EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

CERN-TH.6207/91

Quantum Group Gauge Fields

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CM-P00062488

*Invited talk given at the A.D. Sakharov Memorial Conference,
Moscow, May 1991, to appear in Proceedings of the Conference*

CERN-TH.6207/91
August 1991

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1 Introduction

It is an honour for me to have the opportunity to give a talk at the A.D.Sakharov memorial conference. A.D. Sakharov was always interested in new ideas. Let me recall that in his paper in 1984 [1] he suggested taking into account the fluctuations of the signature. He presented physical arguments why compactified time-like coordinates do not destroy macrocausality and only microcausality is lost. He also argued that manifolds with complicated topology should give essential contributions at the Planck scale.

In our papers [2] we found such a kind of solution in the effective theory of the superstring. It was shown that extra time-like coordinates can solve the cosmological constant problem. To avoid the appearance of massless ghosts we had to put some topological restrictions on the internal space K . One of these restrictions is that K has no Killing vectors at all. The equations of motion showed that K has negative curvature. The simplest example of a manifold with negative curvature is a hyperboloid which is non-compact. In the σ -model approach this means that we have to deal with non-compact σ -models. Such a type of σ -model is a subject of recent investigations in the context of 2-dimensional gravity coupled with $c = 1$ matter [3]. However, one can assume that K is a compact hyperbolic manifold [2]. Such manifolds have rather complicated topology. In the field theories usually adopted, fields take values in the simplest manifolds (line, plane, sphere, group, coset space, etc.). However, adopting the possibility of the signature being the dynamical variable, we have to deal with fields taking values in manifolds with complicated topology. One can go one step further and try to consider field theories in which fields are more complicated objects, for example, elements of the quantum plane or quantum group [4]. Note that there are also trends to consider the quantum plane as space-time varieties. This rather radical proposal may be appropriate in the context of sub-Planckian physics and non-Archimedean geometry [5]. Here I would like to follow a more modest way and deal with a quantum group as a target space for a field which is defined on the usual space-time. Consideration of such models seems to be a natural application of the idea of quantum groups. In this talk I will give some motivations in support of this and also present some recent results in this direction.

In the last two decades we have witnessed success in applying gauge group ideas to the theories elementary particles. However, there are several open problems in the theory of elementary particles based on the standard Yang-Mills gauge fields. Quark confinement in QCD, the ad hoc introduction of Higgs scalars, and difficulties with grand unification are striking examples of such problems. One can think that these problems are not only technical ones but may be caused by the rigid framework of a gauge theory based on Lie groups.

To solve these problems one can try to modify in some way the notion of gauge fields. The deviation from the standard gauge theory should not be too far so that one can keep the beautiful features of the gauge theory, so the proposal to deform somehow the gauge group seems rather natural. A class of deformations of Lie groups (q -deformation) is the subject of recent developments in the so-called quantum groups [6]. Note that in two-dimensional integrable models quantum group ingredients emerged as a technical tool in the study of the usual models.

A new opportunity for using quantum groups was presented in [4]. There quantum groups as gauge groups were considered. Such a new class of field models we call the quantum group gauge fields models. Our goal is to construct such theories in which the parameter q is a running parameter depending on the momentum variable.

To avoid any misunderstanding from the beginning, let me mention that the quan-

tum group is not a group (although there is an analogue of the group operation) and it is not a quantum object in the sense of quantum field theory, so instead of quantum group it would be better to speak about a q -deformed group or pseudogroup. We look for theories which are invariant under the action of quantum groups at the classical level; the quantization of such a theory in the sense of quantum field theory will be the next step.

For simplicity I will consider only the quantum group $SU_q(2)$. In Section 2 an exposition of this group using an analytical formulation [4], more suitable for applications in field theory, is given. The same approach for the quantum groups $SL_q(2)$ and $GL_q(2)$ can be found in [4].

The exponential map between Lie algebras and Lie groups plays an important role in the theory of usual gauge and chiral fields. An analogue of the Lie algebra generator for a quantum group is defined in Section 3.

