**Supersymmetric structure of four-dimensional
antisymmetric tensor fields.**

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Abstract

Antisymmetric tensor fields are discussed. The model, as other topological field theories, possesses a supersymmetric off-shell structure which can be used to define a class of Slavnov invariant observables.

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1. Introduction

The antisymmetric tensor fields have been introduced for many years [1] and are responsible for a large number of interesting problems.

For instance, the presence of zero modes [2], makes the quantization of such a system not straightforward and gives a beautiful opportunity to apply the Batalin-Vilkovisky formalism [3]. More recently the antisymmetric tensor fields have been investigated in the context of topological field theories; indeed one can compute [4] a topological invariant which generalizes the three-dimensional linking number [5].

The aim of this work is to analyse some properties of the non-Abelian case; our main result being the existence of an off-shell supersymmetric structure. The existence of this supersymmetry seems to be a general property of the topological field theories; indeed, such a feature is shared also by the three-dimensional Chern-Simons model [6] and by the three-dimensional Einstein-Hilbert gravity [7]. In the non-Abelian model considered here, the supersymmetry plays a crucial role in solving the local cohomology of the Slavnov operator modulo a total derivative.

The work is organized as follows: in sect.2 we quantize the model following the procedure of Batalin-Vilkovisky [3] and we introduce the on-shell supersymmetry. Sect.3 is devoted to the study of the Slavnov identity and to the off-shell susy algebra. In sect.4 we discuss the constraints which characterize the classical action and which will be the starting point for a quantum algebraic extension of the model [8]. Finally, in sect.5 we show how the supersymmetric structure can be used for constructing examples of Slavnov invariant observables.

2. The model

We start with the action:

$$S_{\text{inv}} = -\frac{1}{4} \int_M d^4x \, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a, \quad (2.1)$$

where M is the flat four-dimensional Minkowski space,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c, \quad (2.2)$$

and $B_{\rho\sigma}^a = -B_{\sigma\rho}^a$ is a two-form.

Notice that the expression (2.1), being metric independent, has a topological interpretation [4]. The action (2.1) possesses two kinds of invariance given by:

$$\begin{aligned} \delta^{(1)} A_\mu^a &= -(\partial_\mu \theta)^a = -(\partial_\mu \theta^a + f^{abc} A_\mu^b \theta^c) \\ \delta^{(1)} B_{\mu\nu}^a &= -f^{abc} B_{\mu\nu}^b \theta^c, \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} \delta^{(2)} A_\mu^a &= 0 \\ \delta^{(2)} B_{\mu\nu}^a &= -(D_\mu \varphi_\nu - D_\nu \varphi_\mu)^a. \end{aligned} \quad (2.4)$$

The quantization of the model, due to the presence of zero modes in the transformations (2.4) [2], is not straightforward and requires the procedure developed by Batalin and Vilkovisky [3]. Following [3], the gauge-fixing action is:

$$\begin{aligned} S_{gf} = \int d^4x \, (& b^a \partial A^a + \bar{c}^a \partial^\mu (D_\mu c)^a + h^{a\nu} (\partial^\mu B_{\mu\nu}^a) + \omega^a \partial \xi^a + h_\mu^a (\partial^\mu e^a) \\ & + \omega^a \lambda^a + (\partial_\mu \bar{\xi}^{a\mu}) \lambda^a - (\partial^\mu \bar{\phi}^a) (D_\mu \phi)^a + f^{abc} \bar{c}^b \xi_\mu^c \\ & - (\partial^\mu \bar{\xi}^{a\nu}) (D_\mu \xi_\nu)^a - (D_\nu \xi_\mu^a) + f^{abc} B_{\mu\nu}^b c^c \\ & + \frac{1}{2} f^{abc} \epsilon^{\mu\nu\rho\sigma} (\partial_\mu \bar{\xi}_\nu^a) (\partial_\rho \bar{\xi}_\sigma^b) \phi^c), \end{aligned} \quad (2.5)$$

where $(c^a, \bar{c}^a, b^a)$, $(\xi_\mu^a, \bar{\xi}_\mu^a, h_\mu^a)$, are respectively the ghosts, the antighosts and the lagrangian multipliers for the transformations (2.3) and (2.4), $(\phi^a, \bar{\phi}^a, \omega^a)$ the ghosts, the antighost and the lagrangian multiplier for the zero modes of (2.4) and (e^a, λ^a) a pair of fields which takes into account a further degeneracy associated with h_μ^a .

