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MEMBRANE DYNAMICS IN THE INTRINSIC LIGHT-FRONT COORDINATES

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ABSTRACT

We study the dynamics of the membrane, using internal light-front (LF) coordinates. The set of constraints, although equivalent to the standard one, is different. The intrinsic LF gauge is defined. Four additional, alternative gauge-fixing conditions are analysed. Two of them polynomialize the system, whilst the other two are convenient for studying the initial-value problem. In particular, one of them is also extrinsically (i.e. in the ambient space) light-front. In this gauge, the system is shown to be consistently reduced to attain a canonical form in terms of pure transverse variables.

Two constraints on these variables still hold, clearly showing the presence, as they must, of $D-3$ degrees of freedom. Finally, the initial-value problem in this intrinsic-extrinsic LF gauge is solved.

Although the paper is based on the first-order action, the LF-Hamiltonian approach is discussed too.

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1. INTRODUCTION

Since the paper of Hughes, Liu and Polchinsky [1], a vast amount of research has been carried out on the subject of supermembranes as models for elementary particles. It was shown that a Green-Schwarz supermembrane couples consistently to an 11D supergravity [2] background. As a theory *per se*, it can be thought of as an alternative theory to that of string models. It can also be regarded as a master theory for understanding strings through dimensional reduction [3]. However, it is by now recognized that supermembranes are neither a straightforward generalization of superstrings nor membranes of strings.

Whilst strings are essentially linear extended objects, membranes are very non-linear. The question, for instance, of what are the massless states of the theory, is still open for supermembranes [4].

Our feeling is that the major part of the difficulties we face when dealing with supermembranes stem from their highly non-linear bosonic structure and are only peripherally related to their fermionic hair.

This is the main motivation for undertaking the analysis of the membrane. Since the membrane is a relativistic object—its action already given by Dirac [5] a long time ago—we will try to discover what it looks like when described by intrinsic light-front (LF) coordinates, because we know that relativistic systems become simpler, i.e. much less kinematically coupled, where represented by LF coordinates. This is true for D-dimensional gravity [6], strings [7], and supergravity [8].

In the case of strings, the extrinsic LF gauge was crucial for ‘solving’ the system. As there remains an additional coordinate once $x^+ \sim \tau$, there is no need to go any further with the analysis and choose also intrinsic LF coordinates. However, it is possible to do so [9] and the theory takes on a very simple aspect.

In the next section we will describe the geometry and kinematics of the intrinsic LF coordinates and introduce the standard first-order action. A new representation of the membrane constraints is found. In Section 3 we define the intrinsic LF gauge and study some geometrical properties of the residual gauge group, and check that the two constraints still present in the action form a 2D subalgebra.

Then, in Section 4, we discuss the Hamiltonian approach to membranes in the intrinsic LF gauge, which is a singular point in the context of the canonical approach. Section 5 is dedicated to discussing additional gauge fixing in the framework of the intrinsic LF gauge. Four different gauge-fixing conditions are studied. Two of them polynomialize the system; the third one turns out to be convenient for studying the initial-value problem, and the fourth one, extrinsic LF, is the one that allows us to write down the field equations in canonical form in the ambient transverse space. Then,

in Section 6, we study the dynamics and the conservation of the first-class and second-class constraints in the four gauges introduced earlier. We show how, in the double (extrinsic and intrinsic) LF gauge, the system can be described in terms of its two (D–2) canonical variables restricted with 1 + 1 constraints, in such a way that it is explicit that the membrane has D–3 independent degrees of freedom. In Section 7 we analyse the initial-value problem for the double LF and the γ -gauges. We show that in both of them the initial-value problem for the membrane is well posed. In Section 8, the last one, we discuss the results.

2. GEOMETRY AND MEMBRANE ACTION IN THE INTRINSIC LF COORDINATES

The intrinsic LF coordinates are the null coordinates induced on the 3D membrane world volume by the light cone of the ambient space-time. They are particularly well described in terms of six parameters e, N, N', m, n , and n' [10], giving for the membrane metric

$$ds^2 := g_{ij} d\sigma^i d\sigma^j = e^2 (d\sigma + N du + N' dv)^2 - 2m^2 (1 - nn')^{-2} (du - n' dv)(dv - n du), \quad (2.1)$$

where $i = (0, 1, 2) = (u, v, \sigma)$ are the three coordinates needed to represent the embedding of the membrane into space-time. From Eq. (2.1) we can see that there exists a null triad:

$$\begin{aligned} e_{+i} dx^i &:= m(1 - nn')^{-1} (dv - n du), \\ e_{-i} dx^i &:= m(1 - nn')^{-1} (du - n' dv), \\ e_{1i} dx^i &:= e(d\sigma + N du + N' dv), \end{aligned} \quad (2.2)$$

in the sense that $e_+^2 = e_-^2 = 0$, $e_+ \cdot e_- = -1$, $e_\pm \cdot e_1 = 0$, $e_1^2 = 1$. It turns out that $u = \text{const}$ ($v = \text{const}$) is a characteristic surface iff $n' = 0$ ($n = 0$). In terms of these variables, the 3D volume element is

$$(-\det g_{ij})^{\frac{1}{2}} = (1 - nn')^{-1} em^2. \quad (2.3)$$

The 3D covariant action for the membrane we start with is

$$S = -(4\pi^2 \beta)^{-1} \int d^3\sigma \{ \pi^i x_i^r - 2^{-1} g_{ij} \pi^i \pi^j (-g)^{-\frac{1}{2}} - 2^{-1} (-g)^{\frac{1}{2}} \}, \quad (2.4a)$$

where π^i , x^r , and g_{ij} are independent variables, $x^r(\sigma^i)$ belongs to the D-dimensional ambient space, which we assume throughout this paper to be flat, and β is the membrane constant. Varying π^i leads to $\pi^i = (-g)^{\frac{1}{2}} g^{ij} x_j^r$, which, if introduced into Eq. (2.4), gives the standard so-called first-order action with a cosmological term

$$S = -(8\pi^2 \beta)^{-1} \int d^3\sigma \{ g^{ij} x_i^r x_j^r - 1 \} (-g)^{\frac{1}{2}}, \quad (2.4b)$$

equivalent to the Dirac one [5]. For the sake of simplicity, we do not consider the presence of the 3D Einstein action, even if it is mandatory on physical grounds.

To analyse action (2.4a) in the intrinsic LF coordinates, we write it in terms of LF variables (2.1),

$$S = -(4\pi^2\beta)^{-1} (\pi^u \cdot x_u + \pi^v \cdot x_v + \tilde{\pi}^1 \cdot x_1 - 2^{-1}e(1 - nn')m^{-2}(\tilde{\pi}^1)^2 - ne^{-1}(1 - nn')^{-1}(\pi^u)^2 - 2^{-1}m^2e(1 - nn')^{-1} - n'e^{-1}(1 - nn')^{-1}(\pi^v)^2 + (1 + nn')e^{-1}(1 - nn')^{-1}\pi^u \cdot \pi^v), \quad (2.5a)$$

where $\tilde{\pi}^1 := \pi^1 + N\pi^u + N'\pi^v = e^{-1}e_{,1}\pi^1$ is a convenient new variable; ambient space indices have been omitted and $d_\sigma x := x_{,1}$. Independent variations with respect to it yield $\tilde{\pi}^1 = m^2e^{-1}(1 - nn')^{-1}x_{,1}$. After insertion of this value of $\tilde{\pi}^1$ into Eq. (2.5a), we obtain

$$S = -(4\pi^2\beta)^{-1} (\pi^u \cdot x_u + \pi^v \cdot x_v - N\pi^u \cdot x_1 - N'\pi^v \cdot x_1 + 2^{-1}m^2(1 - nn')^{-1}[e^{-1}x_{,1}^2 - e] - ne^{-1}(1 - nn')^{-1}(\pi^u)^2 - n'e^{-1}(1 - nn')^{-1}(\pi^v)^2 + (1 + nn')e^{-1}(1 - nn')^{-1}\pi^u \cdot \pi^v). \quad (2.5b)$$

Note that m^2 is a multiplier associated with the algebraic constraint $e^2 = x_{,1}^2$. Variations with respect to e give the value of m^2 :

$$m^2 = e^{-2}(\pi^u - n'\pi^v) \cdot (\pi^u - \pi^v). \quad (2.5c)$$

In order to guess what would be the constraints arising from the action related to the 3D-diffeomorphism gauge group, we compute the conserved, vanishing components T_j^i of the energy momentum tensor T_j^i . The Bianchi identities arise owing to the diffeomorphism invariance of the action. The T_j^u turn out to be

$$T_1^u \sim \pi^u \cdot \tilde{\pi}^1 = 0, \quad (2.6a)$$

$$T_v^u \sim e^2 N' \pi^u \cdot \tilde{\pi}^1 + m^2(1 - nn')^{-2}[2n' \pi^u \cdot \pi^v - (1 + nn')(\pi^u)^2] = 0, \quad (2.6b)$$

$$T_u^u \sim 2e^2 N \pi^u \cdot \tilde{\pi}^1 + 2m^2(1 - nn')^{-2}[n(\pi^u)^2 - n'(\pi^v)^2] + e^2(1 - nn')^{-2}m^4 - e^2(\tilde{\pi}^1)^2. \quad (2.6c)$$

Owing to the above-mentioned value of $\tilde{\pi}^1$, Eq. (2.6a) is equivalent to $\pi^u \cdot x_{,1} = 0$. It will be shown to be one of the relevant differential first-class constraints in the intrinsic LF coordinates.

If the intrinsic LF choice $n' = 0$ is made, as it will be throughout almost the rest of the paper, then $T_v^u = 0$ implies that $(\pi^u)^2 = 0$ and Eq. (2.6c) determines e ; it becomes equivalent to $e^2 = x_{,1}^2$.

