies with $N_{me} = 1200$. For the eigenfrequencies the error varies as N_{me}^{-2} , as expected. Thus to decrease the eigenfrequency error below 0.1 μHz would require about 7000 points. On the other hand the error in both the Richardson and the variational frequencies decreases much more rapidly with increasing N_{me} . At $N_{me} = 1200$ the errors in the Richardson frequencies are well below 0.05 μHz , and the errors in the variational frequencies are smaller than 0.1 μHz . Thus we may estimate that the errors in these quantities for $N_{me} = 2400$ are at or below 0.01 μHz , which should be adequate to match the observational accuracy within the foreseeable future.

For comparison we also computed eigenfrequencies by using the constant coefficient method outlined in equation (5.3); only radial modes, and modes of degree $l \geq 20$ where the Cowling approximation is sufficiently accurate, were considered. The resulting errors are shown in Figure 15d. Here even 600 points are sufficient to achieve errors below 0.1 μHz . As an indication of the consistency of the different integration methods it might also be mentioned that with $N_{me} = 2400$ the Richardson and the constant coefficient frequencies agree to within 0.005 μHz .

As mentioned above, comparison of the eigenfrequencies and the variational frequencies provides a check of the accuracy of the equilibrium model. Figure 16a shows differences between the Richardson extrapolation and the variational frequencies, computed for Model 1 of J. Christensen-Dalsgaard (1982), interpolated to a mesh with 2400 points. In accordance with the discussion in section 4.3 the differences have been scaled with the energy ratio Q_m . There are evidently discouragingly large differences, indicating that the errors in the model are unacceptably large. Furthermore, the behaviour for $l = 0$ and $l > 0$ is different, due to the different formulations in equations (4.17) and (4.19), which are equivalent only if the model is consistent. This difference causes problems if variational frequencies are used to compute the low-degree frequency separations in equation (3.29) (Fossat 1985; Christensen-Dalsgaard 1986). In contrast, results for a model computed with substantially higher numerical accuracy are shown in Figure 16b. Although there are still differences, and a different behaviour of the radial and nonradial modes, the effect is now reduced to an acceptable level. It should also be noticed that for $l > 0$ the scaled differences depend little on l , indicating that the error in the model is predominantly located near the surface. Further

analysis of results such as these is clearly of considerable importance for the optimization of the calculation of solar models.

6. Sensitivity of the frequencies to the structure of the Sun.

To analyse observations of solar oscillations, it is common to study the differences between the observed frequencies and those computed for a variety of solar models. Given that significant differences exist, the goal is evidently to identify the likely sources of the errors in the models. To do so, one must consider the sensitivity of the computed frequencies to changes in the model.

As discussed in Chapter 14, it is possible to obtain kernels that relate, say, changes $\delta\rho$ in density and $\delta\Gamma_1$ in the adiabatic exponent Γ_1 to the changes in the frequencies. Inspection of these kernels then immediately shows how different modes respond to modifications in particular regions of the model. Furthermore, from the differences between observed and computed frequencies one may in principle obtain the corrections $\delta\rho$ and $\delta\Gamma_1$ to the assumed solar model by means of inverse analysis.

This technique permits the determination of important aspects of the internal structure of the Sun. Of more fundamental interest, however, are the material properties and other features that determine the structure. The properties of matter in the Sun (equation of state, opacity and nuclear reaction rates) are required in computations of solar models, as are assumptions about the features that need to be included in the computation. By studying solar oscillations we may hope to obtain information about matter under solar conditions, and to uncover, or constrain, possible additional features that might be required in the computation. Information of this type is evidently important for the computation of models of other stars; in addition it allows us to use the Sun as a test of the basic physical description of plasmas under conditions that are difficult to achieve in the laboratory (see Chapter 4).

A helioseismic determination of ρ and Γ_1 , as discussed above, can evidently be used to constrain solar models in this manner. Nevertheless, it is of obvious interest to consider more directly the relation between the assumptions underlying the model computation and the oscillation frequencies. Through a suitable parametrization of some of these assumptions, keeping the others fixed, it is possible to obtain kernels that relate aspects, say, of the physics to the oscillation frequencies. An interesting example is the analysis by Korzenik & Ulrich (1989) of the effects of the opacity. However, a commonly used, although less general, technique is to compare frequencies of specific models computed under different assumptions. Examples are the calculations by Scuflaire *et al.* (1975), Iben & Mahaffy (1976), Christensen-Dalsgaard *et al.* (1979), Ulrich & Rhodes (1983), Shibahashi, Noels & Gabriel (1983), Noels, Scuflaire & Gabriel (1984), and Christensen-Dalsgaard (1984b, 1986). More recently, Cox, Guzik & Kidman (1989) studied the effects of microscopic diffusion, and Christensen-Dalsgaard, Däppen & Lebreton (1988) compared frequencies computed with different assumptions about the equation of state.

To be useful, calculations of this nature require considerable care. It is essential that the two models considered differ only in the particular property which is under investigation. Also, given that the radius and luminosity of the Sun are known with high accuracy, the two models should be calibrated (by adjusting the convective efficiency and the initial composition) to have the same radius and luminosity; this of course excepts calculations aimed specifically at testing the effect of radius or luminosity changes. Numerical errors in the calculation of the models and the frequencies should either be negligible, or have the same effect for the two models. Finally, the presentation of the results should provide a reasonably comprehensive overview of the effect of the modification on a representative set of modes, and should illustrate the relationship between the changes in the model and the corresponding frequency changes. A convenient format is provided by plots of relative differences between relevant quantities in the two models, and of the frequency differences. In particular it should be noticed from equations (3.19) and (3.20) that the change in the frequencies of p modes, with which we are mainly going to be concerned, is largely determined by the change in the sound speed.

A detailed discussion of the effects of various modifications to the solar model was presented by Christensen-Dalsgaard (1988c). Of major concern are clearly the effects of the uncertainties in the physics. The effect of the treatment of the equation of state on the oscillation frequencies was considered in Chapter 4. It may also be noticed that changes in the nuclear reaction rates have very small effects on the frequencies of at least the five minute oscillations (e.g. Christensen-Dalsgaard 1988c). Indeed, it follows from the high temperature sensitivity of the reactions, and the requirement that the model have the correct luminosity, that a change in the overall energy generation rate, as controlled by the cross section of the p+p reaction, can be compensated by a much smaller change in temperature; changes in the rates of the other reactions in the pp chains are evidently entirely insignificant for the frequencies (Bahcall & Ulrich 1988).