Two formulations of quantum group gauge fields are discussed in Section 4. In the first one the gauge fields take values in the quantum universal enveloping algebra of $SU(2)$. The second formulation is based on the quantum group analog of the Lie algebra. As a first step in dealing with q -deformed systems, the dynamics of the simplest q -deformed systems [7] are presented in Section 5. Their quantization in the sense of $\hbar$ -deformation is also discussed. Finally, a generalization of the notion of the quantum plane for an infinite number of degrees of freedom is briefly sketched.

2 Quantum group $SU_q(2)$

We consider as an example $SU_q(2)$. Let us start with the usual group $SU(2)$. As is known, any element of $SU(2)$ can be presented in the form

$$g = \begin{pmatrix} a & -c^* \\ c & a^* \end{pmatrix}, \quad (1)$$

where a and c are complex numbers such that the unitarity condition is satisfied

$$gg^* = g^*g = I, \quad (2)$$

i.e.

$$aa^* + c^*c = 1. \quad (3)$$

Let us deform this representation substituting in (1), instead of c^* , the expression qc^* with q being a real number and preserving the unitarity condition. This can be done only if a, c, a^* and c^* are not numbers, but are elements from some non-commutative algebra $\mathcal{A}$. By analogy with (1), (2) we define the quantum group $SU_q(2)$ as a set of elements with the following properties.

Definition 1 An element of the quantum group $SU_q(2)$ is a 2×2 matrix

1) with the form

$$g = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \quad (4)$$

$$a, c \in \mathcal{A},$$

(such a form we call the canonical form); and

2) with unitarity condition

$$gg^* = g^*g = I. \quad (5)$$

The requirements (5) lead to the following conditions on elements a and c

$$aa^* + q^2 c^* c = 1, \quad (6)$$

$$a^* a + cc^* = 1, \quad (7)$$

$$ca^* = qa^* c, \quad (8)$$

$$ac = qca, \quad (9)$$

$$c^* c = cc^*. \quad (10)$$

The group operation is defined only for matrices from $SU_q(2)$ with commuting entries. Let

$$g = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}, \quad h = \begin{pmatrix} \tilde{a} & -q\tilde{c}^* \\ \tilde{c} & \tilde{a}^* \end{pmatrix} \in SU_q(2),$$

and $[g_{ij}, h_{kl}] = 0$. Then

$$gh = \begin{pmatrix} a\tilde{a} - qc^*\tilde{c} & -qa\tilde{c}^* - qc^*\tilde{a}^* \\ c\tilde{a} + a^*\tilde{c} & -qc\tilde{c}^* + a^*\tilde{a} \end{pmatrix} \in SU_q(2), \quad (11)$$

i.e. gh has the canonical form (38) and the unitarity condition (5) is satisfied.

The multiplication (11) can be expressed in the Hopf algebra language as the co-product. The above analytical definition is more suitable for our applications.

Equations (6)-(10) coincide with the relations which are emphasized in the papers [6] in the definition of the $SU_q(2)$ in the Hopf algebra language. Note, however, that we consider the elements a and c as variable elements from $\mathcal{A}$ by analogy with the variable complex numbers in (1). Equation (3) defines a manifold in C^2 . In a similar way equations (6)-(10) define a non-commutative manifold in $\mathcal{A}^2$.

3 Quantum Lie algebra

As is known the Lie algebra for a given Lie group G can be defined in the following way. Let $g(t)$ be a function of a real variable t and take values in G , with $g(0) = g$. Then a tangent vector $L = g^* \frac{dg(t)}{dt} |_{t=0}$ is an element of the corresponding Lie algebra. We consider here the same approach for a quantum group.