It is convenient to introduce $\tilde{\pi}^v := +m^{-1}(1 - nn')\pi^1 e_{,1} = (\pi^v - n\pi^u)$. Up to an overall factor, $\tilde{\pi}^v$ is the projection of the covariant momentum density along the null vector e_+ . Action (2.5b) becomes

$$S = -(4\pi^2\beta)^{-1} (\pi^u \cdot x_u + \tilde{\pi}^v \cdot (x_v - N'x_{,1}) + n\pi^u \cdot (x_v - N'x_{,1}) - N\pi^u \cdot x_{,1} + 2^{-1}m^2(1 - nn')^{-1}[e^{-1}x_{,1}^2 - e] - e^{-1}(1 - nn')^{-1}n'(\tilde{\pi}^v)^2 + e^{-1}\pi^u \cdot \tilde{\pi}^v). \quad (2.5b')$$

We want to show that there exists an LF representation of the standard system of membrane constraints [11] which, outside the LF gauge, constitute an equivalent set. Variations of m say that $e^2 = x_{,1}^2$. Introducing this value for e^2 into the above action, varying $\tilde{\pi}^v$ and inserting its value into it, leads to

$$S = -(4\pi^2\beta)^{-1} (\pi^u \cdot x_u - N\pi^u \cdot x_{,1} + 4^{-1}e^{-1}n'^{-1}[\pi^u + eD_u x]^2 - 4^{-1}e^{-1}n[\pi^u - eD_u x]^2), \quad (2.6)$$

where the notation $D_u x := x_{,v} - N'x_{,1}$, $D_u x := x_{,u} - Nx_{,1}$ has been used.

This expression clearly shows that $\pi^u x_{,u}$ is the typical $\sim pq$ part of the Hamiltonian formulation and, in addition, that N, n, n'^{-1} ($n' \neq 0$) are Lagrange multipliers. Their respective associated constraints are one bilinear and the remaining two quadratic:

$$\pi^u \cdot x_{,1} = 0, \quad (2.7a)$$

$$(\pi^u \pm eD_u x)^2 = 0. \quad (2.7b, c)$$

This set is equivalent to the 2 + 1 inspired traditional set of constraints [11]. It is the LF representation of the 3D Virasoro algebra. In fact, variations of N' in action (2.6) yield the value

$$N' = e^{-2}x_v \cdot x_{,1} + e^{-3}(1 + nn')(1 - nn')^{-1}\pi^u \cdot x_{,1}. \quad (2.8)$$

After insertion of this value of N' in Eq. (2.7a), Eqs. (2.7b,c) turn out to be

$$(\pi^u)^2 + e^2x_{,v}^2 - (x_v \cdot x_{,1})^2 \pm 2e\pi^u \cdot x_v = 0. \quad (2.7b', c')$$

Their addition is $\propto (\pi^u)^2 + x_{,1}^2x_v^2 - (x_v \cdot x_{,1})^2$ and their difference $\propto \pi \cdot x_v$. Together with Eq. (2.7a) they are the standard membrane constraints. Looking at Eq. (2.6), it is also evident why the intrinsic LF choice $n' = 0$ has to be analysed separately.

Since this gauge choice will be heavily used in the following sections, it is worth pointing out that its infinitesimal residual invariance group (i.e. the group that leaves $n' = 0$ invariant) is

$$(u, v, \sigma) \rightarrow [u + \xi^u(u), \quad v + \xi^v(u, v, \sigma), \quad \sigma + \xi'(u, v, \sigma)]. \quad (2.8)$$

This can be seen by looking at the infinitesimal variations of $g^{uu} = -2n'm^{-2}$.

3. INTRINSIC LIGHT-FRONT GAUGE

As already mentioned, this gauge is defined by imposing $n' = 0$. It amounts to stating that the 2D world-volume surfaces, $u = \text{constant}$, are characteristics. The dynamics of the membrane is then described by action (2.5b'), where n' has been set to zero plus the additional equation corresponding to performing independent variations of n' and then setting n' equal to zero, i.e.

$$S = -(4\pi^2\beta)^{-1} (\pi^u \cdot x_u + \pi^v \cdot (D_v x + e^{-1}\pi^u) - N\pi^u \cdot x_1 + n\pi^u \cdot D^v x), \quad (3.1a)$$

and

$$(\pi^v)^2 = 0, \quad (3.1b)$$

where $\tilde{\pi}^v \rightarrow \pi^v$.

This action has an almost typical Hamiltonian structure: the first term has exactly the $p\dot{q}$ form; the second term might (and in fact it does) represent multipliers π^v associated with second-class constraints, and the last two terms represent the two remaining first-class constraints. What is less standard is the presence of Eq. (3.1b), which is a constraint on the set of multipliers.

It is amusing to see that, even if action (2.6) is singular when $n' \rightarrow \infty$, it almost leads to the right one [Eq. (3.1a)] when this limit is taken. We do independent variations of π^u, π^v in action (2.5b'), and then realize that

$$\pi^u + eD_v x = (1 - nn')^{-1} 2n' \pi^v. \quad (3.2)$$

Then, the infinitesimal behaviour when $n' \rightarrow 0$ of the singular term $\sim (n')^{-1} [\pi^u + eD_v x]^2$, is the same as $2^{-1}\pi^v \cdot (D_v x + e^{-1}\pi^u)$. In this limit, action (2.6) almost becomes Eqs. (3.1a,b), where we have used $\pi^u \sim -eD_v x + O(n')$ in order to get from $-4^{-1}e^{-1}n[\pi^u - eD_v x]^2 \rightarrow n\pi^u \cdot D_v x$. This apparent difference is physically irrelevant, since after a redefinition this 2^{-1} gets absorbed with π^v .

It is straightforward to check that the two constraints $\phi_v := \pi^u \cdot x_v$ and $\phi_1 := \pi^u \cdot x_1$ satisfy the expected 2D subalgebra in terms of the equal (null) time u Poisson brackets

$$\begin{aligned} \{\phi_v(v, \sigma), \phi_v(v^1, \sigma^1)\} &= (\phi_v(v, \sigma) + \phi_v(v^1, \sigma^1)) \partial_v \delta^2[(v, \sigma) - (v^1, \sigma^1)], \\ \{\phi_1(v, \sigma), \phi_1(v^1, \sigma^1)\} &= (\phi_1(v, \sigma) + \phi_1(v^1, \sigma^1)) \partial_1 \delta^2[(v, \sigma) - (v^1, \sigma^1)], \\ \{\phi_1(v, \sigma), \phi_v(v^1, \sigma^1)\} &= \phi_v(v, \sigma) \partial_1 \delta^2[(v, \sigma) - (v^1, \sigma^1)] \\ &\quad + \phi_1(v^1, \sigma^1) \partial_v \delta^2[(v, \sigma) - (v^1, \sigma^1)]. \end{aligned} \quad (3.3)$$

The dimensionality two of this first-class algebra stems from the fact that, by imposing $n' = 0$, u has been essentially fixed, whilst the transformations for v and σ have been left free [see Eq. (2.8)].

Independent variations of n, N, N', π^v, π^u , and x give the field equations arising from action (3.1a):

$$\pi^u \cdot D_v x = 0 = \pi^u \cdot x_1 = \pi^v \cdot x_1, \quad (3.4abc)$$

$$D_v x + e^{-1}\pi^u = 0, \quad D_u x + nD_v x + e^{-1}\pi^v = 0, \quad (3.4de)$$

$$\pi^u_{,u} - (N\pi^u)_{,1} + (\pi^v + n\pi^u)_{,v} - [N'(\pi^v + n\pi^u)]_{,1} - [e^{-3}(\pi^u \cdot \pi^v)x_1]_{,1} = 0. \quad (3.4f)$$

This set has to be completed with Eq. (3.1b). In this gauge the value of m^2 Eq. (2.5c) becomes $m^2 = -e^{-2}\pi^u \cdot \pi^v$. We have shown that Eqs. (3.4a,b) are first-class constraints in the dynamics generated by the u evolution. Equations (3.4d) and (3.4e) provide the values of π^u, π^v and, after taking into account the constraints, also the value of $2n$:

$$2n = -(D_u x \cdot D_v x)^{-1} (D_u x)^2. \quad (3.5)$$

The last equation, (3.4f), is the fundamental equation for the membrane. Equation (3.4c) looks like playing a role similar to that of (3.4b). However, it has the peculiarity that it is a differential 'constraint' in terms of the non-canonical (along the u evolution) variables π^v , similarly to what happens with the algebraic 'constraint' $(\pi^v)^2 = 0$.

A more complete understanding of these peculiarities will be conveyed in Sections 6 and 7.

It is interesting to analyse geometrically both the action (3.1) and the field Eqs. (3.4). For simplicity, we consider the infinitesimal group $u \rightarrow u^1 = u, v \rightarrow v^{-1} = v + \xi^v(u, v, \sigma)$, and $\sigma \rightarrow \sigma^1 = \sigma + \xi^1(u, v, \sigma)$ as a large part of the invariance group of $n' = 0$.

With respect to this group, x are scalars: $\delta x = \xi^v x_v + \xi^1 x_1$. The LF parameters (2.1), which determine the metric, transform in a non-geometric way, i.e.

$$\delta e = (\xi^1 e)_{,1} + (\xi^v e)_{,v} - e D_v \xi^v, \quad (3.5a)$$

$$\delta N' = \xi^v N'_{,v} + \xi^1 N'_{,1} + N' D_v \xi^v + D_v \xi^1. \quad (3.5b)$$

These (and the additional) transformation laws can be obtained using the definitions of these parameters Eqs. (2.1) and the geometric transformation laws of both the metric tensor $\delta g_{ij} = \xi^l \cdot \partial_l g_{ij} + \partial_i \xi^l g_{lj} + \partial_j \xi^l g_{li}$ and the covariant momentum density $\delta \pi^v = \partial_j (\xi^j \pi^v) - (\partial_j \xi^j) \pi^v$.