The canonical dimensions of the fields $(A, B, c, \bar{c}, b, \xi, \bar{\xi}, h, \phi, \bar{\phi}, \omega, e, \lambda)$ are respectively $(1, 2, 0, 2, 2, 1, 1, 0, 2, 2, 2, 2)$ and the assigned ghost charges $(0, 0, 1, -1, 0, 1, -1, 0, 2, -2, -1, 0, 1)$. It is easy to prove that the gauge-fixed action

$$S = S_{\text{inv}} + S_{gf}, \quad (2.6)$$

is invariant under the following transformations:

$$\begin{aligned}
sA_\mu^a &= -(D_\mu c)^a \\
sc^a &= \frac{1}{2} f^{abc} b^b c^c \\
s\xi_\mu^a &= (D_\mu \phi)^a + f^{abc} \xi_\mu^b \xi_\mu^c \\
s\phi^a &= f^{abc} b^b \phi^c \\
sB_{\mu\nu}^a &= -(D_\mu \xi_\nu - D_\nu \xi_\mu)^a - f^{abc} B_{\mu\nu}^b c^c + f^{abc} \varepsilon_{\mu\nu\rho\sigma} (\partial^\rho \xi^{b\sigma}) \phi^c \\
s\bar{\xi}_\mu^a &= h_\mu^a, \quad sh_\mu^a = 0 \\
s\bar{c}^a &= b^a, \quad sb^a = 0 \\
s\bar{\phi}^a &= \omega^a, \quad s\omega^a = 0 \\
se^a &= \lambda^a, \quad s\lambda^a = 0,
\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
\{s, s\} B_{\mu\nu}^a &= -2\varepsilon_{\mu\nu\rho\sigma} f^{abc} \phi^c \frac{\delta S}{\delta B_{\rho\sigma}^b} \\
\{s, s\} &= 0 \quad \text{on the other fields,}
\end{aligned} \tag{2.8}$$

i.e., the operation s is nilpotent on-shell.

It should be noted that, contrary to what happens in the ordinary cases, the gauge-fixing (2.5) is not an exact s -variation:

$$\begin{aligned}
S_{gf} &= \int d^4x \left(\bar{c}^a \partial A^a - (\partial^\mu \bar{\xi}^{a\nu}) B_{\mu\nu}^a + \bar{\phi}^a \partial \xi^a + \bar{\xi}^{a\mu} (\partial_\mu c^a) + \bar{\phi}^a \lambda^a \right) \\
&\quad - \int d^4x \frac{1}{2} f^{abc} \varepsilon^{\mu\nu\rho\sigma} (\partial_\mu \bar{\xi}_\nu^a) (\partial_\rho \bar{\xi}_\sigma^b) \phi^c,
\end{aligned} \tag{2.9}$$

however, since the last term in (2.9) is metric independent, it follows that the topological character of the model is not spoiled [9]. The transformations (2.7) do not represent the unique invariance of the model but, as it happens in the three-dimensional Chern-Simons gauge theory [6] and in the three-dimensional Einstein-Hilbert gravity [7], it turns out that the action (2.6) possesses a further global symmetry whose generators carry a Lorentz index

and whose action on the fields is given by:

$$\begin{aligned}
\delta_\mu A_\nu^a &= -\varepsilon_{\nu\mu\tau\rho} \partial^\tau \bar{\xi}^{a\rho} \\
\delta_\mu c^a &= -A_\mu^a \\
\delta_\mu \bar{c}^a &= 0 \\
\delta_\mu b^a &= \partial_\mu \bar{c}^a \\
\delta_\mu B_{\nu\rho}^a &= -\varepsilon_{\nu\mu\tau\rho} \partial^\tau \bar{c}^a \\
\delta_\mu \xi_\nu^a &= -B_{\mu\nu}^a \\
\delta_\mu \bar{\xi}^{a\tau} &= -\delta_\mu^\tau \bar{\phi}^a \\
\delta_\mu h^{a\nu} &= (\partial_\mu \bar{\xi}^{a\nu}) + \delta_\mu^\nu \omega^a \\
\delta_\mu \phi^a &= \xi_\mu^a \\
\delta_\mu \bar{\phi}^a &= 0 \\
\delta_\mu \omega^a &= \partial_\mu \bar{\phi}^a \\
\delta_\mu \lambda^a &= \partial_\mu c^a \\
\delta_\mu c^a &= 0,
\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
\{\delta_\rho, \delta_\nu\} &= 0 \\
\{s, \delta_\rho\} &= \partial_\rho + \text{eq. of motion.}
\end{aligned} \tag{2.11}$$

The equations (2.8), (2.11) show that the algebra given by s and δ_μ closes on-shell on the translations, allowing then for a supersymmetric interpretation of the model.