Here we concentrate on two aspects of the model:

- 1) The opacity.
- 2) The composition profile.

Analysis of these examples illustrates the theoretical properties of the oscillations which have been presented in the preceding sections. Furthermore, it may be argued that, apart from the very uncertain region near the solar surface, the opacity is the least well-determined of the physical properties required to compute a model of the Sun (see Chapter 4 and references therein). In section 6.1 we discuss the effects on the structure and the p-mode frequencies of modifying these two aspects of the model. Section 6.2 deals with the properties of low-degree p modes, which are particularly important in providing information about the solar core. Finally section 6.3 considers the properties of g modes in terms of the asymptotic expression for their frequencies.

6.1 Effects of changes in the model.

We have used a reference model which is similar to Model 1 of J. Christensen-Dalsgaard (1982); the equation of state was calculated according to the Eggleton, Faulkner & Flannery (1973) prescription,

and the opacity was obtained from spline interpolation in the Cox & Tabor (1976) tables. The heavy element abundance was $Z = 0.02$. The initial abundance of hydrogen, and the mixing length parameter, were adjusted to obtain a model at solar age, assumed to be 4.75×10^9 years, with the solar luminosity $3.846 \times 10^{33} \text{ ergs}^{-1}$ and the solar radius $6.96 \times 10^9 \text{ cm}$. However, the numerical precision was considerably higher than for Model 1: the calculation used 600 spatial mesh points, and 26 time steps between the Zero Age Main Sequence and the present Sun. As was illustrated in Figure 16 this leads to a considerable improvement in the consistency of the computed frequencies. – The frequencies were computed with the centred difference technique and Richardson extrapolation, as discussed in sections 5.5 and 5.8.

The modified models were also adjusted to obtain the same radius and luminosity, to within a relative error of less than 10^{-5} . To study the effects of the modifications of the opacity full evolution models were calculated, whereas the modification in the composition profile was studied by means of a static model of the present Sun.

i) Opacity modifications.

To study the effect of the opacity on the structure and frequencies of a solar model, we have computed models where the opacity was artificially modified. Specifically, the opacity κ , as a function of density ρ and temperature T , was determined from

$$\log \kappa = \log \kappa_0 + A f(\log T), \quad (6.1)$$

where the unmodified opacity κ_0 was obtained from the tables of Cox & Tabor (1976). Here A is a constant determining the magnitude of the opacity modification, and the function f is defined by $f(x) = \exp[-(y/\Delta \log T)^2]$, where

$$y = \begin{cases} x - \log T_1 & \text{for } x < \log T_1 \\ 0 & \text{for } \log T_1 < x < \log T_2 \\ x - \log T_2 & \text{for } \log T_2 < x. \end{cases} \quad (6.2)$$

While this definition is evidently entirely arbitrary, it has the advantage of providing an opacity increase over a well-defined region in $\log T$, with a continuous transition to zero on either side.

It should be noticed that the structure of the Sun is independent of the local value of the opacity in the adiabatic part of the convection zone, i.e. approximately between $\log T = 4$ and $\log T = 6.3$. Thus modifications to the opacity in this temperature interval has no effect on the model.

Recent opacity calculations (see for example Cox *et al.* 1989) have shown a very substantial increase, by up to a factor 2, at conditions corresponding to the solar photosphere. To investigate the effects of such a change, we have computed a model with $\log T_1 = 3.5$, $\log T_2 = 3.8$, $\Delta \log T = 0.2$ and $A = 0.3$, thus roughly corresponding in magnitude to the actual increase in the tabulated opacity. Figure 17 shows the resulting differences between the models. In the deeper interior, the

model is virtually unchanged. The photospheric pressure, and hence density, is roughly inversely proportional to the opacity, and is therefore drastically changed. However, there is also a significant change in the sound speed, which leads to substantial changes in the frequencies. These are illustrated in Figure 18. In accordance with the discussion in section 4.4, the differences have been scaled by the ratio Q_{ν} of mode inertias (cf. equation (4.27)). It follows from Figure 13 that for the highest-degree modes the raw differences are larger by a factor of about 5 than those shown. Modes of degree less than about 500 all penetrate well beyond the region where the sound speed is affected (cf. Figure 3), and for these modes the scaled frequency change is mainly a function of frequency, but depends little on the depth of penetration of the mode, and hence on l . At higher degree the modes sample only part of the negative sound-speed difference, and the frequency change is smaller.

Quite apart from the specific modification considered, this example illustrates the important property that changes to the model near the surface lead to scaled frequency changes that depend only on frequency, and not on degree. Also, the frequency change is very small at low frequency. As discussed in section 4.2 the low-frequency modes are evanescent near the surface, and hence their amplitude is small, compared with the amplitude in the interior (see also Figure 9); as a result they are barely affected by changes in the surface region. Qualitatively similar changes result from modifications to the treatment of the upper, significantly superadiabatic part of the convection zone (Christensen-Dalsgaard 1986); more generally, it seems likely that the uncertainties in the treatment of the surface layers (nonadiabaticity of the oscillations; the treatment of convection; possible effects of magnetic fields; etc.) would give rise to a similar behaviour of the frequency changes. Thus in analyzing observed frequencies it is possible to absorb these uncertainties by allowing an undetermined frequency-dependent part of the scaled differences between observed and computed frequencies.

To illustrate the effect of a modification in the opacity near the base of the convection zone we also computed a model with $\log T_1 = 6$, $\log T_2 = 6.6$, $\Delta \log T = 0.15$, and $A = 0.1$, corresponding to a maximum change in opacity at fixed T and ρ of 26 per cent. This increases the depth of the convection zone in the model by about 0.02 R . The resulting changes in the model are shown in Figure 19. A striking feature is the comparatively small change in the sound speed in much of the convection zone. In fact it is easy to show that except in the outer part of the convection zone the sound speed is approximately given by

$$c^2 \approx (\Gamma_1 - 1) R g_s \left(\frac{R}{r} - 1 \right), \quad (6.3)$$

where $\Gamma_1 \approx 5/3$ was assumed to be constant (e.g. Christensen-Dalsgaard 1986). Hence the sound speed is essentially given by the total mass and surface radius of the model, which are fixed. Since the convection zone extends more deeply in the modified model, the temperature gradient is higher (being adiabatic) in this model than in the reference model just beneath the bottom of the convection zone of the latter. This is the reason for the increase in $\delta \ln T$ and $\delta \ln c$ with increasing depth just beneath the convection zone. The changes in the deep interior of the model are more

difficult to describe in simple terms. However, it should be noticed that, due to the high sensitivity of the energy generation rate on temperature, the temperature change in the core has to be small.