It turns out that π^u is a scalar density

$$\delta \pi^u = (\xi^v \pi^u)_{,v} + (\xi^1 \pi^u)_{,1}, \quad (3.6a)$$

as also is $eD_v x$. Notice, however, that $D_v x$ does not transform like a scalar, since Eq. (3.5a) states that e is not a density. The transformation law of π^v is a little bit involved:

$$\delta\pi^v = (\xi^v \pi^v)_{,v} + (\xi^1 \pi^v)_{,1} - (D_v \xi^v) \pi^v - \xi^v_{,1} \tilde{\pi}^1. \quad (3.6b)$$

This result, together with Eqs. (3.4e) and (3.5a), tells us that $eD_v x + enD_v x$ does not transform geometrically. Equation (3.4f) suggests the introduction of the operator $\tilde{D}_v := (\cdot)_{,v} - (N^1 \cdot)_{,1}$, which will act, in general, on densities. One might wonder what would be the transformation law of the object $\tilde{D}_v \pi^v$. It is almost that of a density:

$$\sigma \tilde{D}_v \pi^v = (\xi^v \tilde{D}_v \pi^v)_{,v} + (\xi^1 \tilde{D}_v \pi^v)_{,1} - \tilde{D}_v (\xi^v_{,1} \tilde{\pi}^1). \quad (3.7a)$$

There is another term appearing in Eq. (3.4f) whose transformation law is straightforward to calculate:

$$(e^{-3} \pi^u \cdot \pi^v x_{,1})_{,1} = -(e^{-1} m^2 x_{,1})_{,1} = -\tilde{\pi}^1_{,1} \quad (3.7b)$$

$$\delta \tilde{\pi}^1_{,1} = (\xi^v \tilde{\pi}^1_{,1})_{,v} + (\xi^1 \tilde{\pi}^1_{,1})_{,1} + \tilde{D}_v (\xi^v_{,1} \tilde{\pi}^1) + (e^{-1} m^2 \xi^v_{,1} D_v x)_{,1}. \quad (3.7b)$$

Closing these geometrical considerations, we introduce $\tilde{D}_u(\cdot) := (\cdot)_{,1} - (N \cdot)_{,1}$.

Even if neither $\tilde{D}_u \pi^u + \tilde{D}_v \cdot n \pi^u$ nor $\tilde{D}_v \pi^v$ are geometric quantities, as a consequence of Eqs. (3.7a,b) and (3.4f) they are very useful for studying the system, as will be seen in the following. Equation (3.4f) takes the form

$$\tilde{D}_u \pi^u + \tilde{D}_v \cdot n \pi^u + \tilde{D}_v \pi^v - (e^{-3} (\pi^u \cdot \pi^v) x_{,1})_{,1} = 0. \quad (3.8)$$

Observe that ξ^v, ξ^1 generate contributions that are qualitatively different from the transformation laws. The terms that prevent $\pi^v, \tilde{D}_v \pi^v$, and $\tilde{\pi}^1_{,1}$ from being geometric quantities depend only on ξ^v and not on ξ^1 [see Eqs. (3.6b) and (3.7)]. We are now prepared to turn our attention to the Hamiltonian aspects of the membrane dynamics in the intrinsic LF gauge.

4. HAMILTONIAN INTRINSIC LF APPROACH

In this section we will consider the intrinsic LF canonical approach to membrane theory. Before that, we will briefly discuss the same approach for gauge theories.

The standard covariant second-order Yang-Mills action is

$$S = -4^{-1} \langle \text{tr } F_{rs} F^{rs} \rangle = -4^{-1} \langle -2F_{+-}^2 - 4F_{+i} F_{-i} + F_{ij}^2 \rangle, \quad (4.1)$$

where $F_{rs} = \partial_r A_s - \partial_s A_r - [A_r, A_s]$ and $x^r = (x^+, x^-, x^i)$, $x^\pm := 2^{-\frac{1}{2}}(x^0 \mp x^{D-1})$. The conjugate momenta with respect to the x^+ evolution are

$$p^+ := \frac{\delta S}{\delta \partial_+ A_+} = 0, \quad p^- := \frac{\delta S}{\delta \partial_+ A_-} = F_{+-}. \quad (4.2a, b)$$

$$p^i := \frac{\delta S}{\delta \partial_+ A_i} = F_{-i}. \quad (4.2c)$$

Equation (4.2a) is the standard primary constraint in the $(D-1)+1$ canonical approach to Yang-Mills theory. Equation (4.2c) is the additional primary constraint in the LF approach. After elimination of p^+ , A_+ becomes non-canonical. The Hamiltonian density reduces to

$$\mathcal{H} = 2^{-1}(p^-)^2 - (\mathcal{D}_- p^- + \mathcal{D}_i p^i) \cdot A_+ + 4^{-1} F_{ij}^2 + \lambda_i (p_i - F_{-i}),$$

where $\mathcal{D}_-, \mathcal{D}_i$ indicate covariant gauge derivatives.

The conservation of Eq. (4.2a) yields

$$\mathcal{D}_- p^- + \mathcal{D}_i p^i = 0. \quad (4.3)$$

Its conservation is trivially satisfied. Conservation of Eq. (4.2c) determines $\partial_- \lambda_i$. Equation (4.3) is a set of first-class constraints (the Gaussian ones), whilst Eq. (4.2c) is second class. The LF gauge condition $A_- = 0$ is a canonical gauge-fixing condition in this approach.

In particular,

$$\delta A_- = \{A_-, (\epsilon(\mathcal{D}_- p^- + \mathcal{D}_i p^i))\} = -\mathcal{D}_- \epsilon, \quad (4.4)$$

allowing the gauge condition $A_- = 0$ to be implemented. The functional integral measure is given by

$$\begin{aligned} \int Dp^- Dp^i D A_- D A_i \delta(A_-) \delta(\mathcal{D}_- p^- + \mathcal{D}_i p^i) \det \{A_-, \mathcal{D}_- p^- + \mathcal{D}_i p^i\} \delta(p^i - F_{-i}) \times \\ \times \det^{\frac{1}{2}} \{p^i - F_{-i}, p^j - F_{-j}\} \exp i(p^- \partial_+ A_- + p^i \partial_+ A_i - 2^{-1} p^{-2} - 4^{-1} F_{ij}^2) = \\ = N \int D A_+ D A_- D A_i \delta(A_-) \det \{A_-, \mathcal{D}_- p^- + \mathcal{D}_i p^i\} \exp i(-4^{-1} F_{rs}^2). \end{aligned} \quad (4.5)$$

Following the canonical approach we thus regain the Faddeev-Popov functional measure in the LF gauge. Integrating, now, on A_+ and A_- , we obtain the standard functional integral in terms of the physical modes A_i .

We emphasize that the LF gauge condition is canonical in this approach, allowing the functional measure in Eq. (4.5) to be completely justified.

We may now analyse the membrane theory following this LF Hamiltonian approach. The action we start with is

$$S = 2^{-1} \int d^3\sigma (-g)^{\frac{1}{2}} \{g^{ij} x_{,i} x_{,j} - 1\} . \quad (4.6)$$

The conjugate momenta with respect to the u evolution are given by

$$\pi^u = (-g)^{\frac{1}{2}} g^{uj} x_{,j} , \quad (4.7a)$$

where $j = u, v, 1$. We then have

$$\pi^u = (-g)^{-\frac{1}{2}} \{-G_{uv} D_u x - G_{uu} D_v x\} , \quad (4.7b)$$

where we have introduced $G_{uu} := \tilde{x}_{,u} \cdot \tilde{x}_{,u} = gg^{vv} = (1 - nn')^{-2} 2ne^2 m^2$, $G_{vv} := \tilde{x}_{,v} \cdot \tilde{x}_{,v} = gg^{uu} = (1 - nn')^{-2} 2n'e^2 m^2$, $G_{uv} := -\tilde{x}_{,u} \cdot \tilde{x}_{,v} = gg^{uv} = (1 - nn')^{-2} (1 + nn')e^2 m^2$, $\tilde{x}_{,u(v)} := eD_{u(v)} x$, and $N = e^{-2} x_{,u} \cdot x_{,1}$, $N' = e^{-2} x_{,v} \cdot x_{,1}$. From Eq. (4.7b) we obtain the primary constraints

$$\phi_v := \pi^u \cdot x_{,1} = 0 = \pi^u \cdot x_{,v} =: \phi_1 , \quad (4.8a, b)$$

$$\phi_2 := (\pi^u)^2 + G_{vv} = 0 . \quad (4.8c)$$

It is convenient to rewrite Eq. (4.7b) as

$$\phi_u := \pi^u + eD_v x + \alpha(D_u + \beta D_v)x = 0 , \quad (4.9a)$$

where α is an independent variable. Equations (4.8) and (4.9a) imply $\alpha = (-g)^{-\frac{1}{2}} G_{vv}$ and

$$\beta = G_{uu}(G_{uv} + (G_{uv}^2 - G_{uu}G_{vv})^{\frac{1}{2}})^{-1} . \quad (4.9b)$$

When $G_{vv} = 0$, Eq. (4.9a) becomes a primary constraint.

In terms of the LF variables, Eq. (4.7b) has the form

$$\phi_u = \pi^u + eD_v x + (1 - nn')^{-1} 2en'(D_u + nD_v)x = 0 , \quad (4.9c)$$

where α, β can be directly read off.