3. The off-shell algebra

To write down a Slavnov identity corresponding to the transformations (2.7) we introduce a set of external sources coupled to the non-linear variations of (2.7):

$$\begin{aligned}
S_s &= \int d^4x \left(\gamma^{a\mu\nu} (s B_{\mu\nu}^a) + \Omega^{a\mu} (s A_\mu^a) + L^a (s c^a) + D^a (s \phi^a) \right. \\
&\quad \left. + \rho^{a\mu} (s \xi_\mu^a) + \frac{1}{2} f^{abc} \varepsilon_{\mu\nu\rho\sigma} \gamma^{a\mu\nu} \gamma^{b\rho\sigma} \phi^c \right),
\end{aligned} \tag{3.1}$$

then the complete action

$$\Sigma = S_{inv} + S_{gf} + S_s, \tag{3.2}$$

satisfies the Slavnov identity

$$S(\Sigma) = 0, \quad (3.3)$$

where

$$\begin{aligned} S(\Sigma) = \int d^4x \left(\frac{\delta \Sigma}{\delta \gamma^{\alpha\mu\nu}} \frac{\delta \Sigma}{\delta B_{\mu\nu}^a} + \frac{\delta \Sigma}{\delta \Omega^{\alpha\mu}} \frac{\delta \Sigma}{\delta A_\mu^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta \Sigma}{\delta c^a} + \frac{\delta \Sigma}{\delta D^a} \frac{\delta \Sigma}{\delta \phi^a} \right. \\ \left. + \frac{\delta \Sigma}{\delta \rho^{\alpha\mu}} \frac{\delta \Sigma}{\delta \xi_\mu^a} + h_\mu^a \frac{\delta \Sigma}{\delta \xi_\mu^a} + b^a \frac{\delta \Sigma}{\delta c^a} + \omega^a \frac{\delta \Sigma}{\delta \phi^a} + \lambda^a \frac{\delta \Sigma}{\delta e^a} \right). \end{aligned} \quad (3.4)$$

It is now straightforward to verify that the linearized operator

$$\begin{aligned} B_\Sigma = \int d^4x \left(\frac{\delta \Sigma}{\delta \gamma^{\alpha\mu\nu}} \frac{\delta B_{\mu\nu}^a}{\delta A_\mu^a} + \frac{\delta \Sigma}{\delta \Omega^{\alpha\mu}} \frac{\delta \gamma^{\alpha\mu\nu}}{\delta A_\mu^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta \Sigma}{\delta c^a} + \frac{\delta \Sigma}{\delta D^a} \frac{\delta \Sigma}{\delta \phi^a} + \frac{\delta \Sigma}{\delta \rho^{\alpha\mu}} \frac{\delta \Sigma}{\delta \xi_\mu^a} \right. \\ \left. + \frac{\delta \Sigma}{\delta c^a} \frac{\delta L^a}{\delta \phi^a} + \frac{\delta \Sigma}{\delta \phi^a} \frac{\delta D^a}{\delta \rho^{\alpha\mu}} + \frac{\delta \Sigma}{\delta \rho^{\alpha\mu}} \frac{\delta \Sigma}{\delta \xi_\mu^a} + h_\mu^a \frac{\delta \Sigma}{\delta \xi_\mu^a} + b^a \frac{\delta \Sigma}{\delta c^a} + \omega^a \frac{\delta \Sigma}{\delta \phi^a} + \lambda^a \frac{\delta \Sigma}{\delta e^a} \right), \end{aligned} \quad (3.5)$$

is nilpotent:

$$\{ B_\Sigma, B_\Sigma \} = 0. \quad (3.6)$$

The nilpotency of B_Σ is due to the presence, in the source term (3.1), of a non standard quadratic term in the external sources $\gamma^{\alpha\mu\nu}$. This term is responsible for the validity of (3.3) and then for the nilpotency of B_Σ . Notice that the introduction of analogue terms already exists in the literature [6, 10, 11] and, in practice, is the only way which allows to build a nilpotent operator. It is important, at this level, to spend some words on the role of the external sources and come into contact with the so-called anti-fields of the Batalin-Vilkovisky formalism [3]. The introduction of the external sources is, as pointed out in [12], an essential step for a correct quantum treatment of a non-linear symmetry. Indeed, as clearly explained in [13], the external sources take into account the fact that, in the case of a non-linear symmetry, the symmetry itself has to be defined at the quantum level; a counterterm in the external sources is nothing but the renormalization of the composite operators entering the symmetry transformations. It is easily recognized that the external sources we introduced play the same role of the anti-fields of the master-equation [3]. Normally these anti-fields are eliminated in favour of an on-shell nilpotent operator. Probably, as noticed also in [11], the anti-fields should not be eliminated, indeed the master-equation contains already all the source terms needed for defining a nilpotent

operator which can be taken as the starting point for a quantum algebraic characterization of the model.

