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ALGEBRAIC ASPECTS OF THE HIGHER-SPIN PROBLEM

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ABSTRACT

A general algebraic construction is established, which underlies previously proposed consistent equations of interacting gauge fields of all spins in 3+1 dimensions. This construction makes verification of the consistency (gauge invariance) of the higher-spin equations trivial and indicates how these equations can be generalized to higher dimensions and/or conformal-type higher-spin theories.

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1. Introduction

Recently, totally consistent equations of motion of interacting gauge fields of all spins in $3+1$ dimensions were proposed in Refs. [1,2]. These equations describe infinite chains of massless fields with spins from 0 to ∞ . They are explicitly general co-ordinate invariant, possess all necessary gauge symmetries associated with spins $s > 1$ and contain Einstein-Yang-Mills equations in the sector of spins 1 and 2. As shown in Refs. [3,2], there exists a broad class of higher-spin gauge theories which correspond to classical sequences of Yang-Mills gauge groups. Although the higher-spin equations were cast into relatively compact form [1,2], the explicit verification of the consistency of these equations outlined in Appendix of Ref. [2] turns out to be rather involved. The main goal of this article is to bring the higher-spin equations of Ref [2] into a form in which the consistency of these equations becomes algebraically trivial. Perhaps, it is even more important that the proposed algebraic construction suggests some natural ways to generalize the higher-spin equations to higher dimensions $D > 4$ as well as to the case of the so-called conformal higher-spin theories [4,5] and their further generalizations.

2. Preliminaries

In Ref. [1], the dynamics of gauge higher-spin fields was described in terms of "generating functions" $W(Z, Y; K|x)$ and $B(Z, Y; K|x)$ which depend on space-time co-ordinates x^ν ($\nu = 0 - 3$), auxiliary Klein-type operators $K = (k, \bar{k})$ and twistor variables $Z = (z_\alpha, \bar{z}_{\dot{\alpha}})$ and $Y = (y_\alpha, \bar{y}_{\dot{\alpha}})$. The labels $\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$ are interpreted as indices of two-component spinors. The variables Z and Y are mutually commuting. By definition, the Klein operators k and $\bar{k}$ possess the properties

$$\{k, z_\alpha\} = \{k, y_\alpha\} = 0, [k, \bar{z}_{\dot{\alpha}}] = [k, \bar{y}_{\dot{\alpha}}] = 0, k^2 = 1; \quad (2.1a)$$

$$[\bar{k}, z_\alpha] = [\bar{k}, y_\alpha] = 0, \{\bar{k}, \bar{z}_{\dot{\alpha}}\} = \{\bar{k}, \bar{y}_{\dot{\alpha}}\} = 0, \bar{k}^2 = 1, [\bar{k}, \bar{k}] = 0 \quad (2.1b)$$

where $\{, \}$ and $[,]$ denote commutator and anticommutator, resp. i.e., k anticommutes with undotted spinors while $\bar{k}$ anticommutes with dotted spinors.

Physical higher-spin fields are identified with the coefficients of the expansions of generating functions W and B in powers of the auxiliary variables Z , Y and K . It is assumed that W is a one-form with respect to the space-time co-ordinates (i.e., $W(\dots) = dx^\nu W_\nu(\dots)$) while B is a zero-form (function). In fact, W serves as a generating function for gauge higher spins, while B involves non-gauge lower spins and on-mass-shell non-trivial higher-spin curvatures generalizing the gravitational Weyl tensor. In accordance with the normal relationship between spin and statistics the fields W and B (i.e., the physical fields which are identified with the coefficients of the expansion of W and B in powers of the auxiliary variables) are assumed to be (anti)commuting if they carry (odd)even numbers of spinorial indices or, equivalently, if the parities $\pi(W)$ and $\pi(B)$ defined by the relationship

$$W(-Z, -Y, K|x) = (-1)^{\pi(W)} W(Z, Y, K|x) \quad (2.2)$$

are equal to $(1)0$.

In Ref. [2], it was argued that in order to have the possibility of working with other useful forms of equations, which are connected with the equations of Ref. [1] through some non-linear field redefinitions, it is convenient to introduce an additional generating function $s(Z, Y; K|x)$ which behaves as a one-form with respect to additional differentials $dZ = (dz_\alpha, d\bar{z}_{\dot{\alpha}})$, i.e.,

$$s = dz^\alpha s_\alpha(Z, Y; K|x) + d\bar{z}^{\dot{\alpha}} \bar{s}_{\dot{\alpha}}(Z, Y; K|x). \quad (2.3)$$

By definition, the differentials dz^α and $d\bar{z}^{\dot{\alpha}}$ anticommute with themselves and with the corresponding Klein operators,

$$\{dz^\alpha, dz^\beta\} = 0, \{dz^\alpha, d\bar{z}^{\dot{\beta}}\} = 0, \{d\bar{z}^{\dot{\alpha}}, d\bar{z}^{\dot{\beta}}\} = 0, \quad (2.4)$$

$$\{dz^\alpha, k\} = [dz^\alpha, \bar{k}] = 0, [d\bar{z}^{\dot{\alpha}}, k] = \{d\bar{z}^{\dot{\alpha}}, \bar{k}\} = 0. \quad (2.5)$$

As for the choice of commutation relations between dZ and dx^ν , this is not essential. In Ref. [2], it was assumed that dx^ν and dZ anticommute. Here it is more convenient to require dx^ν to commute with all auxiliary variables dZ, Z, Y and K .