Scaled frequency differences between the modified model and the reference model are shown in Figure 20. These differences can be understood relatively simply in terms of the differences in sound speed shown in Figure 19, by using equation (3.19) or (3.22), and the behaviour of the turning point illustrated in Figure 3. At very low degree the modes penetrate almost to the centre, and the frequency change is given by the weighted average in equation (3.22), which is dominated by the region of positive δc beneath the convection zone. As l increases to 10 the turning point moves out through the region of slightly negative δc near the core, and $Q_{\nu} \nu_{nl}$ increases. At higher degrees, beginning at low frequency for $l = 20$, and at increasing ν when l increases to 50, the modes become largely confined within the convection zone. Here δc is negative and significant only near the surface; thus the frequency differences are negative, depend little on l , and are furthermore small at low frequency. It should be noticed that the negative δc near the surface has a substantial effect on the frequencies, despite its insignificant appearance in Figure 19. The reason is the weighting with c^{-1} (cf. equation (3.22)) which makes the frequencies very sensitive to changes in the model near the surface.

To illustrate the effect of a presumably realistic change in the opacity, we finally consider the difference between the reference model, computed with the Cox & Tabor (CT; 1976) opacity, and a model computed with Los Alamos Opacity Library tables (LAOL; Huebner *et al.* 1977), combined with a separate opacity calculation for conditions corresponding to the solar atmosphere (Cox *et al.* 1989). Figure 21 shows the differences between these models. The LAOL opacities are considerably higher in the solar core than the CT values, and as a consequence the initial hydrogen abundance required to obtain the proper luminosity is decreased from 0.732 in the CT model to 0.694 in the LAOL model. On the other hand the opacity in the LAOL tables at the temperatures corresponding to the base of the convection zone is smaller than the CT values, leading to a shallower convection zone in the LAOL model, and consequently to the negative $\delta c/c$ in the region just below the convection zone. The scaled frequency differences shown in Figure 22 reflect the difference in sound speed. At high degree the behaviour is similar to that shown in Figure 18, since it is dominated by the opacity increase in the atmosphere. However, the sound-speed variation at the base of the convection zone causes a substantial variation in $\delta \nu$ with degree at degrees between 20 and 40.

It should be pointed out that from inversion of the frequencies of the five minute oscillations the depth of the solar convection zone has been determined as 0.287 R (Christensen-Dalsgaard, Gough & Thompson 1990; see also Chapter 14). This is significantly greater than the depth of 0.267 R obtained in the LAOL model. Thus there are indications that the LAOL opacities are actually somewhat too small at temperatures of a few million degrees. A similar conclusion was reached by Cox *et al.* (1989); Korzenik & Ulrich (1989) also found an opacity increase in this temperature interval in one of their inversions for the opacity.

ii) Effects of partial mixing.

A strong assumption in normal computations of solar models is the absence of material mixing except in the outer convection zone. As a result, the variation of the hydrogen abundance X with position is solely determined by the rate of nuclear reactions. There is an evident interest in considering the effects on the oscillation frequencies of relaxing this assumption.

Several different instabilities, mostly associated with solar rotation, have been identified that may cause some degree of mixing of the solar core (for reviews, see for example Zahn 1983, Spruit 1984, as well as Chapter 7). However, it appears difficult to achieve substantial mixing across the gradient in molecular weight that arises due to nuclear burning. Without going into such details, Schatzman *et al.* (1981) considered the effects of an assumed "turbulent diffusion", characterized by a so-called critical Reynolds number Re^* which they took to be constant. Lebreton & Maeder (1987) later considered the, presumably more realistic, case of a non-constant Re^* , taking into account the stabilizing effect of the molecular weight gradient.

Regardless of their physical plausibility, the Schatzman *et al.* models are interesting as examples of the consequences of fairly major modifications to the hydrogen abundance profile. To make a careful comparison with the normal model considered here, we estimated the hydrogen profile $X(q)$, q being the mass fraction, for the $Re^* = 100$ model shown in Figure 2 of Schatzman *et al.*, and recomputed the model by scaling this $X(q)$ to get the proper solar luminosity. The procedure was discussed in more detail by Christensen-Dalsgaard (1986); relative to that work we have used a slightly modified $X(q)$, taking particular care to obtain reasonable behaviour near the solar centre.

Differences between the partially mixed and the normal model are shown in Figure 23. The dominant effects are evidently in the core, where the hydrogen abundance is increased by the mixing near the centre, leading to a decrease in the mean molecular weight and consequently to an increase in the sound speed (*cf.* equation (2.32)). This trend is partially reversed in the outer parts of the core. Also, since the convection zone is slightly less deep in the mixed model, there is a jump in the sound-speed difference near the base of the convection zone, similar to that found in Figure 19, although of opposite sign.

The scaled frequency differences, shown in Figure 24, correspond closely to the difference in sound speed. Only the lowest-degree modes penetrate into the region of increased sound speed in the core; this effect, however, is compensated by the negative sound-speed difference in the rest of the radiative interior, which dominates for modes of somewhat higher degree. For degrees between 20 and 50 there is again, as in Figure 20, a transition to modes that are essentially trapped in the convection zone, and for which the frequency differences are dominated by the small region of positive sound-speed difference near the surface.

6.2 The frequency separation between low-degree modes.

It was argued in section 3.2 that for low-degree modes the frequency difference $\delta\nu_n = \nu_n - \nu_{n-1} + \frac{1}{2}$

is a measure of conditions in the solar core. To illustrate this, we consider the quantity

$$d_n = \frac{3}{2l+3} \delta\nu_n, \quad (6.4)$$

which, according to equation (3.29), is expected to depend on frequency roughly as ν^{-1} and to be independent of l . Figure 25 shows d_n for $l = 0$ and 1, both for the normal and the partially mixed model, as well as for the observed frequencies compiled by Duvall *et al.* (1988). Similar results were presented by Fossat (1985), Palé *et al.* (1986) and Cox *et al.* (1989). It is evident that, although decreasing with increasing ν , d_n does not precisely follow the expected ν^{-1} dependence. Also, there is a significant difference between the values at $l = 0$ and 1 for the normal model. In fact, as argued in section 3.2, the asymptotic theory is not sufficiently accurate to represent the finer details of the low-degree frequencies; in particular the effect of the gravitational potential depends strongly on both frequency and degree, and has a significant effect on d_n . On the other hand there is a substantial difference between the results for the normal and the partially mixed model. Faulkner, Gough & Vahia (1986) showed that this is entirely consistent with the asymptotic expression (3.29): in the normal model there is a considerable increase in c with r in the core of the model, due to the increase in the hydrogen abundance and the resulting decrease in the mean molecular weight; according to equation (3.29) this gives rise to a negative contribution to $\delta\nu_n$ in the partially mixed model the variation in X is smaller, c is everywhere a decreasing function of r , and consequently $\delta\nu_n$ is higher. Also, it should be noticed that for the normal model d_n is roughly consistent with the observed values (except at the highest frequencies, where the observational errors are rather large); this, however, is certainly not the case for the mixed model. Thus mixing as severe as that proposed by Schatzman *et al.* appears to be ruled out by the observed frequencies (see also Cox & Kidman 1984; Provost 1984; Christensen-Dalsgaard 1986; Berthomieu & Provost 1987).