Proposition 1 *Let $g(t)$ be a function of a real variable t taking values in $SU_q(2)$, with $g(0) = g$. Then $L = g^* \dot{g}(t) |_{t=0}$ has the form*

$$L = \begin{pmatrix} l_1 & l_0 \\ -l_0^* & \tilde{l} \end{pmatrix}, \quad (12)$$

with l_1, l_0, l_0^* and $\tilde{l}$ satisfying the relations

$$l_0^* c^* - \tilde{l} a = a l_1 + q c^* l_0^* \quad (13)$$

$$q l_1 c^* + q l_0 a = a l_0 - q c^* \tilde{l}. \quad (14)$$

The matrix L is anti-Hermitian, i.e.

$$L + L^* = 0, \quad \text{i.e. } l_1^* = -l_1; \quad \tilde{l}^* = -\tilde{l}.$$

Assuming the linear independence of the generators l_0, l_0^*, l_1 , one gets from the relations (13) and (14) and from the consistency of the algebra (6)-(10) with (13)-(14),

$$al_0 = ql_0a, \quad cl_0 = ql_0c, \quad (15)$$

$$al_1 = -\kappa l_1a, \quad cl_1 = -\kappa l_1c, \quad (16)$$

$$al_0^* = ql_0^*a, \quad cl_0^* = ql_0^*c. \quad (17)$$

Definition 2 Let g be an element of $SU_q(2)$ in the canonical form (4) and L be a matrix of the form

$$L = \begin{pmatrix} l_1 & l_0 \\ -l_0^* & \kappa l_1 \end{pmatrix}. \quad (18)$$

Such a pair is called the (g, L) -pair if the relations (15)-(17) are satisfied.

The importance of the exponential map from Lie algebra to Lie group is well known. We are going to define the exponential map for the quantum group.

In the Lie group case, given g and an element of the Lie algebra L we can define $g(t) = ge^{tL}$, which belongs to the Lie group and satisfies the equation $\frac{dg(t)}{dt} = g(t)L$, with the initial condition $g(0) = g$.

For a given (g, L) -pair let us investigate the requirements on the entries of g and L which guarantee that $g(t) = ge^{tL}$ belongs to $SU_q(2)$ for real t . For this purpose one must check two conditions: i) the canonical form (38) and ii) unitarity (5) for $g(t)$. The unitarity condition follows from antihermiticity of L : $(ge^{tL})(ge^{tL})^* = I$.

Investigation of the canonical form of $g(t) = ge^{tL}$ in the series expansion

$$ge^{tL} = g(1 + tL + \frac{t^2}{2}L^2 + \dots)$$

gives the following result.

Proposition 2 Let a pair (g, L) be given, i.e. the relations (15)-(17) are satisfied, and the matrix L is parametrized by elements of the quantum superplane

$$l_0^2 = 0, \quad l_1^2 = 0, \quad (19)$$

$$l_1l_0 = q^2l_0l_1, \quad l_0^*l_0 = q^2l_0l_0^*, \quad (20)$$

(recall that $l_1^* = -l_1$). Then $g \exp(tL)$ belongs to $SU_q(2)$ for any real t and has the form

$$ge^{tL} = \begin{pmatrix} a \frac{\sinh t\sqrt{A}}{\sqrt{A}} l_1 + a \cosh t\sqrt{A} + qc^* \frac{\sinh t\sqrt{A}}{\sqrt{A}} l_0^* & a \frac{\sinh t\sqrt{A}}{\sqrt{A}} l_0 - q\kappa c^* \frac{\sinh t\sqrt{A}}{\sqrt{A}} l_1 - qc^* \cosh t\sqrt{A} \\ c \frac{\sinh t\sqrt{A}}{\sqrt{A}} l_1 + c \cosh t\sqrt{A} - a^* \frac{\sinh t\sqrt{A}}{\sqrt{A}} l_0^* & c \frac{\sinh t\sqrt{A}}{\sqrt{A}} l_1 + \kappa a^* \frac{\sinh t\sqrt{A}}{\sqrt{A}} l_1 + a^* \cosh t\sqrt{A} \end{pmatrix} \quad (21)$$

where $A = -l_0l_0^*$.

In other words we have a one-parameter group of automorphisms of the quantum group $SU_2(2)$

$$a \rightarrow a(t - \frac{t^3}{3!}l_0l_0^*)l_1 + a(1 - \frac{t^2}{2}l_0l_0^*) + qtc^*l_0^*, \quad (22)$$

$$c \rightarrow c(t - \frac{t^3}{3!}l_0^*l_1) + c(1 - \frac{t^2}{2}l_0^*) - ta^*l_0^*. \quad (23)$$

To construct a representation of the (g, L) pair one can use the method of Woronowicz [6].