The algebra of functions $g(Z, Y; K)$ can be endowed with the associative product law $\star$ that plays a basic role in our construction,

$$(g \star f)(Z, Y_0; K) = \int d^4 Y_1 d^4 Y_2 \exp[-i \sum_{\substack{n, m=0 \\ n > m}}^2 (-1)^{n+m} (Y_m, Y_n)] \quad (2.6)$$

$$\times g(Z, Y_1; K) f(Z + Y_0 - Y_2, Y_2; K)$$

where the symplectic form (Y_m, Y_n) is defined as follows:

$$(Y_m, Y_n) = y_{m\alpha} y_n^\alpha + \bar{y}_{m\dot{\alpha}} \bar{y}_n^{\dot{\alpha}}, \quad (2.7)$$

$$y_{m\alpha} = \epsilon_{\beta\alpha} y_m^\beta, \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \epsilon_{12} = 1. \quad (2.8)$$

In the definition (2.6), it is assumed that the Klein operators k and $\bar{k}$ anticommute with all undotted and dotted spinorial variables, respectively. The functions g and f in Eq. (2.6) are allowed to be differential forms, i.e., to depend on the variables dx^μ , dz^α and $d\bar{z}^{\dot{\alpha}}$ which can be thought of as some external parameters (however, taking into account Eq. (2.5)).

The associativity of the product law (2.6) can easily be checked directly. This product law admits [2] a simple interpretation in terms of the operator realization of higher-spin superalgebras of Ref. [6]. Namely, the associative algebra A constituted by functions (power series) $f(Z, Y; K)$ with the product law (2.6) is of the structure $A_\ell \otimes A$, where the associative algebra $A_\ell(A)$ is generated by the variables z, y and $k(z, \bar{y}$ and $\bar{k})$. A_ℓ can be defined as the associative algebra with generating elements $\hat{z}_\alpha, \hat{y}_\alpha$ and $\hat{k}$ obeying the relations

$$[\hat{z}_\alpha, \hat{z}_\beta] = 0, [\hat{z}_\alpha, \hat{y}_\beta] = -i\epsilon_{\alpha\beta}, [\hat{y}_\alpha, \hat{y}_\beta] = 2i\epsilon_{\alpha\beta}, \quad (2.9a)$$

$$\{\hat{k}, \hat{z}_\alpha\} = \{\hat{k}, \hat{y}_\alpha\} = 0, \hat{k}^2 = 1. \quad (2.9b)$$

Evidently, one can choose such linear combinations of $\hat{z}_\alpha$ and $\hat{y}_\beta$, say $\hat{a}_\alpha$ and $\hat{b}_\alpha$, that

$$[\hat{a}_\alpha, \hat{a}_\beta] = [\hat{b}_\alpha, \hat{b}_\beta] = 0, [\hat{a}_\alpha, \hat{b}_\beta] = i\epsilon_{\alpha\beta}. \quad (2.10)$$

Analogous definition holds for A_r . Now, Eq. (2.6) can be interpreted as the integral formula for

the product law of symbols of operators of Berezin [7] corresponding to the “normal ordering” in which the operators $\hat{a}_\alpha(\hat{b}_\alpha)$ stand from the left (right) (for more details, see Refs. [2, 6]).

3. Equations of interacting gauge fields of all spins in 3+1 dimensions

A crucial observation of Refs. [1, 2] was that the part of the higher-spin equations which involve space-time differentials of W and B can be cast into the form $\star$

$$dW = W \star W, \quad (3.1)$$

$$dB = W \star B - B \star W. \quad (3.2)$$

Equation (3.1) can be interpreted as the zero-strength equation for the infinite-dimensional Lie superalgebra originating from the associative algebra A of Section 2 after transition to the graded commutator $[f, g] = f \star g - (-1)^{\pi(f)\pi(g)} g \star f$ (the additional sign factor involving parity (2.2) emerges since fermionic physical fields are anticommuting). Equation (3.2) implies that the zero-form B is covariantly constant.

