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BOUNDS ON SHORT-RANGE CORRELATIONS IN BARYONS

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Abstract

The coefficients $\delta_{ij} = \langle \Phi | \delta^{(3)}(r_{ij}) | \Phi \rangle$, which measure the zero-range correlation between two constituents in a bound state, enter into the calculation of many physical quantities. Simple upper bounds are derived for these coefficients in the case of symmetric baryons.

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I. Introduction

Many properties of composite systems involve the probability of finding two constituents at the same position. This is the case for the hyperfine splittings in atoms, molecules, mesons or baryons, when one uses spin-spin forces of Breit-Fermi type, which have zero range. One may also mention some production rates, e.g. for quarkonium formation in electron-positron or proton-antiproton collisions; decay rates such as positronium or positronium ion ($e^+e^-e^-$) annihilation, hadronic decay of quarkonium via two or three gluons; nuclear fusion in muonic molecules like $dd\mu$; parity violation in atoms [1,2], etc...

In the present paper, we discuss some properties of the matrix elements

$$\delta_{ij} = \langle \Phi | \mathcal{G}^{(3)}(r_{ij}) | \Phi \rangle \quad (1)$$

Efficient methods have been developed to calculate the δ_{ij} and already applied to a variety of systems [1,2]. After a short review of the 2-body case, we shall present some new results for baryons in the non-relativistic quark model.

II. The Two-Body Problem with Central Forces

In a 2-body system bound by central forces, one deals with the radial equation

$$u''(r) - \frac{l(l+1)}{r^2}u(r) + m(E - V(r))u(r) = 0 \quad (2)$$

where m is twice the reduced mass and r the relative distance. The reduced radial wavefunction $u(r)$ can be chosen real if V is real. After multiplying by $u'(r)$ and integrating by parts, one easily derives, for angular momentum $l > 0$

$$\int_0^\infty u^2(r) \left[V'(r) - \frac{2l(l+1)}{mr^3} \right] dr = 0 \quad (3)$$

In calculations concerning the hydrogen atom, one often uses the property $\langle r^{-3} \rangle \propto \langle r^{-2} \rangle$. Eq. (3) can also simplify the calculations of the spin structure of quarkonium levels, for which a central potential of the type $V = ar + b - c/r$ is used.

For S -wave states, one obtains

$$\delta_{12} = \frac{u^2(0)}{4\pi} = \frac{m}{4\pi} \int_0^\infty u(r)^2 V'(r) dr \quad (4)$$

This identity, attributed to Schwinger, is a special case of more general sum rules, which are reviewed, e.g. in Ref. [3]. The relation (4) is useful to derive rigorous results [4] as well

as to improve the accuracy and stability of numerical calculations, as seen in the following examples:

- i) For a linear potential $V(r) = \lambda r$, the square of the wave-function at the origin, δ_{12} , is independent of the radial number n . For $m = \lambda = 1$, one has $\delta_{12} = 1/4\pi$. This is seen in the exact solutions, $u_n(r) = A_n(r + a_{n+1})/A'_n(a_{n+1})$, where $a_1, a_2, \dots$ are the (negative) zeros of the Airy function. The result is also obtained from eq. (4) with any approximate solution, provided it is normalized.
- ii) One can also show that δ_{12} increases (decreases) with the radial number n for a concave (convex) potential V [4].
- iii) For a (rescaled) hydrogen atom of hamiltonian $H = \vec{p}^2 - r^{-1}$, the normalized solution of the $1S$ state ($n = l = 0$) is $u(r) = 2^{-1/2} r \exp(-(r/2))$, so $\delta_{12} = 1/8\pi \simeq 0.03979$ is the exact result. The best gaussian approximation $\phi = (\alpha/\pi)^{3/4} \exp(-(\alpha r^2/2))$ is obtained for $\alpha = 4/9\pi$ by minimizing the energy. It leads to a very poor estimate $\delta_{12} = (4/9\pi^2)^{3/2} \simeq 0.00956$ when δ_{12} is computed directly whereas eq. (4) gives $\delta_{12} = 2/9\pi^2 \simeq 0.02252$ which is already much better.