The Hamiltonian density obtained by Legendre transformation is zero. The Lagrangian density in terms of the canonical variables is then given by

$$\mathcal{L} = \pi^u \cdot x_{,u} + \Sigma \lambda_i \phi_i , \quad i = u, v, 1 ,$$

which may be rewritten

$$\mathcal{L} = \pi^u \cdot D_u x + n\pi^u \cdot D_v x + \lambda \cdot [\pi^u + eD_v x + \alpha(D_u + nD_v)x] . \quad (4.10)$$

Note that instead of considering the quadratic constraint (4.8c) we have considered (4.9a) which, when $\alpha \neq 0$, is not technically a constraint ($\alpha D_u x$ are time derivatives). However, it is easy to see that $\mathcal{L}$ [Eq. (4.10)] and the Lagrangian density (2.5b'), where $\pi'e(1 - nn')^{-1} = \alpha'$, are equivalent. The ϕ^u has the role of defining π^u . It turns out that the only first-class constraints are Eqs. (4.8). They generate gauge transformations through

$$\delta = \{\xi^v \phi_v + \xi^1 \phi_1 + \xi \phi_2\} . \quad (4.11)$$

When $\alpha = 0$, i.e. if we are in the intrinsic LF gauge, conservation of $\phi_u = 0$ determines $\partial_u \lambda$. (If $\alpha \neq 0$, $\lambda \sim \pi^v$ is determined by $\delta s/\delta \pi^u$.)

We may now fix canonically the LF gauge $G_{vv} = 0$, or equivalently $n' = 0$ ($\alpha = 0$). The generators of the residual gauge freedom are ϕ_v and ϕ_1 . The canonical field equations obtained from Eq. (4.9) in the intrinsic LF gauge are

$$eD_u x = -neD_v x - \pi^v , \quad (4.12a)$$

$$\tilde{D}_u \pi^u = -\tilde{D}_v(\pi^v + n\pi^u) + [e^{-3}(\pi^u \cdot \pi^v)x_{,1}]_{,1} , \quad (4.12b)$$

$$\phi_v = \pi^u \cdot D_v x = 0 = \pi^u \cdot x_{,1} =: \phi_1 , \quad (4.13a, b)$$

$$eD_v x = -\pi^u , \quad (4.14)$$

$$\psi_u := \pi^v \cdot (D_u + nD_v)x = 0 = \pi^v \cdot x_{,1} =: \psi_1 , \quad (4.15a, b)$$

where $\lambda = e^{-1}\pi^v$ and $\alpha = 0$; $\tilde{D}_u$ and $\tilde{D}_v$ have been defined in the previous section.

The integrability condition arising from Eqs. (4.11a) and (4.13) is

$$\tilde{D}_v \pi^v = 2^{-1}(D_u \tilde{x}_{,v} - D_v \tilde{x}_{,u}) - [e^{-3}(\pi^u \cdot \pi^v)x_{,1}]_{,1} , \quad (4.16a)$$

where

$$e^3(D_u \tilde{x}_{,v} - D_v \tilde{x}_{,u}) = (x_{,1,1} \cdot \pi^v) \tilde{x}_v - (x_{,1,1} \tilde{x}_v) \pi^v + (\pi_{,1}^v \cdot \tilde{x}_v) x_{,1} - (\tilde{x}_{,v,1} \cdot \pi^v) x_{,1} . \quad (4.16b)$$

These results can also be obtained from the conservation of the second-class constraints (4.13).

We notice that these field equations already have the appropriate structure to be considered as a u - v evolving system. In fact, Eqs. (17) are the u -propagating equations, whilst Eqs. (4.13) and (4.15) give the v -propagating ones. Equations (4.12) are the constraints to be imposed on the $u = \text{const}$ null surface, whilst Eqs. (4.14) constitute the constraints to be imposed on $v = \text{const}$. Moreover, it can also be shown that Eq. (4.14) is conserved along the v propagation. Conservation of constraints (4.12) is assured by the dynamics of the system. Equations (4.12) and (4.14) both have to be satisfied only initially (in $u = \text{const}$ and in $v = \text{const}$, respectively). We will go a step further along these lines in Section 7, where we will discuss the initial-value problem for membrane theory in the intrinsic LF gauge. We remark that the set of equations (4.11) to (4.14) agree completely with the set (3.4) obtained in the previous section.

5. GAUGE FIXING

The string metric can be obtained by setting $e = 0$ in Eq. (2.1). After gauge fixing $n' = 0$, $x^+ = c^+(u + v)$, we obtain that, in addition, n can also be set to zero.

In the case of membranes, when the (extrinsic) LF, intrinsic ADM gauge is chosen [12], one starts imposing that $x^+ \sim \tau$, and that the internal space-like coordinates are time-orthogonal, $g_{0a} = 0$, $a = 1, 2$. Then, by virtue of the residual gauge invariance, it is discovered [12] that an additional gauge condition holds throughout the world volume: $(-g_{00})^{-\frac{1}{2}} = (\det g_{ab})^{\frac{1}{2}}$. This condition links the 2D space-like volume with the 1D time length.

The different, intrinsically null ($n' = 0$) gauges that we discuss here do not seem to enjoy this property of providing an additional gauge-condition holding. It is convenient to recall that the membrane field equation can be written, in the $n' = 0$ gauge,

$$\begin{aligned} E^r := \partial_t \{ (-g)^{\frac{1}{2}} g^{ij} x_j^r \} &= \partial_u \{ -e x_v^r + e N' x_1^r \} + \\ &+ \partial_v \{ -e \dot{x}^r - 2n e x^r + e(N + 2n N') x_1^r \} + \\ &+ \partial_1 \{ e N' x_u^r + e(N + 2n N') x_v^r + [m^2 e^{-1} - 2e N'(N + n N')] x_1^r \} = \\ &= -\tilde{D}_u \cdot e \cdot D_v x^r - \tilde{D}_v \cdot e \cdot (D_u + 2n D_v) x^r + \partial_1 e^{-1} m^2 \partial_1 x^r = 0. \end{aligned} \quad (5.1)$$

According to the definitions introduced in Eq. (2.1), e, N, N', m^2 , and n are given by

$$e^2 = x_1^2, \quad e^2 N = x_u \cdot x_1, \quad e^2 N' = x_v \cdot x_1, \quad (5.2a)$$

$$m^2 = -x_u x_v + e^2 N N', \quad 2n = m^{-2} (x_u^2 - e^2 N N'), \quad x^2 = e^2 N'^2 \sim n' = 0. \quad (5.2b)$$

The real complexity of the problem becomes apparent if the above values of e, N, N', m^2 , and n are introduced into Eq. (5.1). The first term in E^r [Eqs. (5.1)] becomes $\partial_u \{ -e x_v^r + e^{-1} (x_v \cdot x_1) x_1^r \}$, where $e = (x_1^r)^{\frac{1}{2}}$ is a non-rational function of the basic field variables $x^r(u, v, \sigma)$. One may notice that the most complex quantity is n [Eq. (5.2b)], which entails a rational dependence upon the scalar variables x^r that is different from the dependence carried through e . This observation leads to the introduction of the polynomial gauges α, β :

$$\alpha := n' = 0, \quad m = 1 = e, \quad (5.3)$$

$$\beta := n' = 0 = n, \quad e = 1. \quad (5.4)$$

They both reduce Eq. (5.1) to a polynomial equation. In the gauge α , Eq. (5.1) becomes of ninth order, whilst, if one chooses β , the basic equation is quintic in the scalars x^r . Another useful gauge γ , of canonical type, is the one defined by

$$\gamma := n = n' = 0 = N'. \quad (5.5)$$

Polynomiality is prevented by the presence of negative odd powers of e .

This gauge is well suited to dealing with the initial-value problem. The fourth gauge, δ , that we will consider is of the extrinsic LF type:

$$\delta := n' = 0, \quad x^+ = c^+(u + v), \quad e = 1. \quad (5.6)$$

In this gauge, Eq. (5.1) becomes rational in the x^r owing to its dependence upon $2n$.

In order to be able to speak about gauges, we have to prove their existence. Let us consider, for instance, the gauge α defined by Eqs. (5.3). Let us assume that we are close to it, i.e. $n' = \nu$, $e = 1 + \varepsilon$, $m = 1 + \mu$, and $\nu, \varepsilon, \mu \ll 1$, and let us compute their first-order variations $\delta n'$, δe , and δm when we perform an infinitesimal variation determined by the parameters ξ^u, ξ^v , and ξ^1 . They induce the variations in g_{ij} :

$$\delta g_{ij} = \xi^L \partial_{igij} + \partial_i \xi^L g_{Lj} + \partial_j \xi^L g_{Li}, \quad (5.7)$$

leading to

$$\delta n' = -D_u \xi^u, \quad \delta m^2 = (D_u + n D_v) \xi^u + D_v \xi^v, \quad (5.8a)$$

$$\delta e^2 = 2(\xi_u^u + \xi_v^v + \xi_1^1) - 2(D_u \xi^u + D_v \xi^v). \quad (5.8b)$$

After the transformation, $n' \rightarrow \nu + \delta n'$, $m^2 \rightarrow 1 + 2\mu + \delta m^2$, and $e^2 \rightarrow 1 + 2\varepsilon + \delta e^2$. An appropriate choice of ξ^u makes $\nu + \delta n'$ vanish. We then go to the second of Eqs. (5.8a) and constrain $2\mu + \delta m^2$ to vanish, in this way obtaining ξ^v , and finally, once ξ^u and ξ^v are determined, Eq. (5.8b) provides ξ^1 imposing that $2\varepsilon + \delta e^2$ goes to zero. The group that leaves this gauge invariant, i.e. $\delta n' = \delta m^2 = 0 = \delta e^2$, has the form

$$\xi_0^u, \xi_0^v + \xi^v(u), \xi_0^1 + \xi^1(u, v), \quad (5.9)$$

where

$$(\xi_0^u, \xi_0^v, \xi_0^1) : D_v \xi_0^u = 0 = D_v \xi_0^v + D_u \xi_0^u = \xi_{0,v}^u = \xi_{0,1}^v + \xi_0^1,$$

and $\{0, \xi^v(u), \xi^1(u, v)\}$ constitute the residual gauge group. A gauge similar to this ($n' = 0$, $m = 1$), although differing in the third condition ($x^{D-1} = \sigma$) which is of the extrinsic LF type, has been considered by Sezgin [13].