The central observation of this paper is that the rest equations of Ref. [2] which govern the dependence on the auxiliary spinorial variables Z , and, in fact, make the total system of equations dynamically non-trivial, can be reduced to the form

$$[Q, W]_\star = ds - [W, s]_\star + [P, W]_\star \star B, \quad (3.3)$$

$$[Q, B]_\star = [s, B]_\star + [P, B]_\star \star B, \quad (3.4)$$

$$\{Q, s\}_\star = s \star s + \{P, s\}_\star \star B. \quad (3.5)$$

Here $[a, b]_\star = a \star b - b \star a$ and $\{a, b\}_\star = a \star b + b \star a$, while Q and P are the following fixed

$\star$ We systematically omit the wedge product sign $\wedge$ because only exterior products of space-time differential forms are used throughout this paper. Also, we often omit the space-time argument x of higher-spin generating functions keeping in mind that the exterior differential $d = dx^\mu(\partial/\partial x^\mu)$ is the only operator which acts non-trivially on the space-time co-ordinates.

operators:

$$Q = -i(dz^\alpha(2z_\alpha + y_\alpha) + d\bar{z}^\alpha(2\bar{z}_\alpha + \bar{y}_\alpha)), \quad (3.6)$$

$$P = \int_0^1 dt [\mu dz^\alpha z_\alpha \exp(2it z_\alpha y^\alpha) \bar{k} + \bar{\mu} d\bar{z}^\alpha \bar{z}_\alpha \exp(2it \bar{z}_\alpha \bar{y}^\alpha) \bar{k}], \quad (3.7)$$

where μ is an arbitrary complex parameter while $\bar{\mu}$ is its complex conjugate. The fact that the left-hand sides of Eqs. (3.3)-(3.5) have the above form was already mentioned in Ref. [2] where it was observed that the inner differentiation $[Q, f]_\star$ is equivalent to the ordinary differentiation $(dz^\alpha(\partial/\partial z^\alpha) + d\bar{z}^\alpha(\partial/\partial \bar{z}^\alpha)) f(Z, Y; K)$. As for the P -dependent terms on the right-hand sides of Eqs. (3.3)-(3.5), it is elementary to verify that they reproduce exactly the terms involving the factors of ∇ in the left form of the higher-spin equations of Refs. [1,2] (to do that, one should use the composition law (2.6) and an integral representation for the function ∇ given in Eq. (5.4) of Ref. [2]). As a result, one concludes that Eqs. (3.1)-(3.5) are equivalent to the equations of motion of massless higher spins proposed in Ref. [2].

In order to prove that Eqs. (3.1)-(3.5) are formally consistent (and, therefore, gauge invariant [8]), one should use the following main properties of the operators Q and P , which are simple consequences of Eqs. (3.6), (3.7):

$$Q \star Q = c_1, \quad \{Q, P\}_\star = -c_2, \quad P \star P = 0 \quad (3.8)$$

where

$$c_1 = i(dz_\alpha dz^\alpha + d\bar{z}_\alpha d\bar{z}^\alpha), \quad c_2 = \frac{1}{2}(\mu \bar{k} dz_\alpha dz^\alpha + \bar{\mu} \bar{k} d\bar{z}_\alpha d\bar{z}^\alpha); \quad (3.9)$$

$$\kappa = k \exp(2i z_\alpha y^\alpha), \quad \bar{\kappa} = \bar{k} \exp(2i \bar{z}_\alpha \bar{y}^\alpha). \quad (3.10)$$

As mentioned in Ref. [2], the functions $\chi = e^{2iz_\alpha y^\alpha}$ and $\bar{\chi} = e^{2i\bar{z}_\alpha \bar{y}^\alpha}$ behave as Klein operators with respect to commuting spinorial variables, i.e., $\chi \star f(z, y; k) = f(-z, -y; k) \star \chi$; $\chi \star \chi = 1$ and analogously for $\bar{\chi}$. Taking into account that, by definition of k and $\bar{k}$, they anticommute

with all spinors (both commuting and anticommuting), one concludes that

$$\kappa dz^\alpha = -dz^\alpha \kappa, \quad \bar{\kappa} d\bar{z}^\alpha = -d\bar{z}^\alpha \bar{\kappa}, \quad \kappa^2 = \bar{\kappa}^2 = I. \quad (3.11)$$

Evidently, κ and $\bar{\kappa}$ commute with all other generating elements Z, Y and K .

It is highly essential that the quantities c_1 and c_2 behave as central elements commuting with everything. As for c_1 , this is trivial. The situation with c_2 is more interesting. For the first sight, the factors of κ and $\bar{\kappa}$ may lead to non-commutativity of c_2 with some of polynomials of dz and $d\bar{z}$. However, all potentially dangerous terms vanish since $dz_\alpha d\bar{z}_\beta dz^\beta \equiv 0$ and $d\bar{z}_\alpha d\bar{z}_\beta d\bar{z}^\beta \equiv 0$ for anticommuting two-component spinors.