It is evident from Figure 25 that there is still considerable scatter in the observed values of d_n . For this reason it is common to carry out comparisons between theory and observations in terms of an average frequency separation, such as the quantity D_0 (*cf.* equation (3.30)), which has been measured with considerable precision (*e.g.* Jiménez *et al.* 1988; Gelly *et al.* 1988). However, since the asymptotic relation (3.29) is not exact, the value of D_0 depends on how it is obtained. Here we use a least squares fit (*cf.* Scherrer *et al.* 1983; Christensen-Dalsgaard 1988a) to the frequencies of modes of degree 0 - 3, including those for which $17 \leq n + \frac{1}{2}l \leq 29$, corresponding in frequency to the range between approximately 2500 and 4100 μHz . For each value of l the frequencies are fitted to a polynomial of second order in $n + \frac{1}{2}l - 23$; the coefficients in this polynomial are then in turn fitted to linear functions of $l(l+1)$, which leads to a determination of D_0 , as well as an average value $\Delta\nu_0$ of the overall frequency spacing $\Delta\nu$ (*cf.* equation (3.26)).

We have applied this analysis to the models discussed in the preceding section, to a model utilizing the so-called MHD equation of state (for Mihalas, Hummer & Däppen; *cf.* Hummer & Mihalas 1988; Mihalas, Däppen & Hummer 1988; Däppen *et al.* 1988; see also Chapter 4), and to a set of observed frequencies. The results are shown in Table II. This immediately confirms that the frequency spacing, as reflected in D_0 , is too large in the Schatzman *et al.* model. For the

remaining models there is some scatter in the values of D_0 , the general tendency being that the computed values are slightly higher than the value obtained from the observations. It should be noted, however, that this tendency is not evident in the comparison of $d_{n,l}$ shown in Figure 25; thus it may to some extent be caused by effects of the fit, possibly related to systematic errors in the dependence of the observed frequencies on l .

It has been pointed out (e.g. Steigman *et al.* 1978; Spergel & Press 1985; Faulkner & Gilliland 1985) that the presence in the solar core of a very small population of hypothetical “weakly interacting massive particles” (WIMPs) could contribute to the energy transport in the core and hence lower the central temperature. The reduction in the central temperature, which occurs without substantial modifications in the composition profile, would also lead to a reduction of the sound speed in the core, and hence, according to equation (3.31), to a reduction in D_0 (Faulkner *et al.* 1986; Däppen, Gilliland & Christensen-Dalsgaard 1986; Gilliland & Däppen 1988). By choosing the parameters for the WIMPs appropriately it is possible to construct models for which the predicted neutrino capture rate is consistent with the observed values. In these models, D_0 is typically reduced by 8 – 15 per cent relative to the corresponding normal models. Faulkner *et al.*, Däppen *et al.* and Gilliland & Däppen argued that this apparently led to an improvement in the agreement between the computed and the observed values of D_0 . In fact, the computed values presented in Table II are somewhat smaller than the values for the normal models used by those authors, the difference being probably due to an improvement in the numerical precision of the model and the frequency computations (cf. Section 5). Although we have not attempted to compute models with WIMPs with comparable precision, it appears likely, based on the changes in D_0 induced by the WIMPs in the earlier calculations, that for such models D_0 would be significantly lower than the observations. Certainly there is no evidence in the present results that modifications of the “standard” model are required to bring theory and observations into agreement on D_0 . The same conclusion was reached by Cox *et al.* (1989) on the basis of detailed model calculations.

The computed values of $\Delta\nu_0$ are generally somewhat higher than the observations. Indeed, for these models the differences between the observed and the computed frequencies at all degrees have a general negative slope as functions of frequency (e.g. Christensen-Dalsgaard & Gough 1984; Christensen-Dalsgaard, Däppen & Lebreton 1988; see also Chapter 4), which is probably associated with errors in the outermost layers of the model; this evidently corresponds to $\Delta\nu_0$ being too large in the models. When the opacity is artificially increased in the surface layers, or when the LAOL opacities are used, the computed $\Delta\nu_0$ is reduced to close to the observed value. As shown in Chapter 4, this also reduces the general frequency dependence of the differences between observations and theory; it should be recalled, however, that there are other serious uncertainties in the treatment of the surface layers, which could affect $\Delta\nu_0$ at this level. In any case it is apparent from these results that $\Delta\nu_0$ is predominantly a measure of conditions in the outer parts of the Sun; this is consistent with the asymptotic expression (3.27), since much of the contribution to the integral comes from near the surface, where the sound speed is low (see also Gough 1986).

6.3 Analysis of g-mode frequencies.

Just as in the case of p modes, it is convenient to analyze computed (or, when available, observed) g-mode frequencies in terms of their asymptotic properties, discussed in section 3.3. In this section we consider to what extent the asymptotic relation for the periods approximates computed periods, and how the parameters of this relation can be determined from a set of periods. To do so we analyze a set of computed g-mode periods for a normal model. Furthermore, results are presented for modified models, to illustrate the sensitivity of the g-mode parameters to the properties of the model.

Estimates of the coefficients P_0 , V_1 , V_2 and θ can be obtained by means of least squares analysis of sets of computed periods. A least squares fit of equation (3.35) to periods of gravity modes at a given degree l , with moderate or large radial order, gives values of P_0 , θ , and $V_1 = L^2 V_1 + V_2$. Given V_1 for several values of l , values of V_1 and V_2 are obtained from a separate fit. In our analysis, we mostly assume θ to be constant; this evidently limits the accuracy of the determination.

