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INTEGRABILITY IN RANDOM MATRIX MODELS^{*}

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ABSTRACT

We prove the equivalence between the recent matrix model formulation of $2D$ gravity and lattice integrable models. For even potentials this system is the Volterra hierarchy, and many properties of the continuum matrix model like the Virasoro conditions on the partition function stem directly from the integrability properties of the lattice model and its Hamiltonian properties.

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1. Introduction

Some aspects of the recently discovered non-perturbative solutions to non-critical strings [1] can be better understood and clarified directly in terms of the integrability properties of the random matrix model used to define the coupling of (p, q) -conformal matter to $2D$ -gravity. Soon after the appearance of the original papers on the subject, Douglas [2] showed that the continuum limit of these models can be described in terms of special representations of the Heisenberg algebra. He also exhibited some interesting connections with the KdV hierarchy [3] which strongly indicate a deep interplay still to be elucidated between string theory and integrable models. This letter is a modest attempt in this direction.

The non-perturbative definition of strings [1] requires two different types of equations. Denoting the string susceptibility by $u(x)$ as a function of x , the cosmological constant, the first equation is

$$-x = \sum_{n=1}^{\infty} (2n+1) t_n \mathcal{R}_n(u) \quad (1.1)$$

involving the Gel'fand-Dickii potentials $\mathcal{R}_n$ (see [1] for details) and providing a non-perturbative characterization of the different critical points of a generic hermitean matrix model

$$Z = \int d\Phi e^{-NT \cdot V(\Phi)} \quad (1.2)$$

with

$$V(\Phi) = \sum_j g_j \Phi^j$$

$$\frac{\partial u}{\partial t_n} = D \mathcal{R}_{n+1}(u) \quad (1.3)$$

The second type of equations describe the renormalization group flows of the models defined by (1.1). These equations coincide with the flows of the KdV hierarchy [3]

The geometrical meaning of (1.1) and (1.3) has been clarified [4] using the Schwinger-Dyson equations for loop functionals [5]. In fact it was proved in [4] that from (1.1) and (1.3) it is possible to derive an equation for the loop functional coinciding with the

double scaling limit of the matrix loop equation [5]

$$V' \left(\frac{\partial}{\partial t} \right) \langle W(t) \rangle = \int_0^1 dt' \langle W(t') \rangle \langle W(t-t') \rangle + \frac{1}{N^2} \int_0^1 dt' \langle W(t') W(t-t') \rangle \quad (1.4)$$

where the loop functional is defined by

$$W(t) = \frac{1}{N} \text{Tr} e^{t\Phi} \quad (1.5)$$

One first analyzes (1.4) in the planar limit in order to find the critical points which fix the scaling behavior of the loop operators and later one takes the double scaling limit [1] maintaining fixed the string coupling constant. It remains an open problem to show that (1.1) and (1.3) follow from the double scaling limit of (1.4) and its resolution would clarify the possible existence of non-perturbative parameters in string theory.

The matrix loop equations (1.4) can be easily derived from the invariance of (1.2) under the infinite set of field redefinitions $\Phi \mapsto \Phi + \epsilon \Phi^{n+1}$ [6]. Using the matrix representation of triangulations of Riemann surfaces it should be possible to interpret these matrix field redefinitions as a discrete version of conformal transformations and the following identities can be thought of as the discrete ancestors to conformal invariance in the sense of a vanishing central extension. The Ward identities induced by the transformations above are :

$$\sum_{k=0}^n \langle tr \Phi^k tr \Phi^{n-k} \rangle = N \langle \sum_{j=1}^{\infty} j g_j \{ tr \Phi^{n+j} \} \rangle \quad (1.6)$$

which are equivalent to (1.4) and can also be written in terms of formal Virasoro generators as follows

$$L_n Z = 0 \quad n \geq -1$$

$$L_n = \sum_k \frac{1}{N^2} \frac{\partial^2}{\partial g_k \partial g_{n-k}} + \frac{1}{N} \frac{\partial}{\partial g_n} + \sum_j j g_j \frac{\partial}{\partial g_{n+j}} + \delta_{-1,n} N g_1 + \delta_{0,n} N^2 \quad (1.7)$$

The double scaling limit of the Virasoro generators (1.7) can be defined following the same rules as for the definition of the continuum limit of (1.4). In this limit they can be interpreted as the components of the energy momentum tensor of a Z_2 -twisted free scalar field [4].