Consistency of the equations of motion (3.1)-(3.5) means that they should respect the identities $d^2 f \equiv 0$ and $[Q^2, f]_\star \equiv 0$ for $f = W, B$ and s . With the aid of the properties (3.8)-(3.10), it is not difficult to verify consistency of Eqs. (3.1)-(3.5). However, the consistency proof can be trivialized even further with the aid of the following field redefinition

$$s = s' + Q - P \star B. \quad (3.12)$$

Indeed, Eqs. (3.3)-(3.5) now take the form

$$ds = [W, s]_\star, \quad (3.13)$$

$$s \star s = c_1 + c_2 B, \quad (3.14)$$

$$[s, B]_\star = 0 \quad (3.15)$$

(neglecting primes).

Higher spin equations of motion in the form (3.1), (3.2), (3.13)-(3.15) do not involve the operators Q and P (3.6), (3.7). These operators reappear, when solving Eq. (3.14) order by order in powers of B . In accordance with Eq. (3.12), Q and $P \star B$ describe zeroth and first order contributions to s' while s in Eq. (3.12) is assumed to involve only highest orders.

Let us stress that the higher-spin equations in the form (3.1), (3.2), (3.13)-(3.15) become dynamically non-trivial (interacting) because the central elements c_1 and c_2 are essentially distinct. Indeed, an attempt to replace c_2 by some expression of the form $\mu dz_\alpha dz^\alpha + \bar{\mu} d\bar{z}_\alpha d\bar{z}^\alpha$ leads to dynamically trivial equations at least in the linear in B approximation. The reason is that, in this case, the form of the equations analogous to Eqs. (3.3)-(3.5) would involve terms of the form $[\bar{Q}, W]_\star \star B$, and $[\bar{Q}, B]_\star \star B$ in place of $[P, W]_\star \star B$ and $[P, B]_\star \star B$ where $\bar{Q} = \mu dz_\alpha(2y^\alpha + z^\alpha) + \bar{\mu} d\bar{z}_\alpha(2\bar{y}^\alpha + \bar{z}^\alpha)$. Such terms do not contribute in the linearized approximation since $[\bar{Q}, W]_\star$ and $[\bar{Q}, B]_\star$ vanish to lowest order in B as a consequence of Eqs. (3.3)-(3.5). Let us emphasize that the central element $c_2 \neq c_1$ exists because the Klein operators k and $\bar{k}$ were introduced.

Equations of motion in the form (3.1), (3.2), (3.13)-(3.15) look very symmetric and are obviously consistent. However, now it is not so immediate to analyze their physical content. This question was fully analyzed in Ref. [8] and in Refs. [2,9] where it was shown that the form of equations of this paper is equivalent to that of Ref. [8]. The final result is that, disregarding auxiliary fields which possess no physical degrees of freedom, Eqs. (3.1), (3.2), (3.13)-(3.15) describe an infinite chain of massless fields of all spins $s = 0, 1/2, 1, \dots, \infty$ in which every spin appears twice. In the sector of one of the two spin-2 fields, these equations reduce to Einstein equations with the cosmological term (unbroken higher-spin gauge symmetries require non-vanishing cosmological constant [10], i.e., one should use anti-de Sitter background space to develop a self-consistent perturbative expansion).

As emphasized in Refs. [8,3], in order to describe systems of massless spins of all spins with non-trivial Yang-Mills internal symmetries it suffices to note that the higher-spin equations remain consistent when all fields, W, B and s , take values in an arbitrary associative algebra, say $\text{Mat}_n(\mathbb{C})$, and then apply one or another truncation procedure based on automorphisms of the higher-spin equations (one can prove consistency of Eqs. (3.1), (3.2), (3.13)-(3.15) without commuting product factors).

So far, we discussed higher-spin equations in 3+1 dimensions. The physical content of these equations is fully clarified due to the detailed perturbative analysis of Refs. [10-13,8]. However, the level of algebraic simplicity of the proposed formulation is so impressive that it becomes

difficult to overcome a desire to guess generalizations of these equations to higher dimensions ($d > 4$) and/or to conformal-type higher-spin theories.

4. Generalizations

A crucial problem of the higher-spin gauge theory consists of generalizations of four-dimensional results to higher dimensions $D > 4$. Needless to say this is highly important for developing a mechanism of spontaneous breakdown of higher-spin gauge symmetries via compactification of extra dimensions. The linearized analysis of Refs. [14,15] indicated that consistent higher-spin gauge theories exist in all dimensions. In Ref. [15], we proposed infinite-dimensional superalgebras for the role of D -dimensional higher-spin superalgebras which at the same time can be interpreted as conformal higher-spin superalgebras in $D-1$ dimensions (close ideas were used in Refs. [4,5]). The key point consists in replacing four-dimensional spinors by D -dimensional ones.