The asymptotic formula is valid in principle only for sufficiently large periods; hence there is a lower limit for the period below which the higher order terms, as well as the frequency dependence of θ , are too large to be neglected. Moreover, to allow the determination of the second order coefficient $V_1 = L^2 V_1 + V_2$ (or V_1 and V_2), the relative departure of P from the first order linear behavior must be sufficiently large relative to the numerical error (or noise in the case of observed periods). This is well satisfied for the computed modes which we consider.

A final limitation of the asymptotic analysis is that it is valid only in the Cowling approximation. In fact, as shown in Figure 26, the effect of the perturbation in the gravitational potential on the period is proportional to $1/P$ and is much smaller than the second order terms. Thus it is reasonable to apply the asymptotic expression to periods computed from the full set of equations, or to observed periods. This is in contrast to the p-mode case, where the effect of the perturbation in the gravitational potential for low-degree modes is comparable with the second-order term in the asymptotic expansion, as measured, for example, by the quantity D_0 (cf. Figure 14).

Since the asymptotic expression is not exactly satisfied, the values of P_0 , V_1 , V_2 and θ derived from least square analysis depend on the set of periods which is used. Thus we need to investigate the sensitivity of the results on the mode selection. We consider sets of modes with degrees $l = 1, \dots, 4$, for various ranges $n_{\min}$ to $n_{\max}$ in the radial order at each l . This gives a collection of estimates of P_0 , V_1 , V_2 and θ ; their scatter around the mean values gives an indication of the extent to which the asymptotic behaviour is satisfied. We have used three different choices of ranges:

Mode set (i): Vary $n_{\min}$ from 10 to 15 and $n_{\max}$ from 27 to 35 for all l ; the resulting mean values of P_0 , V_1 , V_2 and θ are given in Table III, in the line labelled “normal model (*)”.

Mode set (ii): $n_{\min}$ between 10 and 15 and $n_{\max}$ between 27 and 35 for $l = 1$; $n_{\min}$ between 10 and 15 and $n_{\max}$ between 51 and 59 for $l = 2, 3, 4$. In this way the period ranges are more

nearly the same for all l . The resulting mean values of the parameters are labelled "normal model (**)" in Table III.

Mode set (iii): n_{min} between 16 and 20 and n_{max} between 30 and 35 for $l = 1$; n_{min} between 30 and 35 and n_{max} between 51 and 59 for $l = 2, 3, 4$. The resulting mean values of the parameters are labelled "normal model" in Table III.

The quantities P_0 and V_1 are very similar in all three analyses. V_2 and θ are similar for mode sets (ii) and (iii), θ being close to the expected value of $1/6$, whereas they are lower than for mode set (i) by about 8 per cent and 25 per cent respectively. Finally we find that the scatter of the values V_2 and θ are smaller for mode set (iii) than for mode set (ii). Unless otherwise noted mode set (iii) is therefore used for all the other computations reported in this section.

The uncertainty in the estimates of the quantities P_0 , V_1 , V_2 and θ can be visualized by inspecting the histograms of the relative departure of the results from their mean values. They are presented in Figure 27 for a normal model, for an analysis using mode set (iii). This shows that the two global quantities P_0 and V_1 are insensitive to the precise set of modes selected. Moreover the values obtained from the fit are in reasonable agreement with the values $P_0 = 35.56$ minutes and $V_1 = 0.402$ computed from the expressions (3.36) and (3.37).

It should be noted that the coefficient V_1 is sensitive to the description of the very centre of the model, in particular to the smoothness of quantities like the gradient of molecular weight, which affects the behaviour of N . When the physical quantities are well-behaved near the centre, the value of V_1 obtained by this analysis is insensitive to the precise set of periods considered; otherwise there may be a considerable scatter.

The results obtained for V_2 are scattered by about 4 per cent around the mean value 5.7. For θ the scatter is 7 per cent around 0.167 (which is close to the asymptotic limit of $1/6$). As noted above it is substantially smaller than for mode sets including modes of lower radial order. The latter result is due to the variation of θ with frequency and degree. For example, the scatter in V_2 and θ is much larger for mode set (i); in this case the mean value of θ , 0.215, is also quite far from the asymptotic limit. Good results for the fit have been obtained by Ellis (1987, 1988), for modes of lower radial order, but utilizing a simplified model with a polytropic convection zone where $N = 0$. In this case, it can be shown that the variation of θ with ω is not as large (Provost & Berthomieu 1986). Ellis (1988) also studied the effect on the fit of noise in the periods; the conclusion was that the second-order parameters could only be faithfully estimated for a relative noise level below 10^{-4} .

We may also consider, for each value of the degree l , the set of estimates obtained for θ when the range in order is varied, as well as the resulting mean value θ_l which depends on l . This is illustrated in Figure 28a, for mode set (ii), and in Figure 28b, for mode set (iii). Contrary to what might have been expected, the scatter around the mean is generally larger than in the case where all degrees were combined. The dispersion around θ_l is due partly to the dependence of θ on ω , although there may also be a contribution from third order terms. For $l = 1$ the scatter is smaller for mode set (iii), where the radial order is higher; however, somewhat surprisingly, this trend is

reversed for higher l . Also we note that, contrary to expectations based on equation (3.40), the values of θ_l generally decrease with increasing l ; this behavior is less evident for higher values of n_{min} as shown in Figure 28b. There is no obvious correlation between this behaviour and the scatter in the combined case. It is evident that further work is required to interpret these results. However, they indicate that it is necessary to deal with high radial order modes of different degrees to obtain good determinations of the asymptotic constants.

To test the effect of the Cowling approximation, a least squares analysis was performed with a set of modes computed in this approximation. As indicated in Table III this gives essentially the same values of P_0 and V_1 while the values of V_2 is lower by about 9 per cent. Hence the perturbation in the gravitational potential has a moderate effect on the asymptotic fit. This is evidently consistent with the comparison made in Figure 26.

We have considered the effects on the g-mode parameters of the various modifications to the model which were also considered in the preceding sections. The use of the MHD equation of state (cf. section 6.1) leads to a slightly higher value of P_0 , with little change in the remaining parameters. The model with artificially enhanced interior opacity is very close to the normal one, whereas using the LAOL opacities leads to a small decrease in P_0 , and to slight increases in V_1 and V_2 . Thus, it appears that modifications to the physics, within the framework of "standard" models, have little effect on the properties of the g modes.