In this letter we consider a different derivation of (1.1) and (1.3) with the Toda lattice and the Volterra hierarchy for even potentials [7]. The two basic steps in this approach are: i) to show the connection between discrete integrable models and the matrix models defined in terms of orthogonal polynomials, and ii) understand the extension of the double scaling limit to non-linear lattices. The outputs of this construction are (1.1) and (1.3) where (1.1) appears as an initial condition of the non-linear lattice.

An interesting fact about the Volterra equation which gives rise to some interesting speculation is its relation to the Liouville model on a regular lattice discussed in [8]. In fact the equations of motion for the energy-momentum tensor of the discretized Liouville theory coincide with the Volterra equations of motion. The existence of an infinite number of conserved charges in involution for the Volterra equation defines an infinite set of hamiltonian equations and a collection of compatible hamiltonian structures which in the continuum limit reproduce the well-known Gelfand-Dickii bihamiltonian properties of the KdV equation [9]. A natural question to ask about the double scaling limit of this lattice system is if it can be directly related to the underlying Liouville theory. It is tempting to speculate that the double scaling limit of these non-linear lattices describes in an effective form a string phase of the quantum Liouville theory.

2. Volterra Hierarchies and String Equations

For simplicity we consider only the case of one matrix models with even potentials. Similar arguments can be carried out with arbitrary potentials. One can obtain Ward identities similar to those in (1.6) valid for even and general potentials. For even potentials we take the set of field redefinitions $\phi \mapsto \phi(1 + \epsilon \text{tr} \phi^n)$ leading to the relations

$$(N + \frac{2l}{n}) \frac{\partial Z}{\partial g_l} = \sum_j \frac{2j}{N} g_j \frac{\partial^2 Z}{\partial g_j \partial g_l} \quad (2.1)$$

Notice that for even potential the field redefinitions leading to (1.6) do not give rise to a set of Virasoro constraints. In fact the expectation values appearing in the left hand side of (1.6) cannot in general be represented in terms of derivatives with respect to the couplings appearing in an even potential. This is due to terms of the form $(\text{tr} \phi^{2l} \text{tr} \phi^{2n-k})$ with k odd. In the planar limit these terms factorize and vanish for even potentials. In this section we will show the equivalence between matrix models with even potentials and Volterra hierarchies. If one wants to work with arbitrary potentials one is led to

Toda hierarchies [10] and most of what we shall present generalizes straightforwardly to that case. The generic even potential is

$$V(M) = \sum_i g_i M^{2i} \quad (2.2)$$

if one defines orthogonal polynomials with respect to the measure

$$d\mu(\lambda) = d\lambda e^{-\beta V(\lambda)} \quad (2.3)$$

subject to the normalization condition

$$P_n(\lambda) = \lambda^n + \dots$$

$$\int d\mu(\lambda) P_n(\lambda) P_m(\lambda) = h_n \delta_{nm} \quad (2.4)$$

the partition function of the one matrix model can be written as [11]

$$Z(g) = h_0 h_1 \dots h_{N-1} \quad (2.5)$$

the orthogonal polynomials satisfy the two step recursion relation

$$\lambda P_n = P_{n+1} + R_n P_{n-1} \quad (2.6)$$

and

$$h_n = R_n h_{n-1} \quad (2.7)$$

hence

$$Z(g) = \prod_i R_i^{N-i} \quad (2.8)$$

changing basis from the P_n 's to an orthonormal basis,

$$\mathcal{P}_n = h^{-1/2} P_n \quad (2.9)$$

now the operation of multiplication by λ becomes

$$\lambda \mathcal{P}_n = \sum L_{nm} \mathcal{P}_m \quad (2.10)$$

where L is the Jacobi matrix

$$L = \sum \sqrt{R_k} E_{k,k+1}^+ \quad (2.11)$$

$$E_{k,k+1}^\pm = E_{k,k+1} \pm E_{k+1,k}$$

and E_{ij} is the matrix whose only non-vanishing entry is a 1 in position (i,j) . As we