Let us consider the case of even dimensions $D = 2k$. In place of bosonic spinorial variables Z and Y used in the previous sections, we introduce two pairs of spinorial variables $(\hat{Y}_A, \hat{Y}^A)$ and $(\hat{Z}_A, \hat{Z}^A)$ with $A = 1, 2, \dots, 2^k$, obeying the commutation relations

$$[\hat{Z}_A, \hat{Y}^B] = [\hat{Z}_A, \hat{Z}^B] = \delta_A^B. \quad (4.1)$$

Generally, spinors with upper and lower indices are assumed to be independent (in that case, it is assumed that the right-hand sides of all other commutation relations vanish). However, if there exists an invariant antisymmetric charge conjugation matrix C_{AB} ($D = 2, 4 \pmod{8}$) for even D), we impose the Majorana conditions $\hat{Z}_A = C_{BA} \hat{Z}^B$ and $\hat{Y}_A = C_{BA} \hat{Y}^B$. Note that C_{AB} is forced to be antisymmetric as the left-hand sides of Eq. (4.1) involve commutators.

Let γ_a^B be a set of Dirac matrices in D dimensions $\{\gamma_a, \gamma_b\} = 2\eta_{ab}I$ where η_{ab} is a flat Minkowski metric. Generators of AdS algebra $\mathfrak{o}(2, D-1)$ admit a realization

$$L_{ab} = [\gamma_a, \gamma_b]_B^A (\hat{Z}^B \hat{Z}_A + \hat{Y}^B \hat{Y}_A), \quad (4.2a)$$

$$P_a = \gamma_{aB}^A (\hat{Z}^B \hat{Z}_A + \rho \hat{Z}^B \hat{Z}_A), \quad (4.2b)$$

where L_{ab} and P_a are generators of Lorentz transformations and AdS translations, resp., while $\rho = \pm 1$. Let us introduce the matrix $\Gamma = (-1)^{D(D-1)/2} \gamma_0 \dots \gamma_{D-1}$ possessing the properties $\Gamma^\dagger = \Gamma$, $\Gamma^2 = I$, $\{\Gamma, \gamma_a\} = 0$, as well as the related chiral projectors $\Pi_\pm = 1/2(1 \pm \Gamma)$ and chiral spinors,

$$\psi_{\pm B} = (\Pi_\pm \psi)_B, \quad \psi_\pm^B = (\psi \Pi_\pm)^B. \quad (4.3)$$

Evidently, the generators L_{ab} connect the same chiralities while P_a connect opposite chiralities,

$$L_{ab} = [\gamma_a, \gamma_b]_B^A (\hat{Z}^B \hat{Z}_{+A} + \hat{Z}^B \hat{Z}_{-A} + \hat{Y}_+^B \hat{Y}_{+A} + \hat{Y}_-^B \hat{Y}_{-A}), \quad (4.4a)$$

$$P_a = \gamma_{aB}^A (\hat{Z}^B \hat{Z}_{-A} + \hat{Z}^B \hat{Z}_{+A} + \rho (\hat{Y}_+^B \hat{Y}_{-A} + \hat{Y}_-^B \hat{Y}_{+A})). \quad (4.4b)$$

Klein operators $\hat{k}_\pm$ can now be defined by the relations

$$\{\hat{k}_\pm, \hat{Z}_\pm\} = \{\hat{k}_\pm, \hat{Y}_\pm\} = [\hat{k}_\pm, \hat{Z}_\mp] = [\hat{k}_\pm, \hat{Y}_\mp] = 0, \quad (4.5)$$

$\hat{k}_\pm^2 = I$, $[\hat{k}_+, \hat{k}_-] = 0$, ($\hat{Z}_\pm$ ($\hat{Y}_\pm$) denote both $\hat{Z}_{\pm A}$ and $\hat{Z}_\pm^A$ ($\hat{Y}_{\pm A}$ and $\hat{Y}_\pm^A$)). Thus, the Klein operators $\hat{k}_\pm$ commute with the Lorentz generators and anticommute with P_a . This property is highly important for the formulation of the higher-spin dynamics along the lines of Ref. [8].