On the other hand P_0 and V_1 are very sensitive to the behavior of the composition, through its effect on the Brunt-Väisälä frequency, near the centre of the Sun (Berthomieu, Provost & Schatzman 1984a,b). Thus observational determination of these quantities would provide two constraints on the deep interior of solar models. This is potentially very important in allowing a distinction to be made between the different non-standard models which have been constructed to satisfy the neutrino constraint. As an example Table III shows results for the partially mixed model, approximating the $Re^* = 100$ model of Schatzman *et al.* (1981), which was discussed in section 6.1 (cf. Figure 23). Here P_0 is increased to 55 minutes, whereas V_1 is very substantially reduced. These results are related to the behaviour of the quantity N/x which enters in the determination of P_0 and V_1 in equations (3.36) and (3.37): the central value decreases from about 90 in the normal model to about 28 in the partially mixed one. Moreover, V_2 is much lower in the partially mixed model, while θ , which is predominantly determined by conditions near the base of the convection zone, is barely affected.

The value of V_1 is extremely sensitive to the details of the assumed hydrogen abundance profile near the centre. We have found that slight variations in the $X(q)$ which was fitted to the Schatzman *et al.* curve resulted in substantial changes in V_1 . This can in fact be understood from the asymptotic expression in equation (3.37), since V_1 is computed as the difference between two quantities that are singular at the centre. Thus an observational determination of V_1 would provide a very sensitive probe of the evolutionary history of the Sun.

Models with WIMPs (cf. section 6.2) have a quasi-isothermal central core; relative to a normal model this increases N in the core, and hence the value of P_0 is lower. Faulkner *et al.* (1986) and

Däppen *et al.* (1986) obtained values around 29 minutes.

Considerable uncertainty surrounds the region just beneath the convection zone. This includes the extent of the overshooting region, and the possible presence of a magnetic field concentration. Thus observational information about this region is badly needed. The asymptotic value of θ is directly a measure of ζ and thus of the variation of the temperature gradient. In all the models considered here, N^2 varies linearly with r just below the convection zone, and hence $\zeta = 1$. Despite the uncertainty in the determination of θ from the computed periods, the estimates are in reasonable agreement with the predicted value of $1/6$, whereas they are much further from, for example, the value of 0.125 which would correspond to a parabolic behavior of N^2 at the lower boundary of the convection zone. It would be of obvious interest to consider models with a different behaviour of N^2 in this region, due to, for example, the presence of a magnetic field.

7. Conclusion.

The principal feature of the theory of adiabatic solar oscillations is its simplicity. For a given solar model it is straightforward to compute oscillation frequencies with a precision that at least approaches the precision attainable in the observed frequencies. Furthermore, it is possible to relate the behaviour of the frequencies, as functions of mode degree and order, to the properties of the solar interior. It is these features that make the oscillation frequencies such powerful probes of the solar interior.

It is true that the theory of solar oscillations, as presented here, is oversimplified. We have neglected effects of nonadiabaticity and other processes that damp or excite the modes; these may change the frequencies by several microherz (e.g. Christensen-Dalsgaard & Frandsen 1983; Cox *et al.* 1989). Furthermore, there are bound to be dynamical interactions between the oscillations and convection in the uppermost layers of the convection zone, where the convective timescale is comparable with the oscillation period, and where the Mach number of convection is not small. Finally the inhomogeneity of the solar atmosphere, and the presence of magnetic field concentrations, may well have significant effects on the frequencies. However, for the study of the solar interior these effects may to a first approximation be eliminated. They occur in a region whose vertical extent is small compared with the horizontal wavelength of most of the modes observed, and where these modes propagate predominantly vertically. Hence they affect all modes of a given frequency equally, regardless of the degree l , provided that the l -dependence of the mode inertia is taken into account. Thus the uncertainties near the solar surface translate into an undetermined frequency-dependent component of the frequencies, which is common to all modes; if allowance is made for this, the dependence of the frequencies on l can be used to study how the properties of the solar interior depend on radius.

Irrespective of this, it is evidently important to improve our understanding of the uppermost layers of the Sun and their influence on the frequencies. Some aspects of this problem are discussed in Chapter 16. Encouraging progress has been made recently both in the application of simplified

models of convection to the study of oscillations (Balmforth & Gough 1988, 1990), and in detailed hydrodynamical simulations of granulation (Stein & Nordlund 1989; Stein, Nordlund & Kuhn 1988, 1989). Eventually we may hope to use observations of solar oscillation frequencies, line widths, amplitudes and phase relations to test models for the excitation of oscillations and for the interaction between oscillations and convection. This would be of obvious benefit to studies of other types of pulsating stars.

Within the framework of "standard" solar evolution theory, the structure of the model, and hence the oscillation frequencies, are determined by the assumed physics of the solar interior, as well as by the global parameters of the model. Thus in this sense the frequencies can be regarded as providing information about the physics. As such, they are sensitive measures. It was demonstrated in Section 6 that changes in opacity that are well within the range of uncertainty in current opacity calculations lead to frequency changes which are large compared with the internal precision in the measurement or the calculation of the frequencies, and which have a recognizable signature. Similarly it has been found that improvements in the equation of state may have substantial effects on the frequencies, generally leading to an improved agreement between theory and observation (Christensen-Dalsgaard, Däppen & Lebreton 1988; see also Chapter 4). Thus it appears that, particularly with the vastly improved observational data that can be expected in the coming decade, we may be able to use the Sun as a laboratory for testing our ideas about the properties of matter. This is of obvious importance for the application of these properties to the computation of models of other stars, but it may also provide basic physical insight into the behaviour of hot and dense plasmas.

Of course this assumes that the Sun is similar to a normal solar model. It is possible that a combination of non-standard features could mimic an error in the physics of a standard model, and hence fool us completely. More likely, however, substantial departures from the standard assumptions would be recognizable in the observed frequencies. For instance, departures from a spherically symmetric structure would give rise to a definite signature in the m -dependence of the frequencies. Nevertheless, when interpreting the observed data we should evidently keep in mind that the Sun may have surprises in store for us.

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Table I.

	c_1	c_2	c_3
p-mode mesh	10.	0.01	0.015
g-mode mesh	0.025	0.1	0.0001

Parameters in equation (5.12) for the determination of meshes suitable for computing p and g modes, respectively, in a model of the present Sun.

Table II.