4 Quantum group gauge fields

Usually one assumes that there is a symmetry, for example gauge symmetry or even supersymmetry, at the fundamental level (at high energy) and this symmetry somehow is broken when the energy becomes lower. The most respectable mechanism for symmetry breaking is the Higgs mechanism which is in particular used in the Glashow-Weinberg-Salam theory of the electroweak interactions. This mechanism works by the introduction of the scalar Higgs bosons. The introduction of Higgs scalars is the most ad hoc feature of the spontaneously broken gauge theories. Perhaps it is worth considering in this context q -deformation of the gauge theory. Other problems where q -deformation apparently could be used are quark confinement and grand unification. In this paper, by discussing possible formulations and the simplest models of quantum group gauge theory, we are taking the first step in this direction.

The idea can be symbolically expressed in the following way:

$$A_\mu^i(x)X_i \xrightarrow{q\text{-deformation}} A_\mu^i(x)X_i^{(q)} \xrightarrow{\hbar\text{-deformation}} \hat{A}_\mu^i(x)X_i^{(q)},$$

where A_μ^i are the usual complex-valued gauge fields, X_i are generators of a Lie algebra, and $X_i^{(q)}$ denote q -deformed generators, i.e. quantum algebra generators. $\hbar$ -deformation means the quantization along the line of the quantum field theory leading to quantum operators $\hat{A}_\mu^i(x)$.

The standard formulas for the Yang-Mills gauge theory

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \quad (24)$$

$$S = \int \langle F_{\mu\nu}, F_{\mu\nu} \rangle dx \quad (25)$$

and for the gauge transformation

$$A_\mu(x) \rightarrow g^* A_\mu(x) g + g^* \partial_\mu g, \quad (26)$$

$$F_{\mu\nu}(x) \rightarrow g^* F_{\mu\nu} g \quad (27)$$

obviously have a meaning for $A_\mu(x)$ belonging to an algebra $\mathcal{B}$ with involution, and equipped with some topology and an invariant form $\langle \cdot, \cdot \rangle$. If $\{e_a\}$ is a basis in $\mathcal{B}$ one has

$$A_\mu(x) = A_\mu^a(x)e_a, \quad g(x) = \exp(\omega^a(x)e_a),$$

where $A_\mu^a(x)$ and $\omega^a(x)$ are complex-valued functions.

From this point of view it is rather natural to introduce the notion of quantum group gauge field in at least two ways. In the first one can consider the q -deformation of the enveloping of the Lie algebra as a basic algebra for gauge fields. In this case one gets a gauge field with an infinite number of components. In the second one may attempt to consider a gauge field which takes values in an analogue of the Lie algebra for the quantum group.

We have seen in Section 3 that an element of the Lie algebra analogue for the quantum group $SU_q(2)$ has the form

$$L = \begin{pmatrix} l_1 & l_0 \\ -l_0^* & i \end{pmatrix} = l_1 \begin{pmatrix} 1 & 0 \\ 0 & \kappa \end{pmatrix} + l_0 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + l_0^* \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \quad (28)$$

where l_0, l_1, l_0^* are attached to a given element $g \in SU_q(2)$. One can multiply l_0, l_1, l_0^* by a complex number preserving the basic relations (15)-(20). Since it is not an algebra it cannot be used immediately as the algebra $\mathcal{B}$ in the gauge theory. We take $\mathcal{B}$ to be equal to an enveloping algebra. Then

$$A_\mu(x) = \quad (29)$$

$$A_\mu^1(x)l_1 \begin{pmatrix} 1 & 0 \\ 0 & \kappa \end{pmatrix} + A_\mu^0(x)l_0 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + A_\mu^{0*}(x)l_0^* \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} + A_\mu^{01}(x)l_0l_1 \begin{pmatrix} 0 & \kappa \\ 0 & 0 \end{pmatrix} + \dots$$

A more general approach would be to take $\mathcal{B}$ as a deformation of the universal enveloping algebra of the Lie algebra $su(2)$ denoted by $U_q(su(2))$. It is dual to the quantum group $SU_q(2)$. It is instructive to write first the corresponding formulas for the case when $\mathcal{B}$ is an enveloping algebra of $su(2)$, i.e. $U(su(2))$. In this case the gauge field has the form

$$A_\mu(x) = A_\mu^0(x) + A_\mu^i(x)X_i + \sum_{i_1 \leq \dots \leq i_k} A_\mu^{i_1 \dots i_k}(x)X_{i_1} \dots X_{i_k}, \quad (30)$$

where $X_i, i = 1, 2, 3$ are generators of $su(2)$.