We have introduced double the number of spinor generating elements, $\hat{Z}$ and $\hat{Y}$, assuming that a dependence on $\hat{Z}$ will be governed by certain equations analogous to Eqs (3.3)-(3.5). As before, we introduce anticommuting spinorial differentials dZ_A and dZ^B as well as the corresponding "gauge one-form" $s = dZ_A s^A + dZ^A s_A$. Now the problem reduces to establishing an appropriate generalization of Eq. (3.14) while Eqs. (3.1), (3.2), (3.13) and (3.15) generalize trivially. A crucial observation is that there exists a unique natural generalization of the central element c_2 ,

$$c_2 = \mu_+ c_{+2} + \mu_- c_{-2}, \quad c_{\pm 2} = \kappa_\pm (dZ_{\pm A} dZ_\pm^A)^{\eta(D)} \quad (4.6)$$

where $\eta(D)$ is a maximal degree such that $c_2 \neq 0$, i.e., $\eta(2k) = 2^{k-1}$ for $k = 1, 2 \pmod{4}$ when spinors are Majorana, and $\eta(2k) = 2^k$ otherwise. As previously, κ_+ and κ_- are assumed to

anticommute with dZ_+ and dZ_- , resp., while commuting with all other generating elements (κ_+ and κ_- admit representations analogous to Eq. (3.10)). μ is an arbitrary complex deformation parameter.

Thus, with respect to the differentials dZ , the right-hand side of Eq. (4.6) is a degree $2\eta(D)$ -form. On the other hand, there are some indications that s should be linear in dZ for all D . This fixes a form of a counterpart of Eq. (3.14) unambiguously,

$$(\star s)^{2\eta(D)} = c_1 + c_2 B \quad (4.7)$$

with

$$c_1 = (dZ_{+A} dZ_+^A)^{\eta(D)} + (dZ_{-A} dZ_-^A)^{\eta(D)}. \quad (4.8)$$

The rest equations are of the same form $\star$ as for $D = 4$,

$$dW = W \star W, \quad dB = [W, B]_\star, \quad ds = [W, s]_\star, \quad \{s, B\}_\star = 0. \quad (4.9)$$

It should be noted, however, that a space of dynamical variables can be restricted by requiring all generating functions to commute with some set of elements a_i of the associative algebra A in which these generating functions take their values,

$$[a^i, W]_\star = 0, \quad [a^i, B]_\star = 0, \quad [a^i, s]_\star = 0. \quad (4.10)$$

(The possibility of imposing additional conditions of this type was suggested by Fradkin and Linetsky in the context of conformal higher-spin superalgebras [5] - see also below.) These conditions are formally consistent with Eqs. (4.7), (4.9) since $c_{1,2}$ belong to the centre of A . However, to avoid a dynamically trivial situation, one should in addition require a^i to commute

$\star$ Strictly speaking, there is another way for generalizations by allowing the field s to involve highest degrees of dZ that enables one to decrease a degree of s on the left-hand side of Eq. (4.7). However, this may lead to imposing too stringent restrictions on W due to the conditions $ds = [W, s]$. A possible way out may consist of allowing W to depend on dZ as well. Also one cannot a priori exclude a possibility that W will involve highest space-time differential forms for $D > 4$. We hope to analyze these possibilities elsewhere.

with the AdS generators (4.2) and with a linearized part of s . There are three types of bilinear operators a^i possessing this property,

$$a^0 = \hat{Y}_A \hat{Y}^A, \quad a^1 = \tilde{C}_{AB} \hat{Y}^A \hat{Y}^B, \quad a^{-1} = \tilde{C}^{-1AB} \hat{Y}_A \hat{Y}_B. \quad (4.11)$$

Note that all operators (4.11) trivialize for Majorana spinors. Operators $a^{\pm 1}$ make sense when there exists an invariant symmetric “charge conjugation matrix” $\tilde{C}_{AB} = \tilde{C}_{BA}$ ($D = 0, 6, 7 \pmod{8}$). In addition, we expect that in many cases, some conditions (4.10) should be imposed with AdS invariant operators a^i involving highest degrees of $\hat{Y}$. It is worth mentioning that in presence of fermions one has to modify operators a^i in an appropriate way with the aid of the Klein operators, to guarantee fermions to survive Eq. (4.10) (e.g., $a^0 \rightarrow \hat{Y}_A \hat{Y}^A + n/2 \hat{k} \cdot \hat{k}$ for some integer n). Analogously to the case of $D = 4$, higher-spin systems with non-trivial internal symmetries can now be introduced by virtue of embedding all quantities in appropriate matrix algebras [8, 3]. A detailed discussion of all these questions will be given elsewhere, as well as the analysis of the higher-spin equations for odd D , which differs essentially from the above analysis of even D .