change the couplings g_i ; the polynomials change as well as the matrix elements of L . Using simple homogeneity arguments and a theorem of Moser [12] it is not difficult to show that the matrix L changes with the couplings according to

$$\frac{\partial L}{\partial g_i} = [L_+^{2i}, L] \quad (2.12)$$

in (2.12) L_+^{2i} is defined as the antisymmetric matrix such that the strictly upper triangular parts of L_+^{2i} and L^{2i} coincide. In the continuum limit L becomes a Schrödinger operator after some simple renormalization, and L_+^{2i} is the discrete analogue of the operation of taking the differential part of a pseudodifferential operator. The matrices L_+^{2i} in our case are the analogues of taking the differential part of the operators $(-d^2/dx^2 + u(x))^{n+1/2}$, $n = 0, 1, 2, \dots$. In the theory of integrable systems [3] an equation of the form (2.12) is called a Lax equation and its complete integrability is a consequence of the existence of an infinite number of conservation laws in involution (with vanishing Poisson brackets). For the system (2.12) this means

$$\frac{\partial M_j}{\partial g_i} - \frac{\partial M_i}{\partial g_j} - [M_i, M_j] = 0 \quad M_n = L_+^{2n} \quad (2.13)$$

Notice that in this context the flows are generated only by the even powers of L_+^{2i} . The form of the matrix (2.11) implies that the only non-vanishing entries of the matrix L^n appear in positions $(i, i+n)$, $(i, i+n-2), \dots$ and their transpose, and furthermore in $[L_+^{2i}, L]$ the only non-vanishing entries appear at positions $(i, i+1)$ and $(i+1, i)$ whereas for $[L_+^{2n+1}, L]$ only the diagonal elements are different from zero. Consequently L_+^{2n} generates flows and L_+^{2n+1} provides initial conditions. This cannot be seen so clearly in the continuum limit where the form of the operators generated by $[L_+^{2n}, L]$ and $[L_+^{2n+1}, L]$ are essentially the same.

If the Volterra flows are associated to a potential with a finite number of non-zero couplings, the operator $d/d\lambda$ is also represented by a Jacobi matrix P [13] such that $P_{ij} \neq 0$ for $|i-j| \leq 2k-1$ if the potential has degree $2k$. The matrix P can be chosen to be antisymmetric and it obviously satisfies $[P, L] = 1$. Since P is a Jacobi matrix and $[P, L]$ is diagonal, we can write P as a linear combination of L_+^{2n+1} 's

$$P = \sum_{j=1}^k 2j c_j L_+^{2j-1} \quad [P, L] = 1 \quad (2.14)$$

Conversely, it is possible to prove using Favard's theorem [14] that beginning with a Volterra hierarchy together with the initial condition (2.14) for the flows, the set

of polynomials (2.6) defined by the operator L are an orthogonal set with respect to the measure (2.3). With this equivalence theorem it is possible to make a further approximation and instead of studying a Volterra hierarchy on the semi-infinite line one can set the problem of a finite lattice with the initial condition

$$[P, L] = \begin{pmatrix} 1 & 0 & \dots \\ 0 & 1 & \dots \\ \vdots & \vdots & \ddots \\ 0 & 0 & \dots & -N+1 \end{pmatrix} \quad (2.15)$$

since (2.12) is also completely integrable for a finite lattice with open boundary conditions it should be interesting to study the type of motion associated to the initial condition (2.14) as the lattice becomes bigger and denser approaching the continuum limit.