Frequencies	$\Delta\nu_0$	D_0
Normal model	136.46 μHz	1.511 μHz
MHD equation of state (<i>cf.</i> Chapter 4)	136.68 μHz	1.519 μHz
Opacity increase at surface	135.70 μHz	1.505 μHz
(<i>cf.</i> Figure 17)		
Opacity increase in interior	136.65 μHz	1.481 μHz
(<i>cf.</i> Figure 19)		
LAOL opacity	135.24 μHz	1.496 μHz
(<i>cf.</i> Figure 21)		
Schatzman <i>et al.</i> mixed model	136.68 μHz	1.976 μHz
(<i>cf.</i> Figure 23)		
Duvall <i>et al.</i> observed frequencies	135.15 μHz	1.487 μHz

Average frequency separations (*cf.* equations (3.26) and (3.30)) for a number of solar models, as well as for the compilation of observed frequencies by Duvall *et al.* (1988).

Table III

Model	P_0	V_1	V_2	θ
Normal model (*)	35.58	0.428	6.18	0.215
Normal model (**)	35.56	0.432	5.71	0.169
Normal model	35.55	0.441	5.68	0.167
Normal model	35.55	0.441	5.25	0.167
with Cowling approx.				
MHD equation of state	35.95	0.437	5.56	0.168
Opacity increase	35.54	0.435	5.58	0.161
in interior				
LAOL opacity	35.45	0.458	6.06	0.162
Schatzman <i>et al.</i>	54.36	0.173	3.78	0.157
partially mixed model				

Asymptotic g-mode parameters obtained from least-squares fitting to computed periods for modes of degree 1 – 4. The cases “normal model (*)” and “normal model (**)” used mode sets (i) and (ii), respectively, whereas for the remaining cases mode set (iii) was used.

Figure captions:

Figure 1. Adiabatic oscillation frequencies for a normal model of the present Sun, as functions of the degree l . For clarity points corresponding to modes with a given radial order have been connected by straight lines. Only g modes with radial order less than 40 have been included.

Figure 2. (a) Characteristic acoustic frequency S_l (cf. equation (2.33); -----), labelled by the values of l , buoyancy frequency N (cf. equation (2.34); —————), and acoustical cut-off frequency ω_{co} (cf. equation (3.5); - - - - -), shown in terms of the corresponding cyclic frequencies, against fractional radius r/R . The heavy horizontal lines indicate the trapping regions for a g mode with frequency $\nu = 100 \mu\text{Hz}$, and for a p mode with degree 20 and $\nu = 2000 \mu\text{Hz}$. (b) Acoustical cut-off frequency in the region where the p modes are reflected. Illustrated are ω_{co} (-----), as well as the Lamb acoustical cut-off frequency $\omega_{co,p}$ for an isothermal atmosphere (cf. equation (3.7); - - - - -). The thin continuous line shows $N/2\pi$ in the solar atmosphere.

Figure 3. The turning point radius r_t (a) and the penetration depth $R - r_t$ (b), in units of the solar radius R , as a function of degree l for three values of the frequency ν . The curves have been calculated from equation (3.11) for a normal model of the present Sun. In b) the curves terminate at the degree where the frequency equals the f-mode frequency; for p modes the degree is below this value at a given frequency (see also Figure 1).

Figure 4. Schematic illustration of the propagation of sound waves in a star. Due to the increase of the sound speed with depth, the deeper parts of the wave fronts move faster. This causes the refraction of the wave described by equation (3.16). Notice that waves with a smaller wavelength, corresponding to a higher value of the degree l , penetrate less deeply.

Figure 5. Observed frequencies of solar oscillations, plotted according to the Duvall relation (3.17). (Note that $\omega/k_k = \omega R/L$ has been used as abscissa instead of ω/L). The value 1.58 of the constant α was obtained by minimizing the spread of the points around the mean curve. (From Christensen-Dalsgaard *et al.* 1985).

Figure 6. Effective Duvall phase α_{nl} (cf. equation (3.24)); here L has been replaced by $L_0 = l + 1/2$ for modes of a normal solar model. Each curve corresponds to a fixed value of l ; the curves are for

$l = 0 - 5, 10, 20, 30, 40, 50, 70$ and 100 , with α_{nl} increasing with l at fixed frequency.

Figure 7. Plot of the g-mode periods (multiplied by L) of a normal solar model as a function of $|n| + l/2 - 1/4 - \theta$, with $\theta = 1/6$, for $l = 1$ (•) and $l = 4$ (★). The straight line depicts the linear approximation. The two hyperbolic curves indicate the asymptotic approximation up to the second order.

Figure 8. Eigenfunctions for p modes, with a) $l = 0, n = 23, \nu = 3310 \mu\text{Hz}$; b) $l = 20, n = 17, \nu = 3373 \mu\text{Hz}$; c) $l = 60, n = 10, \nu = 3233 \mu\text{Hz}$. The arrows mark the asymptotic location of the turning points r_t (cf. Figure 3).

Figure 9. Eigenfunctions of p modes with $l = 1$, plotted against fractional radius r/R in the outermost parts of the Sun. The cases shown are: $\nu = 1604 \mu\text{Hz}$ (-----) $\nu = 2286 \mu\text{Hz}$ (-----) $\nu = 3653 \mu\text{Hz}$ (—————).

Figure 10. Eigenfunctions for g modes, with a) $l = 1, n = -5, \nu = 108 \mu\text{Hz}$; b) $l = 2, n = -10, \nu = 101 \mu\text{Hz}$; c) $l = 4, n = -18, \nu = 103 \mu\text{Hz}$.

Figure 11. Normalized inertia E_{nl} (cf. equation (4.1)) for f and p modes of the present Sun, as function of the cyclic frequency ν . For clarity points corresponding to modes with a given degree l have been connected. Selected curves are labelled with l .

Figure 12. Normalized inertia E_{nl} (cf. equation (4.1)) for f and g modes of the present Sun, plotted against frequency ν . For clarity points corresponding to modes with a given degree l have been connected. Selected values of l are indicated in the figure.

Figure 13. The inertia ratio Q_{nl} , defined in equation (4.27), against frequency ν , for f and p modes in a normal solar model. Each curve corresponds to a given degree l , selected values of which are indicated.

Figure 14. (a) Differences between frequencies $\nu^{(C)}$ computed in the Cowling approximation and

frequencies $\nu^{(F)}$ resulting from the full equations, for p modes, plotted against frequency. All modes with degree $l = 0 - 20$ are shown. Each curve corresponds to a given value of l , which has been indicated on the first few curves.

(b) Relative differences between $\nu^{(C)}$ and $\nu^{(F)}$ for g and f modes, plotted against mode order. Each curve corresponds to a given value of l , which has been indicated.