Now we are in a position to discuss generalizations of Eqs. (4.6)-(4.11) which, in particular, may describe conformal higher-spin theories investigated in Refs. [4, 5] as well as their further extensions. Such generalizations are based on enlarged sets of generating elements. The simplest possibility consists of tensoring the original algebra by endowing all generating elements, $\hat{Z}^u$, $\hat{Y}^u$, dZ^u , with an additional label $u = 1, \dots, n$, assuming all variables to be mutually commuting for distinct values of u . Since $c_{\pm 1, 2}^u$ of the form (4.6), (4.8) commute with everything for every value of u , one can now use Eqs. (4.7), (4.9), (4.10) with $c_{1, 2}$ replaced by arbitrary functions $\sigma_{1, 2}(c_{\pm 1, 2}^u)$ without spoiling consistency. For example, one can set $\sigma_{1, 2} = \sum_{u=1}^n c_{1, 2}^u$. It is worth mentioning that for some functions $\sigma_{1, 2}$, Eq. (4.8) may turn out to be invariant under some of the automorphisms for the original associative algebra A that gives a possibility to truncate the equations in the same fashion as discussed in Refs. [2, 3, 8]. Specifically, if $\sigma_{1, 2}$ do not depend on some of the Klein operators, one can truncate the system by setting these Klein operators equal to unity. Note also that there is a subclass of consistent equations which result when Eqs. (4.6)-(4.11) written originally in $D + \ell$ dimensions are applied to the case of D -dimensional

space-time (the label ν of dx^ν runs from 0 to $D - 1$). Since $D + \ell$ -dimensional spinors can be interpreted as a set of D -dimensional spinors, this possibility is a particular case of those mentioned above.

For definiteness, let us focus on the $D = 4$ conformal higher-spin theories introduced by Fradkin and Linetsky in Ref. [5] where these theories were analyzed in the lowest order in interactions. A first step towards totally consistent equations consists of doubling the auxiliary variables, Z , Y , $K \rightarrow Z^u$, Y^u , K^u with $u = 1, 2$. Then, there are just two possibilities: (i) $c_1^{\text{conf}} = \sum_{u=1}^2 (c_{\pm 1}^u + c_{\mp 1}^u)$, $c_2^{\text{conf}} = \sum_{u=1}^2 (\mu + u c_{+2}^u + \mu - u c_{-2}^u)$, and (ii) $c_1^{\text{conf}} = c_{+1}^1 c_{-1}^2 + c_{-1}^1 c_{+1}^2$, $c_2^{\text{conf}} = \mu + c_{+2}^1 c_{+2}^2 + \mu - c_{-2}^1 c_{-2}^2$ where $c_{\pm 1, 2}^u$ are assumed to be of the form (4.6), (4.8) for every value of u . The degree $\eta(D)$ on the left-hand side of Eq. (4.7) should be set equal to 1 or 2 for the cases (i) and (ii) resp. In addition, one can require all quantities to commute with the operator $a = Y_{1A} Y_2^A + \frac{1}{2} n (k_+^1 k_-^1 k_+^2 k_-^2)$, n being an arbitrary fixed integer, which generalizes the “particle number operator” T introduced in Ref. [5] (a coincides with T when $n = 0$ or 1 for purely bosonic or supersymmetric case resp; for $n > 1$ one arrives at new conformal higher-spin theories which are not, however, superconformal in the usual sense). To this end, it should be noted that the condition (4.10) with a in place of a^i probably leads to too strong of a restriction on the field variables for the case (i), because a does not commute with the Klein operators $k_{\pm 1}^u$ and $k_{\pm 2}^u$ which play essential role in the formulation (i). The formulation (ii) is more natural in this respect (and looks therefore as more promising) since one can now truncate equations by requiring all field variables to depend only on the combinations $k_{\pm 1}^1 \cdot k_{\pm 2}^2$ which commute with a .

One can continue this model building process by taking more and more species Z^u , Y^u and K^u with $u = 1, \dots, \ell$. The operator a generalizes to the set of operators $a^{uv} = -a^{vu} = Y^{uA} Y_A^v + \psi^u \psi^v$ constituting the $\mathfrak{o}(\ell)$ Lie algebra, while the corresponding conditions (4.10) now imply that physical fields are $\mathfrak{o}(\ell)$ -singlets. Here ψ^u are additional Clifford generating elements, $\{\psi^u, \psi^v\} = \delta^{uv}$, which should be introduced in order to allow fermion physical fields to survive. (This construction generalizes to higher ℓ that proposed in Ref. [5] for the case $\ell = 2$, which in its turn is equivalent to the construction of a with the Klein operators described above.)

As a result, one arrives at the class of generalized higher-spin theories parametrized by ℓ . For $\ell = 1$, we get normal higher-spin equations which generalize Einstein equations. For $\ell = 2$,

the above construction very likely leads to the Fradkin-Linetsky conformal higher-spin theories which generalize Weyl gravity and involve higher derivatives in the field equations (which in turn serve as a source of negative-norm states upon quantization). One can speculate that the order of derivatives in the field equations will increase with ℓ and therefore only $\ell = 1$ higher-spin theories are expected to be self-consistent quantum-mechanically.

5. Conclusion

The equations of motion of interacting gauge higher spins in the form (3.1), (3.2), (3.13)-(3.15) are very compact and essentially geometrical in their form that gives a hope that the proposed approach will be both technically efficient and ideologically fruitful. To illustrate that this hope is real enough, let us note that, at least locally, the problem of solving the higher-spin equations (3.1), (3.2), (3.13)-(3.15) reduces to some purely algebraic problem.