III. Symmetric baryons

For non-central forces or bound states containing more than 2 constituents, eq. (4) is generalized as [1,2]

$$\delta_{12} = \frac{m}{4\pi} \langle \vec{r}_{12} \cdot \frac{\partial V_T}{\partial \vec{r}_{12}} - \frac{2l^2}{mr_{12}^3} \rangle \quad (5)$$

where V_T is the total potential energy. Let us now consider a symmetric three-body bound state, for instance the baryon $\Omega^- (sss)$ bound by pairwise, central forces

$$V_T = V(r_{12}) + V(r_{23}) + V(r_{31}) \quad (6)$$

If one introduces the usual Jacobi variables [5]

$$\vec{\rho} = \vec{r}_2 - \vec{r}_1 \quad \vec{\lambda} = \frac{2\vec{r}_3 - \vec{r}_1 - \vec{r}_2}{\sqrt{3}} \quad (7)$$

so that

$$\vec{r}_{12} = \vec{\rho} \quad \vec{r}_{23} = -\frac{1}{2}\vec{\rho} + \frac{\sqrt{3}}{2}\vec{\lambda} \quad \vec{r}_{31} = -\frac{1}{2}\vec{\rho} - \frac{\sqrt{3}}{2}\vec{\lambda} \quad (8)$$

the probability for quarks 1 and 2 to be on the top of each other is

$$\delta_{12} = \frac{m}{4\pi} \left(V'(r_{12}) - \frac{1}{2}\vec{r}_{23} \cdot \vec{r}_{12} V'(r_{23}) - \frac{1}{2}\vec{r}_{31} \cdot \vec{r}_{12} V'(r_{31}) - \frac{2l^2}{m\rho^3} \right) \quad (9)$$

This expression was already written down by Cohen and Lipkin [6] in a different context. They neglected the small orbital term, and, using the virial theorem, arrived at interesting conclusions on the mass dependence of the correlation coefficients. For instance, in a (smooth) flavour-independent potential, $\delta_{\pi\pi}$ is larger in sqg than in sgg , with the consequence that the hyperfine splitting $\Xi^* - \Xi$ is larger than $\Sigma^* - \Sigma$, as observed. An empirical systematics of this Cohen-Lipkin effect is presented in Ref. [7]. Our aim here is quite different. We wish to use eq. (9) to derive rigorous bounds on δ_{12} .

Consider first a purely linear potential $V(r) = \frac{1}{2}\lambda r$. One can use the positivity of $\tilde{r}_p^2$ and that of $(\tilde{r}_{12} + \tilde{r}_{23} + \tilde{r}_{31})^2$, which ensures that $\langle \tilde{r}_{23} \cdot \tilde{r}_{12} \rangle \geq -\frac{1}{2}$, when the average is taken for a symmetric wave-function, to obtain

$$\delta_{12} \leq \frac{3}{16} \frac{m\lambda}{\pi} \quad (10)$$

For $m = \lambda = 1$, the bound $\delta_{12} \leq 0.05968$ is not too far from the exact values obtained by numerical methods [6], which are

$$\begin{aligned} l = 0, n = 0 & \quad \delta_{12} = 0.05689 \\ l = 0, n = 1 & \quad \delta_{12} = 0.05663 \end{aligned} \quad (11)$$

Here, l denotes the total angular momentum of the 3-body system and n , which labels the successive eigenstates of given l and permutation properties, corresponds to the number of nodes in the dominant hyperscalar component of the wave-function. Note that δ_{12} depends slightly on n , unlike in the 2-body case. The state with $n = 1$ contains more contributions from higher hyperspherical harmonics, or, say, from configurations with internal angular momenta $l_p = l_1 > 0$ (still coupled by $\tilde{l}_p + \tilde{l}_1 = 0$), and this reduces the value of δ_{12} . For a collective linear potential $V_T \propto (r_{12}^2 + r_{23}^2 + r_{31}^2)^{1/2}$, which is exactly hyperscalar, the correlation coefficient δ_{12} would be strictly independent of n .