Similar arguments also show that β, γ, δ are alternative possible gauge choices. The β -gauge establishes a complete symmetry between u and v . Consequently, v becomes a null coordinate and acquires a 'temporal' character. In this gauge

$$\delta n' = -D_v \xi^u, \quad \delta n = -D_u \xi^v, \quad \delta e^2 = -2(D_u \xi^u + D_v \xi^v) + 2(\xi_u^u + \xi_v^v + \xi_1^1). \quad (5.10)$$

The invariance group for this β -gauge has the form $\xi_0^u + \xi^u(u), \xi_0^v + \xi^v(v), \xi_0^1 + \xi^1(u, v)$, where ξ_0^1 is a solution of the homogeneous part of Eq. (5.10).

On the γ -gauge

$$\delta n' = -D_v \xi^u, \quad \delta n = -D_u \xi^v, \quad \delta N = \xi^u - m^2 e^{-2} \xi_{,1}^v + N' \xi_{,u}^v, \quad (5.11)$$

The residual invariance group is $\xi_0^u + \xi^u(u)$, $\xi_0^v + \xi^v(v)$, $\xi_0^1 + \xi^1(v, \sigma)$, where ξ_0^1 is a solution of the homogeneous sector of Eq. (5.11) and $\xi^u(u)$, $\xi^v(v)$, and $\xi^1(v, \sigma)$ are arbitrary.

Finally, let us study the δ -gauge, which has both an intrinsic ($n' = 0$) and an extrinsic [$x^+ = c^+(u + v)$] LF character. Infinitesimal variations in this case are

$$\delta n' = -D_v \xi^u, \quad \delta e^2 = 2\xi_{,1}^1 + 2N\xi_{,1}^u + 2N'\xi_{,1}^v, \quad \delta x^+ = c^+(\xi^u + \xi^v). \quad (5.12)$$

The residual invariance group has the form $\xi_0^u + \xi^u(u)$, $\xi_0^v - \xi^v(u)$, $\xi_0^1 + \xi^1(u, v)$, where again ξ_0^1 is a solution of the homogeneous sector of the above equations.

It is worth having a first look at the membrane in this δ -gauge. Taking into account Eqs. (5.2), from $n' = 0$, $e^2 = 1$, and $x^+ = c^+(u + v)$ we get

$$x_{,v}^- - x_{,1}^-(\vec{x}_{,v} \cdot \vec{x}_{,1}) + 2^{-1} c^+ x_{,1}^{-2} = \frac{1}{2c^+} [x_{,v}^2 - (\vec{x}_{,v} \vec{x}_{,1})^2]. \quad (5.13)$$

This is a non-linear partial differential equation that determines $x^- = x^-(\vec{x})$; $\vec{x} = x_{,1}^1, \dots, x_{,D-2}^{D-2}$ is the transverse part of x^r .

Using the remaining definitions, Eqs. (5.2), we obtain

$$N = -c^+ x_{,1}^- + \vec{x}_{,u} \cdot \vec{x}_{,1}, \quad N' = -c^+ x_{,1}^- + \vec{x}_{,v} \cdot \vec{x}_{,1}, \quad (5.14a, b)$$

$$m^2 = c^+(x_{,v}^- + x_{,u}^-) - \vec{x}_{,u} \cdot \vec{x}_{,v} + N N', \quad (5.14c)$$

$$2n = m^{-2} [-2c^+ x_{,u}^- + \vec{x}_{,u}^2 - N^2], \quad (5.14d)$$

where $x^-(\vec{x})$ is the value determined by Eq. (5.13). Dynamical evolution of $\vec{x}$ is determined by $\vec{E}$, the transverse part of Eq. (5.1):

$$\vec{E} = -\vec{D}_u D_v \vec{x} - \vec{D}_v (D_u + 2n D_v) \vec{x} + (m^2 \vec{x}_{,1})_{,1} = 0. \quad (5.15)$$

One might think that the D-2 degrees of freedom contained in $\vec{x}$ are independent. This is not so, since the δ -gauge requires $\vec{x}_{,1}^2 = 1$. This condition clearly shows that the number of independent degrees of freedom is D-3. There still remain two equations to be considered: E^+ , $E^- = 0$,

$$\begin{aligned} E^+ &:= \vec{D}_v (1 + 2n) = N_{,1}, \\ E^- &:= -\vec{D}_u D_v x^- - \vec{D}_v [D_u x^- + 2n D_v x^-] + (m^2 x_{,1}^-)_{,1} = 0. \end{aligned} \quad (5.16)$$

Both have to be identities where x^-, N, N', n, m^2 are described in terms of $\vec{x}$ as in Eqs. (5.14). In the next section it will be shown that in fact the system has a consistent evolution along the δ -gauge, and that it has D-3 independent excitations.

6. CONSTRAINTS AND DYNAMICS

In the α -gauge the dynamics is governed by the action

$$S = -(4\pi^2 \beta)^{-1} \langle \pi^u \cdot x_{,u} + \pi^v \cdot (e^{-1} \pi^u + D_v x) + n \pi^u \cdot D_v x - N \pi^u \cdot x_{,1} \rangle \quad (6.1a)$$

jointly with

$$x_{,1}^2 = 1, \quad (\pi^v)^2 = 0, \quad \pi^u \cdot \pi^v = -1. \quad (6.1b)$$

These equations correspond to variations with respect to m, n', e before setting them equal to 1, 0, 1, respectively. The structure of this system is that of a constrained action with Lagrange multipliers n, N, π^v , associated respectively with first-class (n, N) and second-class (π^v) differential constraints. Variations of n, N yield

$$\pi^u \cdot D_v x = 0, \quad \pi^u \cdot x_{,1} = 0, \quad (6.2a)$$

which are good canonical constraints since they depend only upon π_x^u and do not contain time derivatives. Variations of N' give

$$\pi^v \cdot x_{,1} = 0. \quad (6.2b)$$

From the point of view of the u evolution, this 'constraint' does not have a canonical aspect. It contains the non-canonical variables π^v (actually they are multipliers of the second-class constraints). Variations of π^v, π^u, x lead to

$$\pi^u + D_v x = 0, \quad \pi^v + D_u x + n D_v x = 0, \quad (6.3a, b)$$

$$\vec{D}_u \pi^u + \vec{D}_v (\pi^v + n \pi^u) + x_{,1,1} = 0. \quad (6.3c)$$

From Eqs. (6.3a,b), after multiplication by $x_{,1}$, we obtain N, N' :

$$N = x_{,u} \cdot x_{,1}, \quad N' = x_{,v} \cdot x_{,1}. \quad (6.4a)$$

From $(\pi^v)^2 = 0$, we get the value of n , and from $(\pi^u)^2 = 0$ and $\pi^u \cdot \pi^v = -1$ we have three (gauge) conditions that the solutions have to satisfy, i.e.

$$2n = (D_u x)^2, \quad (D_v x)^2 = 0, \quad (D_u x) \cdot (D_v x) = -1. \quad (6.4b)$$

The basic membrane field equation (6.3c) is then clearly polynomial since both the operators $\vec{D}_u, \vec{D}_v$ and the momenta π^u, π^v [Eq. (6.3)] are too. In addition, the three gauge-fixing conditions [$x_{,1}^2 = 1$ and the last two of Eq. (6.4b)] are also polynomials in $x_{,1}^r$. As announced before, the system polynomializes in this gauge. However, Eq. (6.3b) does not have a canonical form in the sense that now this equation has a quadratic dependence on $D_u x$.

We then shift our attention to examining the membrane dynamical behaviour in the β - and the γ -gauges. Since they share the prescriptions $n' = 0 = n$, we shall first analyse the system with these two conditions holding, and then will be more specific. In this case, the action attains the form

$$S = -(4\pi^2\beta)^{-1}(\pi^u \cdot x_u + \pi^v \cdot (D_v x + e^{-1}\pi^u) - N\pi^u \cdot x_1), \quad (6.5a)$$

with additional constraints holding:

$$(\pi^v)^2 = 0 = \pi^u \cdot D_v x = \phi_v, \quad e^2 = x_1^2, \quad m^2 = -e^{-2}\pi^u \cdot \pi^v. \quad (6.5b)$$

The first of these equations corresponds to varying n' in Eq. (2.5b') and then taking $n' = 0$. The second one arises when n is varied and then set to zero. The last two equations come from varying m and e respectively, and then setting both n and n' to zero. Variations of π^u, π^v, x , and N in Eq. (6.5a) lead to

$$\phi_1 := \pi^u \cdot x_1 = 0, \quad \psi_i := \pi^v \cdot x_{i,1} = 0, \quad (6.6a, b)$$

$$e^{-1}\phi_u := e^{-1}\pi^u + D_v x = 0, \quad e^{-1}\pi^v + D_u x = 0, \quad (6.6c, d)$$

$$\tilde{D}_u \pi^u + \tilde{D}_v \pi^v - [e^{-3}(\pi^u \cdot \pi^v)x_{i,1}]_{i,1} = 0. \quad (6.6e)$$

The ϕ_v and ϕ_1 are u -canonical constraints. The $(\pi^v)^2 = 0$ and ψ_i do not have a canonical form (since π^v is a multiplier in the u dynamics). The ϕ_u is a second-class constraint. However, Eqs. (6.6d) and (6.6e) do not match sufficiently well to be a canonical system in the usual form. The first says $\dot{x} \sim \pi^v$ and the second is of the form $\dot{\pi}^u \sim f(x, \pi^u, \pi^v)$. In addition, ϕ_u relates to $\pi^u \sim x_v$ (and not to $\dot{x}$).