Using the Birkhoff-Witt theorem one can represent $A_\mu(x)$ as a field on the space of (x, ξ) variables, where the ξ_i are auxiliary commuting variables, i.e.

$$A_\mu(x, \xi) = \sum A_\mu(x) \xi^{i_1 \dots i_k} \xi_{i_1} \dots \xi_{i_k}. \quad (31)$$

The quantum universal enveloping algebra $U_q(su(2))$ is generated by X_+, X_- and H which obey the relations

$$[H, X_\pm] = \pm X_\pm, \quad [X_+, X_-] = \frac{q^H - q^{-H}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \quad (32)$$

The $U_q(su(2))$ gauge field maintains the same form (30) but now the sector $A_\mu^i(x)X_i$ is not closed under gauge transformations. Hence one has to work with the whole enveloping algebra and with infinite number of additional fields $A_\mu^{i_1 \dots i_k}(x)$. Note, however, that in a finite dimensional representation of the algebra $U_q(su(2))$ only a finite number of fields survives. For example, in the spin $j = \frac{1}{2}$ representation, one has the 2x2 Pauli matrices realization of generators and the $U_q(su(2))$ gauge theory is equivalent to the usual theory with the gauge group $GL(2)$. This group is not a compact one and it leads to difficulties with quantization. To avoid these difficulties in the $j = \frac{1}{2}$ representation one can perform a reduction to $SU(2)$. Jimbo [8] has shown that any irreducible finite-dimensional representation of $U\mathcal{G}$ where $\mathcal{G}$ is a Lie algebra can be deformed to an irreducible representation of $U_q\mathcal{G}$ and there is a corresponding non-degenerate contravariant bilinear form. One can use this representation in the quantum group gauge theory and also in the theory of quantum group chiral fields formulated in terms of currents.

A final remark concerns matter fields. If a two-component field $\phi = (\phi_1, \phi_2)$ takes values in the quantum plane, we have the relation

$$\phi_1(x)\phi_2(y) = q\phi_2(y)\phi_1(x),$$

which may be related to fractional statistics [9, 10].

5 Classical mechanics on the quantum line

Quantum phase space R_q^2 is formed by the generators of position x and momentum p satisfying the relation

$$px = qxp. \quad (33)$$

This is what we mean by saying that x is the coordinate on the quantum line. So the quantum line is a reduction of the quantum phase space. Here the term “quantum space” for R_q^2 is a little misleading. In fact we are considering R_q^2 as a “classical” but q -deformed space and we will quantize the corresponding dynamical system in the sense of $\hbar$ -deformations.

More exactly, one can assume two realizations of the space R_q^2 . Working over the field of real numbers R , we consider an algebra $\mathcal{A}$ with elements p and x belonging to $\mathcal{A}$ and satisfying the relation (33). In this case q is an arbitrary real number. If one works over the field of complex numbers C , then one considers an algebra $\mathcal{A}$ with involution and the elements p and x are Hermitian. In this case it follows from (33) that q is a complex number satisfying the relation $|q| = 1$. The q -deformed symplectic form on R_q^2 is

$$\mathcal{B}_q(z, z') = q^{-1/2}px' - q^{1/2}xp', \quad |q| = 1. \quad (34)$$

Here $z = (x, p)$, $z' = (x', p') \in R_q^2$. The symplectic form B_q is invariant under transformations $z \rightarrow gz$ with the matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belonging to $SL_q(2)$ and with entries commuting with p and x , i.e.