The result (9) seems generalizable to most monotonically increasing potentials V which are used in current phenomenological models. In fact, we shall show that

$$\delta_{12} \leq \frac{m}{8\pi} \left(\sum_{i < j} V'(r_{ij}) \right) \quad (12)$$

provided that $V'(r)/r$ is decreasing. The derivation, at first glance, involves cumbersome trigonometric inequalities. This is avoided in the following proof, due to A. Martin [8].

First, we use the permutation symmetry of the wave-function and the positivity of $\tilde{r}_p^2$ to rewrite eq. (9), with handy notations $v_1 = V'(r_{23})$, $\cos \theta_1 = -\tilde{r}_{31} \cdot \tilde{r}_{12}$ etc., as

$$\delta_{12} \leq \frac{m}{12\pi} \left(\sum_i v_i + \frac{1}{2} \sum_i v_i [\cos \theta_j + \cos \theta_k] \right) \quad (13)$$

where, in the second term, (i, j, k) is an even permutation of $(1, 2, 3)$.

It is sufficient to show that

$$A \equiv \sum_i v_i [1 - \cos \theta_j - \cos \theta_k] \geq 0 \quad (14)$$

everywhere. This is obvious if the triangle $(1, 2, 3)$ is equilateral. Let us now assume that $\theta_1 \leq \theta_2 \leq \theta_3$ in the triangle. Then, from elementary geometry, $\sin \theta_1 \leq \sin \theta_2 \leq \sin \theta_3$, since either $\theta_1 \leq \theta_2 \leq \theta_3 \leq \pi/2$ or $\theta_1 \leq \theta_2 \leq \pi/2$ and $\theta_3 \leq \pi - \theta_2$. Thus $d_1 \leq d_2 \leq d_3$, since $\sin \theta_1/d_1 = \sin \theta_2/d_2 = \sin \theta_3/d_3$. One also that

$$\sum_i d_i [1 - \cos \theta_j - \cos \theta_k] = 0 \quad (15)$$

i.e. that the inequality is saturated for harmonic forces. This implies $(1 - \cos \theta_1 - \cos \theta_2) \leq 0$ and $(1 - \cos \theta_2 - \cos \theta_3) \geq 0$ and suggests rewriting A as

$$\begin{aligned} A = \frac{v_2}{d_2} \left[\sum_i d_i (1 - \cos \theta_j - \cos \theta_k) \right] \\ + \left(v_1 - d_1 \frac{v_2}{d_2} \right) (1 - \cos \theta_2 - \cos \theta_3) + \left(v_3 - d_3 \frac{v_2}{d_2} \right) (1 - \cos \theta_1 - \cos \theta_2) \end{aligned} \quad (16)$$

The desired inequality (12) comes out provided

$$\frac{v_1}{d_1} \geq \frac{v_2}{d_2} \geq \frac{v_3}{d_3} \quad (17)$$

q.e.d.

The following remarks are in order:

i) The inequality has much to do with the recent work of Basdevant *et al.* [9] and Martin [11] on the comparison of mesons and baryons. In Ref. [9], it was shown that

$$E_3(m; V) \geq \frac{3}{2} E_2(\frac{2}{3}m; 2V) \quad (18)$$

which bounds the energy of a baryon (qqq) with quark mass m and potential energy $\sum V(r_{ij})$ in terms of the binding energy of a fictitious meson ($q'\bar{q}'$) with quark mass $m' = 3m/4$ and interquark potential $2V$. Our inequality means that the correlation coefficient in the baryon is smaller than its analog in the fictitious meson. This is fully consistent with the result that the mean separation $\langle r_{12}^2 \rangle^{1/2}$ is larger in the baryon than in the fictitious meson [10].

ii) The inequality holds for any state with a symmetric wave-function. It is also true for an antisymmetric state, like the state called $[20, 1^+]$, $N = 2$ in the harmonic oscillator model [5], but it becomes trivial, since $\delta_{12} = 0$ for this state.

iii) The inequality is saturated in the harmonic oscillator case. This explains why, with general potentials, the actual value is not too far from the bound, as seen previously in the linear case.