3. Hamiltonian Structures and the Scaling Limit

It is very instructive to study the hierarchy (2.12) from a hamiltonian point of view. This approach exhibits the discrete version of the Virasoro conditions found in [4]. The hierarchy (2.12) admits a collection of compatible Poisson structures. The most natural one is [7]

$$\{R_n, R_m\}_1 = R_n R_m (\delta_{n,m+1} - \delta_{n,m-1}) \quad (3.1)$$

and the flows (2.12) are generated by the hamiltonians

$$H_k = \frac{1}{2k} \text{tr} L^{2k}$$

$$H_1 = \sum_n R_n$$

$$H_2 = \sum_n \left(\frac{1}{2} R_n^2 + R_n R_{n+1} \right)$$

$$H_3 = \sum_n \left(\frac{1}{3} R_n^3 + R_n^2 R_{n+1} + R_n R_{n+1} R_{n+2} \right) \quad \dots \quad (3.2)$$

for example the original Volterra equation is obtained from H_1 [7]

$$\frac{\partial R_m}{\partial t} = R_m(R_{m+1} - R_{m-1}) = \{H_1, R_m\}_1 \quad (3.3)$$

It is more convenient for our purposes to introduce a new variable

$$u_m = \log R_m \quad (3.4)$$

u_m will become one of the scaling variables in the continuum limit, and

$$\frac{\partial u_m}{\partial g_n} = \{H_n, u_m\} = \left(\frac{\partial}{\partial u_{m+1}} - \frac{\partial}{\partial u_{m-1}} \right) H_n \quad (3.5)$$

hence

$$\frac{\partial u_m}{\partial g_n} = D F_m^{(n)}$$

where

$$D f_m = f_{m+1} - f_{m-1} \quad F_m^{(n)} = \frac{\partial H_n}{\partial u_m} \quad (3.6)$$

and one can check that the flows (3.5) commute.

The second hamiltonian structure which can be defined for the Volterra hierarchy [8] is

$$\{R_n, R_m\}_2 = R_n R_m (\delta_{n,m+1} - \delta_{n,m-1}) + R_n R_m (\delta_{n,m+2} R_{m+1} - \delta_{n,m-2} R_{m-1}) \quad (3.7)$$

the origin of the Virasoro conditions in [4] lies in the discrete analogue of the Gel'fand-Dickii relation [9] between the two hamiltonian structures of the KdV hierarchy. In the case of the Volterra hierarchy we find

$$\{H_n, R_m\}_2 = \{H_{n+1}, R_m\}_1 \quad (3.8)$$

The proof is as follows: using (3.2) and (3.7) one obtains

$$\frac{\partial R_m}{\partial t_n} = \frac{1}{2} R_m \text{tr} \delta_m^{(1)} L L^{2n-1} \quad (3.9)$$

where

$$\delta_m^{(1)} L = R_{m+1}^{1/2} E_{m+1,m+2}^+ - R_{m-1}^{1/2} E_{m-1,m}^+ \quad (3.10)$$

similarly

$$\begin{aligned} \{H_n, R_m\}_2 &= \frac{1}{2} R_m \text{tr} [R_{m+1}^{1/2} (R_m + R_{m+1}) E_{m+1,m+2}^+ - R_{m-1}^{1/2} (R_m + R_{m-1}) E_{m-1,m}^+ \\ &\quad + R_m R_{m+1} R_{m+2}^{1/2} E_{m+2,m+3}^+ - R_m R_{m-1} R_{m-2}^{1/2} E_{m-2,m-1}^+] L^{2n-1} \end{aligned} \quad (3.11)$$

writing $\partial R_m / \partial g_{n+1}$ as $\text{tr} L \delta_m^{(1)} L L^{2n-1}$ and subtracting it from (3.11) one obtains after some simple algebra that the difference is proportional to the matrix element $[L^2, L^{2n-1}]_{m,m+1}$ and therefore it vanishes. The origin of the Virasoro conditions on the partition function in the continuum limit [4] lies in the discrete Gel'fand-Dickii relation and in the fact that the second hamiltonian structure of the Volterra hierarchy gives in the continuum limit the Virasoro algebra. This discrete version of the Virasoro algebra appeared for the first time in [8] and it is also shown in this reference that the classical equations of motion for the discretized Liouville theory are equivalent to the classical Volterra equation (3.3). One wonders whether there is a direct connection between the discrete Liouville system described in [8] and the matrix model definition of 2D-gravity [1].