Let us try now to cast into a Ward identity the δ_ρ invariance (2.10). Due to the source term (3.1) the operator δ_ρ gets modified; a simple algebraic manipulation shows that the δ_ρ invariance is replaced by the Ward identity

$$\mathcal{W}_\mu \Sigma = \Delta_\mu, \quad (3.7)$$

where

$$\begin{aligned} \mathcal{W}_\mu = \int d^4x \left(-\varepsilon_{\nu\mu\tau\rho} (\gamma^{\sigma\tau\rho} + \partial^\tau \bar{\xi}^{\sigma\rho}) \frac{\delta}{\delta A_\nu^a} - A_\mu^a \frac{\delta}{\delta c^a} - \varepsilon_{\nu\rho\mu\tau} (\Omega^{\sigma\tau} + \partial^\tau c^a) \frac{\delta}{\delta B_{\nu\rho}^a} - B_{\nu\rho}^a \frac{\delta}{\delta \xi_\nu^a} \right. \\ \left. + (\partial_\mu c^a) \frac{\delta}{\delta \bar{\xi}^a} - \bar{\phi}^a \frac{\delta}{\delta \xi^a} + (\partial_\mu \bar{\xi}^{\alpha\mu} + \delta_\mu^{\nu\alpha} \omega^\alpha) \frac{\delta}{\delta h^{\alpha\nu}} + \xi_\mu^a \frac{\delta}{\delta \phi^a} + (\partial_\mu \bar{\phi}^a) \frac{\delta}{\delta \omega^a} \right. \\ \left. + (\partial_\mu e^a) \frac{\delta}{\delta \lambda^a} - D^a \frac{\delta}{\delta \rho^{\mu a}} - L^a \frac{\delta}{\delta \Omega^{\mu a}} - \frac{1}{2} (\delta_\mu^{\sigma\tau} \rho^{\sigma\tau} - \delta_\mu^{\tau\rho} \rho^{\sigma\sigma}) \frac{\delta}{\delta \gamma^{\sigma\tau\rho}} \right), \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} \Delta_\mu = \int d^4x \left(-\gamma^{\sigma\sigma\rho} (\partial_\mu B_{\rho\sigma}^a) - \Omega^{\sigma\tau} (\partial_\mu A_\tau^a) + L^a (\partial_\mu c^a) - D^a (\partial_\mu \phi^a) \right. \\ \left. + \rho^{\sigma\tau} (\partial_\mu \xi_\tau^a) + \varepsilon_{\mu\rho\sigma\nu} \Omega^{\sigma\nu} (\partial^\nu h^{\alpha\mu}) - \varepsilon_{\mu\rho\sigma\nu} \gamma^{\alpha\rho\sigma} (\partial^\nu \bar{\phi}^a) \right). \end{aligned} \quad (3.9)$$

The term Δ_μ , being linear in the quantum fields, is a classical breaking; such a feature seems to be a common property of the models which have a non-linear realization of a supersymmetry [14]. The Ward operator $\mathcal{W}_\mu$ has the pleasant property of being linearly realized and:

$$\begin{aligned} \{ \mathcal{W}_\sigma, \mathcal{W}_\rho \} &= 0 \\ \{ B_\Sigma, \mathcal{W}_\sigma \} &= \mathcal{P}_\sigma, \end{aligned} \quad (3.10)$$

where $\mathcal{P}_\sigma$ is the translation Ward operator

$$\mathcal{P}_\sigma = \sum_{(all \ the \ fields \ \varphi)} d^4x (\partial_\sigma \varphi) \frac{\delta}{\delta \varphi}. \quad (3.11)$$

One sees that the introduction of the external source term (3.1) closes off-shell the algebra (2.8), (2.11) and, in this sense, the external sources play the role of the auxiliary fields of the standard linearly realized supersymmetry.

4. Classical approximation

As in the four-dimensional gauge theories [12], [15], we characterize the gauge-fixing (2.5) by imposing the equations of motion of the lagrangian multipliers:

$$\begin{aligned} \frac{\delta \Sigma}{\delta \theta^a} &= \partial A^a \\ \frac{\delta \Sigma}{\delta h^a} &= \partial^\mu B_{\mu}^a + \partial_\nu e^a \\ \frac{\delta \Sigma}{\delta \omega^a} &= \partial \xi^a + \lambda^a \\ \frac{\delta \Sigma}{\delta \lambda^a} &= -\partial \bar{\xi}^a - \omega^a. \end{aligned} \quad (4.1)$$

Thanks to the Landau-gauge, for the A_μ^a field we can impose, as classical Ward identity, also the integrated equation of motion of the ghost ϕ^a :

$$G^a \Sigma = \Delta^a, \quad (4.2)$$

where

$$G^a = \int d^4x \left(\frac{\delta}{\delta \phi^a} - f^{abc} \bar{\phi}^b \frac{\delta}{\delta \theta^c} \right), \quad (4.3)$$

and

$$\Delta^a = \int d^4x f^{abc} \left(\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \gamma^{b\mu\nu} \gamma^{c\rho\sigma} + \varepsilon_{\mu\nu\rho\sigma} \gamma^{b\mu\nu} (\partial^\rho \bar{\xi}^{c\sigma}) + D^b c^c + \rho^{b\mu} A_\mu^c \right). \quad (4.4)$$