Indeed, locally, Eqs. (3.1), (3.2), (3.13) admit a pure gauge solution

$$W(x) = dg(x) \star g^{-1}(x); \quad (5.1)$$

$$B(x) = g(x) \star B_0 \star g^{-1}(x), \quad s(x) = g(x) \star s_0 \star g^{-1}(x), \quad (5.2)$$

where $g(x)$ is an arbitrary function of the space-time co-ordinates x^ν , which takes its values in the associative algebra A and is invertible in the sense that $g \star g^{-1} = g^{-1} \star g = I$. As for B_0 and s_0 , these are arbitrary x -independent elements of A (i.e., $dB_0 = ds_0 = 0$). Now the remaining equations (3.14), (3.15) reduce to the following

$$s_0 \star B_0 = B_0 \star s_0, \quad s_0 \star s_0 = c_1 + c_2 B_0 \quad (5.3)$$

(recall that c_1 and c_2 belong to the centre of A and therefore commute with $g(x)$).

The freedom in the function $g(x)$ corresponds to the gauge arbitrariness of the theory. Thus, Eqs. (5.3) remain as the only non-trivial equations. Remarkably, these equations do not involve any dependence on the space-time co-ordinates. For first sight, it seems rather strange

that the problem of solving non-linear equations of interacting higher-spin gauge fields reduces to some purely algebraic problem. However, this paradox can be resolved by noting that, as emphasized in Ref. [8], the Weyl zero-forms $B = B(Z, Y; K)$ describe all gauge invariant combinations of derivatives of physical fields which remain non-vanishing when higher-spin equations of motion hold. More precisely, Eqs. (5.3) determine which functions of auxiliary variables B_0 can describe a full set of derivatives of physical fields at some fixed point x_0^ν under the condition that the dynamical equations of motion hold. On the other hand, if all derivatives of certain field $\varphi(x)$ are known for some point x_0 , a space-time dependence of $\varphi(x)$ can be reconstructed in some neighbourhood of x_0 for sufficiently smooth functions $\varphi(x)$. We hope to discuss the questions of completeness and physical interpretation of Eqs. (5.1)-(5.3) in a separate publication.

An intriguing related problem is if it is possible to reformulate in an analogous way conventional relativistic equations, such as Einstein equations, Yang-Mills equations, supergravity equations and others. Although, the results of Ref. [9] indicate that such a reformulation exists, a complete solution of this problem is not yet obtained. This reformulation may be very interesting from the viewpoint of constructing solutions of the conventional lower-spin non-linear equations via solutions of some counterparts of Eq. (5.3). As a matter of fact, the problem reduces to establishing consistent lower-spin truncations of the higher-spin equations of motion (3.1), (3.2), (3.13)-(3.15).

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REFERENCES

1. M.A. Vasiliev, *Phys.Lett.* **B243** (1990), 378.
2. M.A. Vasiliev, Lebedev Institute Preprint (1990) N89.
3. S.E. Konstein and M.A. Vasiliev, *Nucl.Phys.* **B331** (1990), 475.
4. E.S. Fradkin and V.Ya. Linetsky, *Mod.Phys.Lett.* **A4** (1989), 731, *Ann.Phys.(N.Y.)* **198** (1990), 293.
5. E.S. Fradkin and V.Ya. Linetsky, *Ann.Phys.(N.Y.)* **198** (1990), 252, *Phys.Lett.* **B231** (1989), 97; Göteborg Preprint 90-9.
6. M.A. Vasiliev, *Fortschr.Phys.* **36** (1988), 33.
7. F.A. Berezin, The Method of Second Quantization (Nauka, Moscow, 1986) (Second edition, in Russian);
F.A. Berezin and M.S. Marinov, *Ann.Phys.(N.Y.)* **104** (1977), 336, and references therein.
8. M.A. Vasiliev, *Phys.Lett.* **B209** (1988), 491, *Ann.Phys.(N.Y.)* **190** (1989), 59.
9. M.A. Vasiliev, *Nucl.Phys.* **B324** (1989), 503.
10. E.S. Fradkin and M.A. Vasiliev, *Phys.Lett.* **B189** (1987), 89, *Nucl.Phys.* **B291** (1987), 141.
11. M.A. Vasiliev, *Fortschr.Phys.* **35** (1987), 741.
12. E.S. Fradkin and M.A. Vasiliev, *Ann.Phys.(N.Y.)* **177** (1987), 63.
13. M.A. Vasiliev, *Nucl.Phys.* **B307** (1988), 319.
14. V.E. Lopatin and M.A. Vasiliev, *Mod.Phys.Lett.* **A3** (1988), 257.
15. M.A. Vasiliev, *Nucl.Phys.* **B301** (1988), 26.