iv) The condition $d/dr(V'/r) < 0$ is encountered in many other problems dealing with the rigorous properties of the solutions of the Schrödinger equation [4-5, 11]. This is the simplest way of expressing the fact that a potential grows less rapidly than the harmonic oscillator.

v) For a power-law potential $V = \frac{1}{2}gr^\beta$ with $1 \leq \beta \leq 2$, one can get a simple analytic bound on δ_{12} for the ground state by using the virial theorem and the variational principle. The sequence is

$$\begin{aligned} \delta_{12} &\leq \frac{m}{8\pi} \frac{3\beta g}{2} \langle r_{12}^\beta \rangle^{\frac{2-\beta}{2}} = \frac{m\beta}{8\pi} \left(\frac{3g}{2}\right)^{\frac{1}{2}} \langle V_T \rangle^{\frac{2-\beta}{2}} \\ &= \frac{m\beta}{8\pi} \left(\frac{3g}{2}\right)^{\frac{1}{2}} \left[\frac{2}{2+\beta} E \right]^{\frac{2-\beta}{2}} \leq \frac{m\beta}{8\pi} \left(\frac{3g}{2}\right)^{\frac{1}{2}} \left[\frac{2}{2+\beta} E_{\text{var}} \right]^{\frac{2-\beta}{2}} \end{aligned} \quad (19)$$

since the exact energy E is smaller than any variational approximation E_{var} . A trial wave-function of gaussian shape

$$\Psi(\vec{r}, \vec{\lambda}) \propto \exp -\frac{1}{2}\alpha(\vec{r}^2 + \vec{\lambda}^2) \quad (20)$$

leads to

$$E_{\text{var}} = \frac{3}{m} \frac{2+\beta}{\beta} \left[\frac{\beta}{2} \frac{\Gamma(\frac{3}{2} + \frac{\beta}{2})}{mg \Gamma(\frac{3}{2})} \right]^{\frac{2}{2+\beta}} \quad (21)$$

so, finally

$$\delta_{12} \leq (mg)^{\frac{1}{2+\beta}} \frac{\beta}{8\pi} \left(\frac{6}{\beta}\right)^{\frac{2-\beta}{2}} \left(\frac{3}{2}\right)^{\frac{1}{2}} \left[\frac{\beta}{2} \frac{\Gamma(\frac{3}{2} + \frac{\beta}{2})}{\Gamma(\frac{3}{2})} \right]^{\frac{2}{2+\beta} \frac{2-\beta}{2}} \quad (22)$$

vi) If the potential V grows faster than the harmonic oscillator, there are conflicting contributions from the centrifugal barrier in eq. (9) and from A becoming positive in eq. (16). There are cases where the latter effect dominates. For instance, for $V = \frac{1}{2}gr^3$, the problem consists of comparing the zero-range correlation with the mean separation. From an accurate hyperspherical calculation [6], one gets for $m = g = 1$

$$\delta_{12} = 0.2263 \quad \frac{9}{16\pi} \langle r_{12}^2 \rangle = 0.2189 \quad (23)$$

Hence the inequality (12) is clearly violated.

vii) There are in fact good reasons to believe that the inequality (12) is inverted if $d/dr(V'/r) > 0$ everywhere. For perturbation around the harmonic oscillator, i.e. $V =$

$r^2 + \lambda w(r)$, the quantity A in eq. (16) becomes positive already at first order, whereas the expectation value of $\hat{L}_p^2$, in eq. (9), contributes only at second order. A numerical investigation shows that for $V = r^{2.1}$ or even $V = r^{2.01}$, one gets, indeed, the inequality $8\pi\delta_{12} > m \langle \sum_{i < j} V'(r_{ij}) \rangle$.

IV. Outlook

The bound on the correlation coefficients can be applied to estimate the lifetime of the proton, if one assumes that the initial state is described in the non-relativistic quark model. More realistic applications deal with weak decays. It is interesting to compare the role of W -exchange diagrams in mesons and in baryons [12]. This concerns, however, flavoured baryons such as csd or ccs and hence it would be desirable to generalize the present study to the case of unequal quark masses.

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