Notice that Δ^a , being linear in the quantum fields, is a classical breaking. Commuting (4.1), (4.2) with the Slavnov identity (3.3) we get a second set of constraints:

$$\begin{aligned} \partial^\mu \frac{\delta \Sigma}{\delta \Omega^{a\mu}} + \frac{\delta \Sigma}{\delta \bar{c}^a} &= 0 \\ \partial^\mu \frac{\delta \Sigma}{\delta \rho^{a\mu}} - \frac{\delta \Sigma}{\delta \phi^a} &= 0 \\ \frac{\delta \Sigma}{\delta e^a} = -\partial h^a \\ \partial^\mu \frac{\delta \Sigma}{\delta \gamma^{a\mu\nu}} + \frac{\delta \Sigma}{\delta \xi^{a\nu}} + \partial_\nu \lambda^a &= 0, \end{aligned} \quad (4.5)$$

and

$$F^a \Sigma = \Xi^a, \quad (4.6)$$

where

$$\begin{aligned} F^a = \int d^4x f^{abc} \left(-\varepsilon_{\mu\nu\rho\sigma} (\gamma^{b\mu\nu} + \partial^\mu \bar{\xi}^{b\nu}) \frac{\delta}{\delta B_{\rho\sigma}^c} + \rho^{b\mu} \frac{\delta}{\delta \Omega^{c\mu}} - D^b \frac{\delta}{\delta L^c} - c^b \frac{\delta}{\delta \phi^c} \right. \\ \left. - A_\mu^b \frac{\delta}{\delta \xi_\mu^c} - \bar{\phi}^b \frac{\delta}{\delta \bar{c}^c} + \omega^b \frac{\delta}{\delta \theta^c} \right), \end{aligned} \quad (4.7)$$

$$\Xi^a = \int d^4x f^{abc} \varepsilon_{\mu\nu\rho\sigma} \gamma^{b\mu\nu} \partial^\rho h^{c\sigma}. \quad (4.8)$$

Again Ξ^a is a classical breaking. The equations (4.1) can be solved by introducing the action $\hat{\Sigma}$ defined by

$$\Sigma = \hat{\Sigma} + \int d^4x \left(\partial^a \partial A^a + h^{a\nu} (\partial^\mu B_{\mu\nu}^a) + h^{a\nu} (\partial_\nu e^a) + \omega^a \partial \xi^a + \omega^a \lambda^a - \lambda^a \partial \bar{\xi}^a \right). \quad (4.9)$$

This definition implies

$$\begin{aligned} \frac{\delta \hat{\Sigma}}{\delta \theta^a} &= 0, \quad \frac{\delta \hat{\Sigma}}{\delta h^{a\nu}} = 0, \quad \frac{\delta \hat{\Sigma}}{\delta \omega^a} = 0 \\ \frac{\delta \hat{\Sigma}}{\delta \lambda^a} &= 0, \quad \frac{\delta \hat{\Sigma}}{\delta e^a} = 0, \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} \partial^\mu \frac{\delta \hat{\Sigma}}{\delta \Omega^{a\mu}} + \frac{\delta \hat{\Sigma}}{\delta \bar{c}^a} &= 0 \\ \partial^\mu \frac{\delta \hat{\Sigma}}{\delta \rho^{a\mu}} - \frac{\delta \hat{\Sigma}}{\delta \phi^a} &= 0 \\ \partial^\mu \frac{\delta \hat{\Sigma}}{\delta \gamma^{a\mu\nu}} + \frac{\delta \hat{\Sigma}}{\delta \xi^{a\nu}} &= 0, \end{aligned} \quad (4.11)$$

from which it follows that $\hat{\Sigma}$ is independent from $\theta^a, h^{a\nu}, \omega^a, \lambda^a, e^a$ and that the variables $\Omega^{a\mu}, \bar{c}^a, \rho^{a\mu}, \phi^a, \gamma^{a\mu\nu}, \xi^{a\nu}$ enter only through the combinations

$$\begin{aligned} \hat{\Omega}^{a\mu} &= \Omega^{a\mu} + \partial^\mu \bar{c}^a \\ \hat{\gamma}^{a\mu\nu} &= \gamma^{a\mu\nu} + \frac{1}{2} (\partial^\mu \xi^{a\nu} - \partial^\nu \xi^{a\mu}) \\ \hat{\rho}^{a\mu} &= \rho^{a\mu} - \partial^\mu \phi^a. \end{aligned} \quad (4.12)$$

Indeed, a direct inspection shows that $\hat{\Sigma}$ takes the simple form:

$$\begin{aligned}\hat{\Sigma} = & -\frac{1}{4} \int d^4x \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a B_{\rho\sigma}^a \\ & + \int d^4x \left(-\hat{\Omega}^{a\mu} (D_\mu c)^a + L^a \frac{f^{abc}}{2} c^b c^c + D^a f^{abc} c^b \phi^c + \hat{\rho}^{a\mu} ((D_\mu \phi)^a + f^{abc} c^b \xi_\mu^c) \right. \\ & \left. - \hat{\gamma}^{a\mu\nu} ((D_\mu \xi_\nu - D_\nu \xi_\mu)^a + f^{abc} B_{\mu\nu}^b c^c) + \frac{1}{2} f^{abc} \varepsilon_{\mu\nu\rho\sigma} \hat{\gamma}^{a\mu\nu} \hat{\gamma}^{b\rho\sigma} \phi^c \right). \end{aligned} \quad (4.13)$$