$$\mathcal{B}_q(gz, gz') = \mathcal{B}_q(z, z'). \quad (35)$$

5.1 Free particle

For the motion of a free particle on the quantum line our coordinates x and p will depend on the real variable t and one has

$$p(t)x(t) = qx(t)p(t). \quad (36)$$

The equation of motion has the form

$$\ddot{x} = 0 \quad (37)$$

or in the canonical form

$$p = m\dot{x}. \quad (38)$$

$$\dot{p} = 0 \quad (39)$$

It is an important property of this system that *the mass m cannot be a number* because from (36), (38) and (40) one gets

$$mp = qpm. \quad (40)$$

The solution of these equations of motion (37)-(39) is

$$x(t) = x + m^{-1}pt, \quad p(t) = p \quad (41)$$

This solution satisfies the requirement (36). Under the additional relations $x'm^{-1} = qm^{-1}x'$, $p'm^{-1} = qm^{-1}p'$, the q -symplectic form (35) is also left invariant under the evolution (41). Note that the evolution (41) is not a transformation of the form $z(t) = gz$ with the entries of g commuting with the components of z .

Equations (38) and (39) and their solution (41) are written for the case when one deals with real numbers. For the case when one deals with an algebra with involution, i.e. $|q| = 1$ and x and p are Hermitian, it is more convenient to deal with a Hermitian operator m and a slightly modified equation (41)

$$\dot{x} = q^{1/2} m^{-1} p. \quad (42)$$

This equation has the following solution

$$x(t) = x + q^{1/2} m^{-1} p t. \quad (43)$$

Equations of motion (39) and (42) can be written in the Hamiltonian form

$$\dot{x} = \{H, x\}, \quad \dot{p} = \{H, p\} \quad (44)$$

with the following Hamiltonian

$$H = \frac{q}{1 + q^2} p m^{-1} p. \quad (45)$$

Here we introduce q -Poisson brackets. These brackets must satisfy a deformed Leibnitz rule because, assuming linearity of the Poisson bracket, one has from (33)

$$\{x, xp\}_q = q^{-1} x \{x, p\}_q. \quad (46)$$

The q -deformed Poisson bracket is connected with the q -deformed symplectic form

$$\mathcal{B}_q(z, z') = z^A \Omega(q)_{AB} z'^A, \quad \Omega(q)_{AB} = \begin{pmatrix} 0 & q^{1/2} \\ -q^{-1/2} & 0 \end{pmatrix}$$

$$\{f, g\}_q = \frac{\partial f}{\partial z^A} \Omega(q)^{AB} \frac{\partial g}{\partial z^B}, \quad z^A = (x, p), \quad (47)$$

where

$$\Omega(q)^{AB} = \begin{pmatrix} 0 & -q^{1/2} \\ q^{-1/2} & 0 \end{pmatrix}. \quad (48)$$

The derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial p}$ have to satisfy the following rules

$$\frac{\partial}{\partial x} p^n x^m = q^n \frac{1 - q^{2m}}{1 - q^2} p^n x^{m-1}, \quad (49)$$

$$\frac{\partial}{\partial p} p^n x^m = \frac{1 - q^{2n}}{1 - q^n} p^{n-1} x^m, \quad (50)$$

which are the same as in [11], and we also assume

$$\partial_p m = q m \partial_p. \quad (51)$$

The Poisson bracket for the “canonical” variables p and x is

$$\{p, x\}_q = q^{1/2}. \quad (52)$$

Let us stress that the most appealing feature of the quantum line analogue of free motion is that the theory is aimed at yielding a mass operator m which is a non-commuting object.

5.2 q -harmonic oscillator.

For the q -harmonic oscillator one has the equation

$$\ddot{x}(t) = -\omega^2 x(t). \quad (53)$$

or

$$\dot{x}(t) = q^{1/2} \omega m^{-1} p(t) \quad (54)$$

$$\dot{p}(t) = -\frac{\omega^2}{q^{1/2}} m x(t) \quad (55)$$

with the following solutions

$$x(t) = x \cos \omega t + q^{1/2} p(\omega m)^{-1} \sin \omega t, \quad (56)$$

$$p(t) = p \cos \omega t - q^{-1/2} \omega m x \sin \omega t \quad (57)$$

Here ω is a number, but we cannot assume that m is a number; rather we must assume that it is subject to the following relations

$$xm = q^{-1}mx, \quad pm = q^{-1}mp. \quad (58)$$

For derivatives, the relations (49) and (50), as well as

$$\frac{\partial}{\partial x} m = qm \frac{\partial}{\partial x}, \quad (59)$$

are assumed. Under the conditions (33) and (58) the evolution (57),(56) leaves invariant the relation $p(t)x(t) = qx(t)p(t)$ and also the q -symplectic form.