From Eqs. (6.6c,d) and the use of Eq. (6.6a,b), we obtain the values of N, N' :

$$N = e^{-2}x_u \cdot x_1, \quad N' = e^{-2}x_v \cdot x_1. \quad (6.7a, b)$$

They are equivalent to stating the orthogonality of $D_{u,v}x$ to x_1 .

We desire to show that the two constraints ϕ_v and ϕ_1 are conserved. It is straightforward to show that on the field equations (6.6c,d,e),

$$\tilde{D}_u \phi_v \sim \phi_1, \quad (6.8a)$$

where $[D_u, D_v]x = (D_v N - D_u N')x_1$ and $\tilde{D}_{u,v}e = e^{-2}x_{i,1} \cdot \pi^{v,u}$ have been used. Consequently, if initially $\phi_v = 0 = \phi_1$, they will also vanish at any future time.

To deal with ϕ_1 , we need to calculate first the consistency condition for ϕ_u and the value of the world-volume curvature $D_u N' - D_v N$ (which we will do in a moment). It then turns out that

$$\tilde{D}_u \phi_1 = 2^{-1}\pi^v \cdot (e^{-1}x_{i,1})_{i,1}\phi_1, \quad (6.8b)$$

showing that ϕ_1 is also conserved.

Dynamical consistency also requires the vanishing of the time evolution of the second-class constraints $\phi_u = 0$. For second-class constraints, the role that the vanishing of their time derivatives has is to determine π^v . After some algebra, we find

$$\begin{aligned} \tilde{D}_u \phi_u &= \tilde{D}_u \pi^u - \tilde{D}_v \pi^v + e^{-3}(\pi^u \cdot \pi_{i,1}^u - \pi^u \cdot \pi_{i,1}^v)x_{i,1} + \\ &+ e^{-3}[(\pi^u \cdot x_{i,1})\pi^v - (\pi^v \cdot x_{i,1})\pi^u] = 0, \end{aligned} \quad (6.9)$$

where the two terms in the middle represent the value of the (u, v) world-volume curvature,

$$D_u N - D_v N' = e^{-4}\pi^v \cdot \pi_{i,1}^u - e^{-4}\pi^u \cdot \pi_{i,1}^v. \quad (6.10)$$

The interesting fact we have now, is the presence of an additional field equation, Eq. (6.9), whose role, apart from determining π^v , is to provide the system with an almost symmetric dynamical structure in (u, v) . This establishes an intertwined (u, v) evolution. Linear combinations of Eqs. (6.6e) and (6.9) lead to the equivalent formulation of the main part of the system:

$$\begin{aligned} \tilde{D}_u \pi^u &= 2^{-1}e^{-2}\pi^v \cdot (e^{-1}x_{i,1})_{i,1}\pi^u - 2^{-1}e^{-2}\pi^u \cdot (e^{-1}x_{i,1})_{i,1}\pi^v + \\ &+ e^{-2}\pi_{i,1}^u(e^{-1}\pi^v)_{i,1}x_{i,1} + 2^{-1}e^{-2}\pi^u \cdot \pi^v(e^{-1}x_{i,1})_{i,1}, \end{aligned} \quad (6.11a)$$

$$\begin{aligned} \tilde{D}_v \pi^v &= 2^{-1}e^{-2}\pi^u \cdot (e^{-1}x_{i,1})_{i,1}\pi^v - 2^{-1}e^{-2}\pi^v \cdot (e^{-1}x_{i,1})_{i,1}\pi^u + \\ &+ e^{-2}\pi_{i,1}^v(e^{-1}\pi^u)_{i,1}x_{i,1} + 2^{-1}e^{-2}\pi^v \cdot \pi^u(e^{-1}x_{i,1})_{i,1}. \end{aligned} \quad (6.11b)$$

Moreover, the system contains Eqs. (6.6c,d). In this way it has a Hamiltonian null structure.

Its evolution is restricted by the constraints ϕ_v, ϕ_1 and by the remaining two (seemingly anomalous) equations $\psi_u := (\pi^v)^2 = 0, \psi_i := \pi^v \cdot x_{i,1} = 0$. They have a clear interpretation: they are the constraints of the system along its v evolution.

The β -gauge contains the additional prescription $e = 1$, which has not to be introduced into action (6.5a) before doing the variations. It can be introduced into Eqs. (6.5b), (6.6), and (6.9) which, as has been pointed out, lead to a polynomial system. They become quintic in the scalar fields $x(u, v, \sigma)$. The system achieves an almost canonical form. What does not appear in a transparent way is the counting of the number of independent excitations of the system.

The γ -gauge was defined by $N = 0$ in addition to $n' = 0 = n$. Equations (6.5b) are not modified. Before letting N go to zero, we have to add to the system the equation that it generates, i.e. $\phi_1 = 0$. The action takes the form

$$S = -(4\pi^2\beta)^{-1}(\pi^u \cdot x_u + \pi^v \cdot [D_v x + e^{-1}\pi^u]) \quad (6.12)$$

The additional equations arising from this action are Eqs. (6.6) and (6.9) where $D^u x = x_{,u}, \tilde{D}_u \pi^u = \pi_{,u}^u$. It is the more canonical gauge considered up to now, since Eqs. (6.6d) and (6.11a) achieve the form $x_{,u} = -e^{-1} \pi^v$ and $\pi_{,u}^u = f(\pi^u, \pi^v, x)$.

The presence of ϵ impedes polynomiality of the system. Similar to the situation in the β -gauge, counting in the Hamiltonian system is not transparent. At the second of the second-order formulation, it is obvious that the x^r have three restrictions on them: $x_{,u}^2 = 0 = (D_v x)^2 = x_{,u} \cdot x_{,1}$. Therefore there will be D-3 independent excitations, as we know there should be.

Finally, we come to the δ -gauge. We start with action (6.1a), recalling that variations with respect to n' lead to

$$\psi_u := (\pi^v)^2 = 0. \quad (6.13a)$$

Recall that m^2 is given by Eq. (2.5c), $m^2 = -e^{-2} \pi^u \cdot \pi^v$. Variations with respect to N', N , and n yield respectively:

$$\psi_1 := \pi^v \cdot x_{,1} = 0, \quad \phi_i := \pi^u \cdot x_{,1} = 0, \quad \phi_v := \pi^u \cdot x_{,v} = 0. \quad (6.13b, c, d)$$

Varying the main physical variables π^u, π^v, x gives

$$e^{-1}(\pi^v - n\pi^u) + D_u x = 0, \quad e^{-1}\phi_u := e^{-1}\pi^u + D_v x = 0, \quad (6.14a, b)$$

$$\tilde{D}_u \pi^u + \tilde{D}_v(\pi^v + n\pi^u) - [e^{-3}\pi^u \cdot \pi^v x_{,1}]_{,1} = 0. \quad (6.15c)$$

The δ -gauge consists of choosing 3D coordinates (u, v, σ) such that, in addition to satisfying $n' = 0$, one also has $x^+ = c^+(u + v), e = 1 = x_{,1}^2$. Dynamical consistency of second-class constraints ϕ_u is equivalent to integrability of Eqs. (6.14). It reads

$$\tilde{D}_u \pi^u - \tilde{D}_v(\pi^v - n\pi^u) + e^{-3}(\pi^u \cdot x_{,1,1})\pi^v - e^{-3}(\pi^u \cdot x_{,1,1})\pi^u + e^{-3}(\pi^v \cdot \pi_{,1}^u - \pi^u \cdot \pi_{,1}^v)x_{,1} = 0. \quad (6.15b)$$

Equations (6.15) are equivalent to

$$\tilde{D}_u \pi^u + \tilde{D}_v n \pi^u = 2^{-1} \pi^v \cdot x_{,1,1} \pi^u - 2^{-1} \pi^u \cdot x_{,1,1} \pi^v + 2^{-1} \pi^u \cdot \pi^v x_{,1,1} + \pi^u \cdot \pi_{,1}^v x_{,1}, \quad (6.16a)$$

$$\tilde{D}_v \pi^v = 2^{-1} \pi^v \cdot x_{,1,1} \pi^v - 2^{-1} \pi^v \cdot x_{,1,1} \pi^u + 2^{-1} \pi^u \cdot \pi^v x_{,1,1} + \pi^v \cdot \pi_{,1}^u x_{,1}, \quad (6.16b)$$

where ambient space indices are omitted. At this point it is convenient to separate indices into $x^\pm = 2^{-1}(x_{\mp}^0 x^{D-1})$ and $\vec{x} = (x^i, i = 1, \dots, D-2)$, its transverse part.

At this stage, the system in which we are interested consists of Eqs. (6.13), (6.14) ($e = 1$), and (6.16), $x^+ = c^+(u + v)$ and $x_{,1}^2 = 1$. In the following, we will analyse it.

Let us consider the (+)-sector of Eqs. (6.14) and (6.16). The first two equations give

$$\pi^{u+} = -c^+, \quad \pi^{v+} = -(1+n)c^+, \quad (6.17a, b)$$

whilst the latter provide a set equivalent to

$$\tilde{D}_v(1+n) = 2^{-1}(N_{,1} - n'_{,1}), \quad (6.18a)$$

$$\mathcal{F} := (1+n)\pi^u \cdot x_{,1,1} - \pi^v \cdot x_{,1,1} - (N_{,1} - N'_{,1}) = 0. \quad (6.18b)$$

Our point of view is that the physical variables are contained in $\vec{x}, \vec{\pi}^u, \vec{\pi}^v$. Since it is known that a membrane has D-3 degrees of freedom, we expect that at some stage we will encounter restrictions that reduce their degrees of freedom to D-3; $\mathcal{F}$ plays a central role in that respect.