Taking into account the constraints (4.10), (4.11), the Slavnov identity (3.3) becomes

$$\int d^4x \left(\frac{\delta \hat{\Sigma}}{\delta \hat{\gamma}^{a\mu\nu}} \frac{\delta \hat{\Sigma}}{\delta B_{\mu\nu}^a} + \frac{\delta \hat{\Sigma}}{\delta \hat{\Omega}^{a\mu}} \frac{\delta \hat{\Sigma}}{\delta A_\mu^a} + \frac{\delta \hat{\Sigma}}{\delta L^a} \frac{\delta \hat{\Sigma}}{\delta c^a} + \frac{\delta \hat{\Sigma}}{\delta D^a} \frac{\delta \hat{\Sigma}}{\delta \phi^a} + \frac{\delta \hat{\Sigma}}{\delta \hat{\rho}^{a\mu}} \frac{\delta \hat{\Sigma}}{\delta \xi_\mu^a} \right) = 0. \quad (4.14)$$

For the linearized operator associated with (4.14) one has:

$$\begin{aligned}B_{\hat{\Sigma}} = & \int d^4x \left(\frac{\delta \hat{\Sigma}}{\delta \hat{\gamma}^{a\mu\nu}} \frac{\delta}{\delta B_{\mu\nu}^a} + \frac{\delta \hat{\Sigma}}{\delta \hat{\Omega}^{a\mu}} \frac{\delta}{\delta A_\mu^a} + \frac{\delta \hat{\Sigma}}{\delta L^a} \frac{\delta}{\delta c^a} + \frac{\delta \hat{\Sigma}}{\delta D^a} \frac{\delta}{\delta \phi^a} + \frac{\delta \hat{\Sigma}}{\delta \hat{\rho}^{a\mu}} \frac{\delta}{\delta \xi_\mu^a} \right) \\ & + \frac{\delta \hat{\Sigma}}{\delta c^a} \frac{\delta L^a}{\delta D^a} + \frac{\delta \hat{\Sigma}}{\delta \phi^a} \frac{\delta D^a}{\delta \phi^a} + \frac{\delta \hat{\Sigma}}{\delta \hat{\rho}^{a\mu}} \frac{\delta \xi_\mu^a}{\delta D^a} + \frac{\delta \hat{\Sigma}}{\delta \xi_\mu^a} \frac{\delta \hat{\rho}^{a\mu}}{\delta \phi^a}, \end{aligned} \quad (4.15)$$

and, since

$$B_{\hat{\Sigma}} \hat{\Sigma} = 0, \quad (4.16)$$

it follows that

$$\{B_{\hat{\Sigma}}, B_{\hat{\Sigma}}\} = 0. \quad (4.17)$$

On a functional which depends only on the variables $A^{a\mu}$, $B^{a\mu\nu}$, c^a , ϕ^a and $\hat{\Omega}^{a\mu}$, $\hat{\gamma}^{a\mu\nu}$, $\hat{\rho}^{a\mu}$, L^a , D^a , the operators $\mathcal{W}_\mu$, $\mathcal{G}^a$, $\mathcal{F}^a$ become

$$\begin{aligned}\mathcal{W}_\mu & \rightarrow \hat{\mathcal{W}}_\mu \\ \mathcal{G}^a & \rightarrow \hat{\mathcal{G}}^a \\ \mathcal{F}^a & \rightarrow \hat{\mathcal{F}}^a, \end{aligned} \quad (4.18)$$

with

$$\begin{aligned}\hat{\mathcal{W}}_\mu = & \int d^4x \left(-\varepsilon_{\nu\mu\rho\sigma} \hat{\gamma}^{a\rho\sigma} \frac{\delta}{\delta A_\nu^a} - A_\mu^a \frac{\delta}{\delta c^a} - \varepsilon_{\nu\mu\rho\sigma} \hat{\Omega}^{a\rho} \frac{\delta}{\delta B_{\nu\rho}^a} - B_{\mu\nu}^a \frac{\delta}{\delta \xi_\nu^a} + \xi_\mu^a \frac{\delta}{\delta \phi^a} \right. \\ & \left. - D^a \frac{\delta}{\delta \hat{\rho}^{a\mu}} - L^a \frac{\delta}{\delta \hat{\Omega}^{a\mu}} - \frac{1}{2} (\delta_\mu^\sigma \hat{\rho}^{a\sigma} - \delta_\mu^\sigma \hat{\rho}^{a\sigma}) \frac{\delta}{\delta \hat{\gamma}^{a\sigma\tau}} \right), \end{aligned} \quad (4.19)$$