Figure 15. Errors in computed frequencies, estimated as the difference $\nu^{(600)} - \nu^{(2400)}$ between computations using 600 and 2400 mesh points. The results in panels (a) – (c) were obtained with the centred difference algorithm in equation (5.2), whereas panel (d) shows results of using the constant coefficient technique described by equation (5.3). Each curve corresponds to a given value of l . *Heavy lines:* $l = 0$ (—); $l = 1$ (-----); $l = 20$ (-----); $l = 50$ (-----); $l = 100$ (-----). *Thin lines:* $l = 150$ (-----); $l = 200$ (-----); $l = 500$ (-----).

(a) Straight eigenvalues. (b) Eigenvalues corrected with Richardson extrapolation (cf. equation (5.7)). (c) Variational frequencies. (d) Straight eigenvalues with the constant coefficient technique; modes with $l = 1$ are not included.

Figure 16. Differences between the frequencies $\nu^{(R)}$ obtained from Richardson extrapolation and the variational frequencies $\nu^{(V)}$; the differences have been scaled by the inertia ratio Q_{ni} (cf. equation (4.27)). The frequencies were computed on a mesh with 2400 points. The line styles are the same as in Figure 15. (a) Results for Model 1 of J. Christensen-Dalsgaard (1982). (b) Results for a model computed with higher numerical accuracy.

Figure 17. Differences, at fixed fractional radius r/R , between a model of the present Sun where the opacity has been artificially increased by up to about a factor 2 near the surface and a normal model, in the sense (opacity modified) – (normal). The variables shown are $\delta \ln c$ (—), $\delta \ln \rho$ (-----), $\delta \ln T$ (-----), and $\delta \ln \Gamma_1$ (-----).

Figure 18. Differences between the frequencies $\nu_{ni}^{(2)}$ of the model with modified surface opacity and the frequencies $\nu_{ni}^{(1)}$ of a normal model, for selected values of l . The differences have been scaled by the inertia ratio Q_{ni} (cf. equation (4.27)). Points corresponding to a given value of l have been connected, according to the following convention: $l = 0, 5, 10, 20, 30$ (—); $l = 40, 50, 70, 100$ (-----); $l = 150, 200, 300, 400$ (-----); and $l = 500, 600, 700, 800, 900, 1000$ (-----).

Figure 19. Differences, at fixed fractional radius r/R , between the model with modified opacity in the interior and the reference model, in the sense (modified model) – (reference model). The lines have the following meaning:

Thin lines: $\delta \ln q$, where $q \equiv m/M$ is the mass fraction (—); $\delta \ln L$, where L is the luminosity at r (-----); $\delta \ln \tau$ (-----); δX , where X is the hydrogen abundance (-----). *Heavy lines:* $\delta \ln c$ (—); $\delta \ln \rho$ (-----); $\delta \ln T$ (-----).

Figure 20. Scaled frequency differences between the model computed with modified opacity in the interior, (2), and the reference model (1), for selected values of l . Points corresponding to a given value of l have been connected, according to the following convention: $l = 0, 5, 10, 20, 30$ (—); $l = 40, 50, 70, 100$ (-----); $l = 150, 200, 300, 400$ (-----); and $l = 500, 600, 700, 800, 900, 1000$ (-----). In addition a few values of l have been indicated in the figure.

Figure 21. Differences, at fixed fractional radius r/R , between calibrated models of the present Sun computed using the Los Alamos Opacity Library and the Cox & Taber tables, in the sense (LAOL) – (CT). The variables plotted and the line styles are as in Figure 19.

Figure 22. Scaled frequency differences between the LAOL model (2) and the CT model (1) illustrated in Figure 21, for selected values of l . Points corresponding to a given value of l have been connected, according to the following convention: $l = 0 - 5, 10, 20, 30$ (—); $l = 40, 50, 70, 100$ (-----); $l = 150, 200, 300, 400$ (-----); and $l = 500, 600, 700, 800, 900, 1000$ (-----).

Figure 23. Differences, at fixed fractional radius r/R , between the model with a partially mixed core and the reference model, in the sense (modified model) – (reference model). The variables plotted and the line styles are as in Figure 19.

Figure 24. Scaled frequency differences between the model with partially mixed core (2) and the reference model (1), for selected values of l . Points corresponding to a given value of l have been connected, according to the same convention as in Figure 20.

Figure 25. Normalized separation d_{ni} (cf. equation (6.4)) between frequencies of low-degree

modes. Curves are shown for $l = 0$ (—) and $l = 1$ (-----). The heavy lines show results for theoretical frequencies. The lower pair is for a normal solar model, and the upper pair for a partially mixed model. The thin lines are based on observed frequencies, from Duvall *et al.* (1988).

Figure 26. Relative changes in the periods as a function of the radial order n describing the effect of the perturbation of the gravitational potential (lower set of points), and the importance of the second order terms in asymptotic formula (3.35) (upper set of points). The symbols indicate the degrees of the modes: $l = 1$ (Δ); $l = 2$ ($\bullet$); $l = 3$ ($\star$); $l = 4$ ($\circ$).

Figure 27. Determination of (a) P_0 , (b) V_1 , (c) V_2 and (d) θ by least squares analysis from computed g-mode periods. For all these quantities, the ordinate represents the percentage of values falling in a given range, when performing a least squares analysis with n_{max} varying from 30 to 35 and n_{min} from 16 to 20 for $l = 1$ and n_{max} from 50 to 60 and n_{min} from 30 to 35 for $l = 2, 3, 4$, and different weighting functions. The abscissa represents the relative departure from the mean values: $P_0 = 35.55$ minutes, $V_1 = 0.44$, $V_2 = 5.7$ and $\theta = 0.167$.

Figure 28. Determination of θ_l by least squares analysis from computed g-mode periods of given degree l , for the normal solar model: (a) mode set (ii); (b) mode set (iii). The percentage of values falling in a given range, when performing a least squares analysis varying n_{max} , n_{min} is plotted relatively to the value of θ and shows a decrease of the mean value θ_l with l .

Fig. 1

Fig. 2a

Fig. 2b

Fig. 3a

Fig. 3b

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 11

Fig. 10

Fig. 12

Fig. 13

Fig. 14a

Fig. 14b

Fig. 15a

Fig. 15b

Fig. 15d

Fig. 15c

Fig. 16a

Fig. 16b

Fig. 18

Fig. 17

Fig. 19

Fig. 20

Fig. 21

Fig. 22

Fig. 23

Fig. 24

Fig. 25

Fig. 26

Fraction.

Departure from mean.

Fig. 27

Fraction.

Departure from mean.

Fig. 28