From (3.8) we immediately obtain

$$\frac{\partial u_m}{\partial g_{n+1}} = (R_m + R_{m+1}) F_{m+1}^{(n)} - (R_m + R_{m-1}) F_{m-1}^{(n)} + R_{m+1} F_{m+2}^{(n)} - R_{m-1} F_{m-2}^{(n)} \quad (3.12)$$

As we will show later, some linear combinations of the $F^{(n)}$'s become in the continuum limit the Gel'fand-Dickii potentials R_n and (3.12) becomes the ordinary Gel'fand-Dickii relation. In terms of hamiltonian flows the string equations become

$$\sum_j j g_j \left(\frac{\partial H_j}{\partial u_{m+1}} - \frac{\partial H_j}{\partial u_{m-1}} \right) = \beta^{-1} \quad (3.13)$$

and in terms of the partition function (2.8) we obtain

$$\sum_j j g_j \frac{\partial \log Z}{\partial g_j} = \frac{N(N+1)}{\beta} \quad (3.14)$$

using (3.13) and iterating (3.12) to relate higher with lower flows one obtains an infinite number of conditions on the partition function. In the continuum limit they become the Virasoro conditions of [4]. To understand these statements more clearly, we briefly analyze the double scaling limit of the Volterra hierarchy.

We will follow the notation of Gross and Migdal in ref.[1], further details can be found there. The string equation in Douglas form [2] has the discrete form

$$\sum_j 2j g_j [L_+^{2j-1}, L] = 2\beta^{-1} \quad (3.15)$$

since only the (m, m) components of this equation are different from zero, (3.15) is equivalent to

$$\sum_j 2j g_j (R_m^{1/2} L_{m,m+1}^{2j-1} - R_{m-1}^{1/2} L_{m-1,m}^{2j-1}) = \beta^{-1} \quad (3.16)$$

summing the first m equations in (3.16) we obtain

$$\sum_j 2j g_j R_m^{1/2} L_{m,m+1}^{2j-1} = m\beta^{-1} \quad (3.17)$$

going to the continuum limit one introduces the continuous variable $x = n/\beta$ and the k -th critical point is obtained by setting $x = 1 - \beta^{-\frac{2k}{2k+1}}t$ and $R = 1 - \beta^{\frac{2}{2k+1}}f(t)$. By tuning the couplings in (3.17) one obtains the k -th critical point. Let $\bar{g}_j^{(k)}$ be the corresponding values of the couplings, then we know from [1] that

$$\sum_i \mu_i \alpha_i^{(k)}(\beta) \sum_{j=1}^k 2j \bar{g}_j^{(k)} R_m^{1/2} L_{m,m+1}^{2j-1} = m\beta^{-1} \quad (3.18)$$

has a well defined scaling limit describing the perturbations by scaling operators of the k th critical point, with the constants $\alpha_i^{(k)}$ chosen appropriately [1]. Since the hamiltonians (3.2) are given as functionals of R_n which is not a scaling variable in the continuum limit, we have to find the linear combinations of hamiltonians with well defined scaling properties. Writing (3.5) in terms of the matrix elements of the powers of L we obtain

$$\frac{\partial u_m}{\partial g_n} = D \frac{\partial H_n}{\partial u_m}$$

with

$$\frac{\partial H_n}{\partial u_m} = R_m^{1/2} L_{m,m+1}^{2n-1} \quad (3.19)$$

since D has good scaling properties in the continuum limit we can write the string

equation as

$$\sum_i \mu_i \alpha_i^{(k)} \sum_j 2j \bar{g}_j^{(k)} \frac{\partial H_j}{\partial u_m} = m\beta^{-1} \quad (3.20)$$