$$\hat{\mathcal{G}}^a = \int d^4x \frac{\delta}{\delta \phi^a}, \quad (4.20)$$

$$\hat{\mathcal{F}}^a = \int d^4x f^{abc} \left(-\varepsilon_{\mu\nu\rho\sigma} \hat{\gamma}^{b\mu\nu} \frac{\delta}{\delta B_{\rho\sigma}^c} + \hat{\rho}^{b\mu} \frac{\delta}{\delta \hat{\Omega}^{c\mu}} - D^b \frac{\delta}{\delta L^c} - c^b \frac{\delta}{\delta \phi^c} - A_\mu^b \frac{\delta}{\delta \xi_\mu^c} \right), \quad (4.21)$$

and:

$$\begin{aligned}\{\mathcal{W}_\mu, \mathcal{W}_\nu\} & = 0 \\ \{B_{\hat{\Sigma}}, \mathcal{W}_\mu\} & = \mathcal{P}_\mu \\ [\hat{\mathcal{G}}^a, B_{\hat{\Sigma}}] & = \hat{\mathcal{F}}^a \\ [\hat{\mathcal{G}}^a, \mathcal{W}_\mu] & = 0 \\ \{\hat{\mathcal{F}}^a, \mathcal{W}_\mu\} & = 0. \end{aligned} \quad (4.22)$$

Finally, one has to remark that, quite surprisingly, the action $\hat{\Sigma}$ can be written as a $B_{\hat{\Sigma}}$ cocycle, i.e.:

$$\hat{\Sigma} = B_{\hat{\Sigma}} \left(\hat{\gamma}^{a\mu\nu} B_{\mu\nu}^a - 2\hat{\rho}^{a\mu} \xi_\mu^a + 3D^a \phi^a - L^a c^a \right). \quad (4.23)$$

This property, shared also by the four dimensional topological gauge-theories [16], probably reflects the topological character of the model.

5. Observables

The construction of the sector of the observables is, in general, a difficult task which requires a detailed analysis of the symmetries which characterize the model. Our discussion, far from being a complete analysis, will be limited to construct some examples; our aim being that of clarifying the meaning of the supersymmetric structure found in the previous chapters.

Let us begin by defining the local sector of the observables as given by the local cohomology of the operator $B_{\hat{\Sigma}}$ modulo d ; i.e. modulo a total derivative. Usually, this problem boils down a kind of descent equations which involve operators built only with ghosts (see for instance [17] for the Chern-Simons case and [16] for the topological four-dimensional gauge theories). In our case, thanks to the existence of a supersymmetric structure, the operator $\mathcal{W}_\mu$ allows to define in a simple and systematic way a class of observables.

To give an example, let us consider the local operator

$$O(x) = \frac{1}{2} \phi^a(x) \phi^a(x) . \quad (5.1)$$

This operator has zero dimensions, ghost number equal to four and it is $B_{\hat{\Sigma}}$ invariant:

$$B_{\hat{\Sigma}} O(x) = 0 \quad , \quad O(x) \neq B_{\hat{\Sigma}} \hat{O} . \quad (5.2)$$

Let us consider now the operator

$$O_\mu = \dot{\mathcal{W}}_\mu O = \xi_\mu^a \phi^a . \quad (5.3)$$

O_μ has dimension one, number ghost three and, thanks to the algebra

$$\{ B_{\hat{\Sigma}} , \dot{\mathcal{W}}_\mu \} = \mathcal{P}_\mu , \quad (5.4)$$

belongs to the cohomology of $B_{\hat{\Sigma}}$ modulo a total derivative. Notice that O_μ can be interpreted in a geometrical way as a one-form.

One more step, we get the operator

$$O_{\mu\nu} = \dot{\mathcal{W}}_\mu \dot{\mathcal{W}}_\nu O = -B_{\nu\mu}^a \phi^a - \xi_\mu^a \xi_\nu^a . \quad (5.5)$$

This operator has dimension two, ghost number two and, by construction, belongs to the cohomology of $B_{\hat{\Sigma}}$ modulo d . Again, from a geometrical point of view $O_{\mu\nu}$ is a two-form.

By applying one more time the operator $\dot{\mathcal{W}}_\mu$ one has

$$O_{\mu\nu\sigma} = \dot{\mathcal{W}}_\sigma \dot{\mathcal{W}}_\nu \dot{\mathcal{W}}_\mu O , \quad (5.6)$$

which has dimension three and ghost number one. The chain stops when four $\dot{\mathcal{W}}_\mu$ operators are present since

$$\{ \dot{\mathcal{W}}_\mu , \dot{\mathcal{W}}_\nu \} = 0 . \quad (5.7)$$

Continuing the computation one gets:

$$\begin{aligned} O_{BB} &= \int d^4x \, \varepsilon^{\mu\nu\rho\sigma} \dot{\mathcal{W}}_\mu \dot{\mathcal{W}}_\nu \dot{\mathcal{W}}_\rho \dot{\mathcal{W}}_\sigma \left(\frac{\phi^a \phi^a}{6} \right) \\ &= \int d^4x \, (\varepsilon^{\mu\nu\rho\sigma} B_{\mu\nu}^a B_{\rho\sigma}^a - 8L^a \phi^a - 8\hat{\Omega}^{ar} \xi_r^a) . \end{aligned} \quad (5.8)$$

O_{BB} is an integrated local operator with dimension four and zero ghost number. (Notice that O_{BB} cannot appear as counterterm thanks to (4.2), (4.7).)