Expressions (6.17) are useful for solving the four constraints (6.13). Insertion of them yields

$$x_{,1}^- = -(c^+)^{-1} \pi^u \cdot \vec{x}_{,1}, \quad (1+n) = (\pi^u \cdot \vec{x}_1)^{-1} \pi^v \cdot \vec{x}_{,1}, \quad (6.19a, b)$$

and

$$\pi^{u-} = -(2c^+)^{-1} \pi^{u^2}, \quad \pi^{v-} = -[2c^+(1+n)]^{-1} \pi^{v^2}. \quad (6.19c, d)$$

The more delicate part is the (-)-sector. Equations (6.14) provide $x_{,u}^-, x_{,v}^-$:

$$x_{,u}^- = [2c^+(1+n)]^{-1} \pi^{v^2} - N(c^+)^{-1} \pi^u \cdot \vec{x}_{,1} - n(2c^+)^{-1} \pi^{u^2}, \quad (6.20a)$$

$$x_{,v}^- = N'(c^+)^{-1} \pi^u \cdot \vec{x}_{,1} + (2c^+)^{-1} \pi^{u^2}. \quad (6.20b)$$

Equations (6.16) will be discussed shortly. The validity of Eqs. (6.19a) and (6.20) entails that three consistency (integrability) conditions must be satisfied: $I_{uv} \sim (6.20a)_v = (6.20b)_u$, $I_{u1} \sim (6.20a)_1 = (6.19a)_u$, $I_{v1} \sim (6.20b)_1 = (6.19a)_v$; I_{uv} coincides with (6.15b), whilst I_{u1} and I_{v1} have to be calculated. After doing these calculations, we get

$$I_{v1} = \mathcal{F},$$

$$I_{u1} + I_{v1} = -\tilde{D}_v(n\pi^u) \cdot \vec{x}_{,1} + 2^{-1} \pi^u \cdot \vec{x}_1(N_{,1} + N'_{,1}) = 0.$$

Using Eq. (6.18a), we obtain a simpler expression for this last equation:

$$I_{u1} + I_{v1} = -n D_v \pi^u \cdot \vec{x}_{,1} = 0 \rightarrow D_v \pi^u \cdot \vec{x}_{,1} = 0, \quad (6.21)$$

since in general n will not vanish.

At this point we ask ourselves whether Eq. (6.18a) is consistent with the value of n provided by Eq. (6.19b). After making use of the whole dynamical system and of the consistency equation (6.21b), we find that $(1+n)$ [Eq. (6.19b)] satisfies the first-order differential equation (6.18a), thus this latter equation is redundant. The last two equations contained in the (-)-sector are (6.16a,b) $^-$. After making use of the whole system, they become proportional to $\mathcal{F}$, i.e.

$$(6.16a)^-, (6.16b)^- \rightarrow \mathcal{F} \pi_u^2 = 0, \quad \mathcal{F} \pi^{v^2} = 0 \rightarrow \mathcal{F} = 0.$$

They constitute identities once $\mathcal{F} = 0$ is satisfied.

The physics of the membrane is contained in the transverse sector. The transverse variables evolve according to

$$\vec{\pi}^u + D_u \vec{x} = 0, \quad \vec{\pi}^v - n \vec{\pi}^u + D_u \vec{x} = 0, \quad (6.22a, b)$$

$$\begin{aligned} \vec{D}_u \vec{\pi}^u + \vec{D}_v \cdot n \vec{\pi}^u &= \frac{1}{2}(\vec{\pi}^u \cdot \vec{x}_{,1,1} - \pi^{v+} x_{,1,1}^-) \vec{\pi}^u - \frac{1}{2}(\vec{\pi}^u \cdot \vec{x}_{,1,1} - \pi^{u+} x_{,1,1}^-) \vec{\pi}^v \\ &\quad + \frac{1}{2}[\vec{\pi}^u \cdot \vec{\pi}^v - \frac{1}{2}(n+1)6^{-1} \vec{\pi}^{v2} - \frac{1}{2}(n+1) \vec{\pi}^{u2}] \vec{x}_{,1,1} + \\ &\quad + \{\vec{\pi}^u \cdot \vec{\pi}^v - \frac{1}{2}[(n+1)^{-1} \vec{\pi}^{v2}]_{,1} - \frac{1}{2}(1+n) \vec{\pi}^{u2}\} \vec{x}_{,1,1}, \quad (6.23a) \\ \vec{D}_v \vec{\pi}^v &= \frac{1}{2}(\vec{\pi}^u \cdot \vec{x}_{,1,1} - \pi^{u+} x_{,1,1}^-) \vec{\pi}^v - \frac{1}{2}(\vec{\pi}^v \cdot \vec{x}_{,1,1} - \pi^{v+} x_{,1,1}^-) \vec{\pi}^u + \\ &\quad + \frac{1}{2}[\vec{\pi}^u \cdot \vec{\pi}^v - \frac{1}{2}(1+n)^{-1} \vec{\pi}^{v2} - \frac{1}{2}(1+n) \vec{\pi}^{u2}] \vec{x}_{,1,1} + \\ &\quad + \{\vec{\pi}^v \cdot \vec{\pi}^u - (1+n) \vec{\pi}^u \cdot \vec{\pi}^u\} \vec{x}_{,1,1}. \quad (6.23b) \end{aligned}$$

This system still needs to have some additional information introduced into it. The values of N and N' are unknown and $\vec{D}_{u,v}$ contains them. To go in that direction, we take the inner product of $\vec{x}_{,1}$ times Eqs. (6.22a,b)₁. It yields

$$\mathcal{F}_v := N'_1 = \vec{\pi}_{,1}^u \cdot \vec{x}_{,1}, \quad N_1 = \vec{\pi}_{,1}^v \cdot \vec{x}_{,1} - n N'_1 - n_1 \vec{\pi}^u \cdot \vec{x}_{,1}. \quad (6.24a, b)$$

Introducing in this last equation the value of n [Eq. (6.19b)], we obtain $\mathcal{F}_u$:

$$\mathcal{F}_u := N_1 = (1+n) \vec{\pi}^u \cdot \vec{x}_{11} + \vec{\pi}_{,1}^u \cdot \vec{x}_{,1} - \vec{\pi}^v \cdot \vec{x}_{,11}. \quad (6.24b')$$

It can immediately be seen that $\mathcal{F}_u - \mathcal{F}_v = \mathcal{F}$. So, if they hold, then $\mathcal{F}$ holds too; $\mathcal{F}$ ceases to be an independent restriction on the system. Alternatively, $\mathcal{F}_{u,v}$ can be found from the identities $\vec{D}_{u,v} e = -e^1 x_{,1,1} \cdot D_{u,v} x$ and Eqs. (6.14).

The role of the necessary conditions (6.24a,b') is to provide the values of N, N' as functions of the canonical transverse variables.

Thus, we have found how to separate the system in the δ -gauge in a main part, explicitly written purely in terms of the transverse LF canonical variables $\vec{x}, \vec{\pi}^u, \vec{\pi}^v$, with all the remaining quantities $x^-, \pi^{u\pm}, \pi^{v\pm}, n, N, N'$ determined as functions of them.

A unique differential constraint holds on the system $D_v \vec{\pi}^u \cdot \vec{x}_{,1} = 0$. It arises because of integrability conditions for x^- , it is not the transverse part of any of the first-class constraints (6.13). Regarded as a restriction on $\vec{\pi}^u$, it reduces down to D-3 the independent degrees of freedom of $\vec{\pi}^u$. Also, $\vec{x}$ contains D-3 independent components due to the gauge condition $\vec{x}_{,1}^2 = +1$.

7. THE INITIAL-VALUE PROBLEM IN THE INTRINSIC LF GAUGE

The u - v evolving system (4.12) and (4.16) or (6.13) and (6.16) may be decomposed into the u -evolution equations (6.14a) and (6.15a) and constraints (6.13c,d) which have to be imposed only at $u = \text{const}$ (since they are preserved along the u evolution of the system, as was shown in the previous section), and the v -evolution equations (6.14b) and (6.16b) with constraints (6.13a,b), which have only to be imposed at $v = \text{const}$ since they are preserved along their v evolution. These field equations contain, in addition to the canonical fields, the Lagrange multipliers associated with the first-class constraints, which can only be determined by gauge-fixing conditions. In the intrinsic LF gauge, $n' = 0$, these multipliers are n and N . They may be obtained in terms of $x_{,u}$ from Eqs. (6.13a) and (6.14a,b):

$$2n = -\vec{x}_{,u} \cdot \vec{x}_{,u} (\vec{x}_{,u} \cdot \vec{x}_{,v})^{-1}, \quad N = x_{,u} \cdot x_{,1} (x_{,1}^2)^{-1}. \quad (7.1a, b)$$

However, if we insert them in Eqs. (6.13) to (6.16), the system becomes non-canonical. Moreover, it is impossible to rewrite it as a canonical one. We thus consider n, N as independent multipliers, and the system (6.13) to (6.16) as a canonical system in terms of the canonical fields and the unknown Lagrange multipliers.

The gauge transformations of n, N generated by the first-class constraints are

$$\delta n = -D_u \xi^v - n D_v \xi^v + \xi^v n_{,v} + \xi^1 n_{,1}, \quad (7.2a)$$

$$\delta N = D_u \xi^1 + N' D_u \xi^v + \xi^v N_{,v} + \xi^1 N_{,1} + (N N' - e^{-2} m^2) \xi_{,1}^1, \quad (7.2b)$$

showing the typical dependence of the Lagrange multipliers on the 'time' derivative of the corresponding group parameters. However, after having fixed n, N by appropriate gauge-fixing conditions, there will still remain a residual gauge freedom which may be used to impose a further gauge-fixing condition on the initial $u = \text{const}$ surface.