and therefore

$$\mathcal{H}_i = \sum_j 2j \bar{g}_j^{(k)} H_j \quad (3.21)$$

is a combination of hamiltonians with good scaling properties. Therefore in the continuum limit we expect the scaling hamiltonians (3.21) to turn into the conserved charges of the KdV hierarchy. To show that this is indeed the case all we have to do is to show that the continuum limit of (3.8) is the standard Gelfand-Dickii relation [9], or equivalently that the second Poisson structure (3.7) becomes the Virasoro algebra. Although the conserved quantities for the Volterra hierarchy were written explicitly in (3.2), we can read (3.8) as a recursion relation for the conserved quantities of the hierarchy. From H_n we can compute H_{n+1} with a unique result if we require H_n to be homogeneous of degree n in R in order to fix the arbitrary constants appearing in the solution and which do not affect the equations of motion. Therefore, knowing H_1 we can derive all other conserved quantities H_n and they are guaranteed to be commuting due to the compatibility of the Poisson structures (i.e. that an arbitrary linear combination of the first and second Poisson structures is still a Poisson structure, a property easily verified in our case). The continuum limit of the Poisson structures is easier to derive in terms of the variables u_m defined in (3.4). For the first Poisson structure (3.1) we have

$$\{u_n, u_m\} = \delta_{n,m+1} - \delta_{n,m-1} \quad (3.22)$$

in the continuum limit $R_n \mapsto R(x) = 1 - \beta^{\frac{2}{2k+1}} f(t)$ with $x = 1 - \beta^{\frac{2k}{2k+1}}t$ and $u_m = \log R_m$, defining $\lambda = \beta^{\frac{2}{2k+1}}$ we have $u_m = -\lambda^2 f_m \mapsto -\lambda^2 f(t)$. Thus the renormalized first Poisson bracket becomes in the continuum limit

$$\{f(t), f(t')\}_1 = \lim_{\lambda \rightarrow 0} \lambda^2 \{f_n, f_m\} \quad (3.23)$$

Next we define the continuum limit of the second Poisson bracket. The easiest way to take this limit is to first multiply (3.7) by some arbitrary test functions $\alpha_n \beta_m$ and sum over n and m . To obtain a Poisson structure with good scaling properties we have to subtract the first Poisson structure with a coefficient depending appropriately on the scaling parameter λ . This is legitimate because the two hamiltonian structures are

compatible for any value of the cut-off. Up to some trivial numerical factors we can write the result as

$$\{f(t), f(t')\} = \left(\frac{1}{2}D^3 + Df + Df\right) \delta(t - t') \quad (3.24)$$

which is the classical form of the Virasoro algebra. As a consequence of the compatibility of the lattice Poisson structures these two brackets are compatible and as before they can be used to generate all the conserved quantities for the continuum system. Once we have the string equations and the Gelfand-Dickii relations one only needs to repeat the arguments on [4] to see that the infinite number of conditions on the partition function become the Virasoro constraints.

4. Conclusion

We have shown that many of the properties of the recently found non-critical strings follow directly from the complete integrability of the Volterra hierarchy equivalent to the original matrix model formulation. If one considers general potentials instead of even ones the three step recursion relation satisfied by the orthogonal polynomials leads to the Toda hierarchy and the arguments presented go through virtually unchanged. To summarize we have found that the integrability of the Volterra hierarchy together with a very particular initial condition gives the discrete string equations together with an infinite number of conditions on the partition function as a consequence of the discrete version of the Gel'fand-Dickii relations. The continuum limit of these relations are the Virasoro conditions found in [4]. It is worth while to point out that in terms of even potentials the origin of the Virasoro constraints is quite different from the relations (1.6) and also from a recent paper [15] a study of the relation between the Toda lattice and matrix models. From our point of view, it is the compatibility of the two Poisson structures of the Volterra hierarchy which is one of the basic reasons for its integrability and its consequence, (3.12) that explain the infinite number of constraints satisfied by the partition function and expressed in the continuum limit as the Virasoro constraints.

From the point of view of non-linear lattices the double scaling limit provides a novel method to obtain continuum limits for integrable mechanical systems. When the matrix model is seen in terms of the Dyson gas, we learn that the statistical properties of this gas are described by a completely integrable mechanical system.

It is known that many of the known conformal field theories can be obtained as special limits of integrable systems. In the examples described in this paper we have found that the effective action of some (non-unitary) conformal models coupled to

2 - D gravity are described by simple integrable systems subject to particular initial conditions. To what extent a similar procedure can be carried out for other known integrable systems is an interesting question currently under investigation.

Note Added While this letter was being typed we received a paper by E. Martinez [16] where the relation between lattice integrability and matrix models is also discussed.

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