It is very simple, now, to produce other observables by using invariants like

$$f^{abc} \frac{c^a c^b c^c}{6} , \quad d^{abc} \frac{\phi^a \phi^b \phi^c}{6} , \dots \quad (5.9)$$

where d^{abc} is the totally symmetric tensor. It is amusing, for example, to reproduce the Chern-Simons three-form starting from

$$f^{abc} \frac{c^a c^b c^c}{6} , \quad c^a \partial_\mu c^a . \quad (5.10)$$

Observe that the action $\hat{\Sigma}$ can be written as

$$\begin{aligned} \hat{\Sigma} &= \int d^4x \, \varepsilon^{\mu\nu\rho\sigma} \dot{\mathcal{W}}_\mu \dot{\mathcal{W}}_\nu \dot{\mathcal{W}}_\rho \left(\frac{1}{6} \phi^a \partial_\sigma c^a \right) \\ &\quad - \int d^4x \, \varepsilon^{\mu\nu\rho\sigma} \dot{\mathcal{W}}_\mu \dot{\mathcal{W}}_\nu \dot{\mathcal{W}}_\rho \dot{\mathcal{W}}_\sigma \left(\frac{1}{48} f^{abc} \phi^a \phi^b \phi^c \right) , \end{aligned} \quad (5.11)$$

which is easily seen to be a trivial cocycle of $B_{\hat{\Sigma}}$.

One has to remark that the operator $\dot{\mathcal{W}}_\mu$ can be applied also to non-local observables like a Wilson line. Let us consider, for example, the non-local operator (in matrix notation, T^a denoting the generators of the gauge group in the adjoint representation):

$$T\tau((T^a \phi^a)U(x, x)) , \quad (5.12)$$

where

$$U(x, x) = P \exp \int_{\gamma_x} T^a A_\mu^a dy^\mu , \quad (5.13)$$

and γ_x a closed path which starts and ends-up in the point x . Using the properties of the Wilson line, it is easily seen that the expression (5.12) is $B_{\hat{\Sigma}}$ invariant. If one understands the operator (5.13) as a formal power series in A_μ^a one can perform, for each term of the series, the $\dot{\mathcal{W}}_\mu$ operation. In this way one is able to define the non-local observable:

$$O_{\nu\mu}^U = \dot{\mathcal{W}}_\mu \dot{\mathcal{W}}_\nu T\tau((T^a \phi^a)U(x, x)) . \quad (5.14)$$

The operator $O_{\nu\mu}^U$ has dimension two and zero ghost number; it can be considered as a non-local two-form and can be integrated over a two-dimensional surface without boundary. It is now apparent that the operator $\dot{\mathcal{W}}_\mu$ can be used in a systematic way for constructing observables.

To end-up this section let us comment on the definition which we used for the observables. While the classical action (3.2) is characterized by several symmetries and constraints, the definition of the observables we adopted makes use only of the Slavnov invariance. This point of view is motivated by the possibility that quantities which belong to the cohomology of $B_{\mathbb{F}}$ can have a vacuum expectation value with a topological meaning. A criterion for probing the topological character of a certain quantity, modulo problems related to the framing procedure [18], is to introduce in an explicit way a metric tensor $g_{\mu\nu}(x)$ and try to repeat the Witten argument [16]. In our model the introduction of a metric tensor is not completely straightforward; indeed one easily sees that the metric $g_{\mu\nu}$ does not enter only in the gauge-fixing (2.5) but, due to the presence of the $\varepsilon_{\mu\nu\rho\sigma}$ in the transformations (2.7), it appears explicitly also in the s -operation (2.7) (see also [9] in the case of a covariant gauge-fixing). However, as can be checked after a quite tedious computation, it is still true that the energy-momentum tensor can be written as an s -variation. Thanks to this property one can repeat the Witten argument and, at least formally, affirm that quantities which belong to the cohomology of $B_{\mathbb{F}}$ can have a topological meaning. In this optic the operator $\hat{W}_\mu$, whose existence is strongly related to the flatness of the space-time, is regarded as a way of producing $B_{\mathbb{F}}$ -invariant quantities which can be used to compute topological invariants [8] of submanifolds embedded in the flat four-dimensional space-time.

ACKNOWLEDGEMENTS

We wish to thank C. Becchi, A. Blasi, O. Piguet and R. Stora for valuable discussions. The hospitality of CERN Theory Division, the Physics Dept. of the University of Geneva, and the LAPP, where this work has been done, is gratefully acknowledged.

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