In the non-canonical γ -gauge, where $n = 0 = N$, we use the residual gauge freedom to impose

$$x^+ = c^+ v \text{ on } u = 0, \quad x^+ = c^+ u \text{ on } v = 0. \quad (7.3a, b)$$

Following the analysis of Section 5, it can be shown that these restrictions may be imposed. Moreover, there is still a gauge freedom $\xi^1 = \xi^1(v, \sigma)$, which we may use to fix

$$N' = 0 \text{ on } u = 0 \quad (7.4a)$$

and

$$x_{,1}^2 = 1 \text{ on } u = 0, v = 0. \quad (7.4b)$$

In the canonical δ -gauge, where $x^+ = c^+(u+v)$ and $x_1^2 = 1$, the Lagrange multipliers are determined from conservation of gauge-fixing conditions. In fact, taking the u -derivative of these conditions and using Eqs. (6.14a) and (6.15a), we obtain

$$n = -1 - (c^+)^{-1} \pi^{v+}, \quad N_1 = n(D_v x)_1 \cdot x_1 + \pi_1^v \cdot x_1. \quad (7.5a, b)$$

They ensure that, if $x^+ = c^+v$ on $u = 0$ and $x_1^2 = 0$ on $u = 0$, then $x^+ = c^+(u+v)$, $x_1^2 = 1$ for all u .

From the v -conservation of $x_1^2 = 1$ we obtain N' . Taking the v -derivative of this condition and using Eq. (6.14b), we get again Eq. (6.24a):

$$N'_1 = \pi_1^u \cdot x_1. \quad (7.6)$$

Equations (7.5) and (7.6) ensure that $x_1^2 = 1$ is valid for all u, v once it is assumed that $u = v = 0$. This is the virtue of a canonical gauge-fixing condition.

We notice that for both gauge conditions γ and δ , the multipliers n, N have been fixed and we impose the same restrictions on $N = 0$ and $v = 0$. However, in the non-canonical γ -gauge the initial conditions (7.3) and (7.4) are not preserved for all u and v . In the δ -gauge, in addition to Eqs. (7.5a, b), which determine the Lagrange multipliers, we have $x_1^2 = 1$ and $x^+ = c^+(u+v)$ for all u, v . This additional 'gift' always occurs when we have a canonical gauge-fixing condition. Phase spaces in both cases are, of course, completely equivalent. This has been explicitly shown by proving that the initial data are the same.

We have now, both on the γ - and the δ -gauges, a well-posed initial-value problem. The initial data have to satisfy the constraints and gauge-fixing conditions:

$$\begin{aligned} (\pi^u)^2|_{u=0} = 0 = \pi^u \cdot x_1|_{u=0}, \quad x^+ = c^+v|_{u=0}, \quad e D_v x = -\pi^u|_{u=0}, \quad (7.7a-d) \\ (\pi^v)^2|_{v=0} = 0 = \pi^v \cdot x_1|_{v=0}, \quad x^+ = c^+u|_{v=0}, \quad (7.8a, b, c) \\ x_1^2|_{u=0=v} = 1. \quad (7.9) \end{aligned}$$

We first work in the δ -gauge. We use the $\pm$ -components of Eq. (7.7d) to obtain $\pi^{u\pm}$ on $u = 0$,

$$\pi^{u+} = -c^+, \quad \pi^{u-} = -D_v x^-. \quad (7.10a, b)$$

From Eq. (7.7b) we get x^- in terms of $\vec{x}$, as in previous sections,

$$x_1^- = -(c^+)^{-1} \pi^u \cdot \vec{x}_1, \quad (7.11)$$

and Eq. (7.7a) gives a constraint for π^u which reduces to Eq. (6.21):

$$D_v \pi^u \cdot \vec{x}_1 = 0. \quad (7.12)$$

We are thus left, on the null surface $u = 0$, with the restrictions (7.12)|_{u=0},

$$D_v \vec{x} = -\pi^u|_{u=0}, \quad \vec{x}_1^2|_{u=0=v} = 1. \quad (7.13a, b)$$

Given $\vec{x}_0$ at $v = 0$ satisfying Eq. (7.13b), we then determine all the restrictions on the $D_v \pi^u$ derivatives, evaluated at $v = 0$, by applying D_v derivatives to Eq. (7.12) and using Eq. (7.13a) to replace $D_v \vec{x}$ derivatives. In this way Eqs. (7.12) and (7.13) determine a feasible set of π^u for the given initial data $\vec{x}_0$ at $v = 0$. The π^u are then defined at least in the neighbourhood of $v = 0$. Given $\pi^u(v, \sigma)$ in that set, we then use Eq. (7.13a) to propagate $\vec{x}$ to the null surface (at least in the neighbourhood of $v = 0$).

We still have to deal with π^v and constraints (7.8a, b) on Σ_v ($v = 0$). We may take Σ_v as a space-like or null hypersurface. Looking for u, v symmetry on the initial data, we discuss the null-surface case. The space-like one follows along the same lines. We then have on Σ_v ,

$$n|_{\Sigma_v} = 0. \quad (7.14)$$

From Eq. (7.5a) we obtain π^{v+}

$$\pi^{v+}|_{\Sigma_v} = -c^+, \quad (7.15)$$

and from Eq. (6.14a)

$$\pi^{v-}|_{\Sigma_v} = -D_u x^-. \quad (7.16)$$

From Eqs. (7.8a, b) we get, on Σ_v ,

$$x_1^- = -(c^+)^{-1} \pi^v \cdot \vec{x}_1, \quad D_u \pi^v \cdot \vec{x}_1 = 0. \quad (7.17a, b)$$

We are thus left, on Σ_v , with the symmetric problem ($u \leftrightarrow v$) of Eqs. (7.12) and (7.13a, b).

We conclude that the initial data are, as indicated in the figure, π^u on $u = 0$, π^v on $v = 0$, $\vec{x}$ on $u = 0 = v$ restricted by Eqs. (7.12), (7.13b), and (7.17b). To implement the restriction on π^u and π^v , we have to use the propagating equations (7.13a) on Σ_u and (6.14a) on Σ_v , as we have discussed above.

Given these initial data, we propagate π^v on Σ_u ($u = 0$) from Σ_u ($v = 0$) using Eq. (6.16b) and $\vec{x}$ from Σ_v using Eq. (7.13a); x^+, x^-, π^{u+} , and π^{u-} are determined from Eqs. (7.7c), (7.8c), (7.10), and (7.11). We then use Eqs. (6.14a) and (6.16a) to propagate $\vec{x}$ and π^u from Σ_u to a new surface Σ'_u ($u = u_1$). Again Eq. (6.16b) propagates π^v on Σ'_u from Σ_u , and Eqs. (6.14a) and (6.16a) propagate $\vec{x}$ and π^u from Σ'_u to a new surface Σ''_u ($u = u_2$), and so on.

The initial-data problem in the γ -gauge may be analysed in the same way. In this case we obtain

$$\pi^{u+} = -ec^+, \quad \pi^{u-} = -e D_v x^-, \quad x_1^- = -(ec^+)^{-1} \pi^u \cdot \vec{x}_1, \quad (7.18a, b, c)$$

and the constraint for $\vec{\pi}^u$, using the fact that we can set $N' = 0$ on Σ_u , reduces to

$$\partial_v(e^{-1}\vec{\pi}^u) \cdot \vec{x}_{,1} = 0. \quad (7.19)$$

The rest of the analysis follows as in the δ -gauge.

The restriction on $\vec{\pi}^v$ is

$$\partial_u(e^{-1}\vec{\pi}^v) \cdot \vec{x}_{,1}|_{\Sigma_v} = 0. \quad (7.20)$$

The initial data are, as in the figure, replacing Eqs. (7.12) and (7.17b) by Eqs. (7.19) and (7.20).

8. SUMMARY

We have described the relativistic membrane geometry with coordinates that are appropriate for stressing the intrinsic light-front properties, i.e. the presence of characteristic surfaces. A LF representation of the 3D Virasoro algebra was found, starting from the first-order covariant action represented in the intrinsic LF coordinates. We then defined the intrinsic LF gauge and studied some geometrical properties of its residual gauge group. The Hamiltonian approach was also discussed, especially in the LF gauge, and the Lagrangian density was built-up using canonical methods and constraint analysis. This approach leads naturally to the distinction between first- and second-class constraints. In the present case, consistency of the second-class constraints yields a field equation that is crucial for understanding the intertwined u - v dynamics. Four different intrinsic LF gauges were introduced and their existence proved. Two of them polynomialize the system; the third is well suited to treating the initial-value problem; and the fourth, a double LF gauge, has one condition of an extrinsically LF type in addition to the intrinsic LF condition common to all these gauge conditions. This is the most interesting gauge. We have found that the system can be consistently reduced to its transverse sector entirely in terms of the transverse variables $\vec{x}, \vec{\pi}^u, \vec{\pi}^v$. Integrability conditions lead to the discovery of an additional canonical constraint, which shows that $\vec{\pi}^u$, as well as $\vec{x}$, contains D-3 independent degrees of freedom.

Finally, we have studied the initial-value problem in this gauge, where we show that it is well posed, and how the (u, v) -dynamics proceed.

Extension of these results to $p > z$ -branes will be straightforward, although it will take a bit longer to achieve.

The results we have obtained will be useful in three directions: in carrying on the quantization of the membrane; in their generalization to the supermembrane; and in the search for exact solutions of the membrane system.

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