With the Poisson bracket (47) the equations of motion (54) and (55) can be written in the form (44) with the following Hamiltonian

$$H = \frac{q}{1+q^2} pm^{-1}p + \frac{1}{q} \frac{\omega^2}{1+q^2} xmx. \quad (60)$$

6 Quantum mechanics on the quantum line

Now let us quantize our relations (9), substituting a q -commutator instead of the q -Poisson bracket and $-i\hbar$ in the left-hand side. So we have

$$px - qxp = -i\hbar q^{1/2}. \quad (61)$$

We call this relation the $\hbar$ - q -deformed relation. We have $q^{1/2}$ in the right-hand side of (61) in agreement with hermiticity (note that here $|q| = 1$).

An explicit connection between p and x in the relation (61) and the usual operators P and X , satisfying the relation

$$[P, X] = -i\hbar, \quad (62)$$

is the following

$$p = -\frac{\hbar}{2 \sin \theta/2} X^{-1} [e^{-\frac{\theta}{\hbar} XP} - 1] \quad (63)$$

with $x = X$ and $q = e^{i\theta}$ (see for example [12]).

In the usual coordinate representation the formula gives

$$pf(x) = -i\hbar q^{1/2} \frac{f(qx) - f(x)}{(q-1)x}. \quad (64)$$

Let us introduce an operator K :

$$q^K f(x) = f(qx). \quad (65)$$

With the usual definition of scalar product the operator K has the following properties under Hermitian conjugation

$$(K)^* = -K - 1. \quad (66)$$

Thus it is natural to introduce the Hermitian operator $\tilde{K} = K + 1/2$. The commutation relations of the operator $\tilde{K}$ with p and x have the form

$$[\tilde{K}, x] = x, \quad [\tilde{K}, p] = -p. \quad (67)$$

Note that the operators p and x in (61) are Hermitian. This is in contrast with the so-called q -oscillator algebra [13, 14, 15], where creation and annihilations operators satisfying analogous relations are reciprocally-conjugate operators.

6.1 Free particle

Now we are going to consider the $\hbar$ -quantization of the q -deformed free particle, or in other words, specify the evolution of the operators $x(t)$ and $p(t)$ satisfying the commutation relations (61). There are at least two possibilities. In the first one we can try to preserve under quantization the classical equation of motion, i.e.

$$\dot{p} = 0, \quad \dot{x} = q^{1/2} M^{-1} p \quad (68)$$

and the form of the classical solution

$$x(t) = x + q^{1/2} M^{-1} p t, \quad p(t) = p. \quad (69)$$

This evolution cannot be performed by a unitarity operator $U(t) = \exp i H t$, with a Hamiltonian quadratic in p .

In the second one we can assume that there is unitary dynamics which preserves only the first equation (68)

$$i\hbar\dot{p} = [H, p] = 0. \quad (70)$$

In particular $H = \frac{1}{2\mu^2} p^2$ with μ being a number satisfies this requirement. However we have

$$i\hbar\dot{x} = \frac{1}{2\mu} [p^2, x] = i\hbar q(1 + q^2) \frac{1}{2\mu} p + (1 - q^4) \frac{1}{2\mu} x p^2, \quad (71)$$

so $\ddot{x} \neq 0$.

6.2 Harmonic oscillator

To define a natural generalization of the harmonic oscillator Hamiltonian let us introduce the following creation and annihilation operators

$$\alpha^+ = \frac{1}{\sqrt{2}}(q^{\rho K/2} x + i p q^{\nu K/2}), \quad \alpha = \frac{1}{\sqrt{2}}(q^{\rho K/2} x - i p q^{\nu K/2}). \quad (72)$$

From the conjugation rules (66) it follows that $(\alpha^+)^* = \alpha$. The relations (61), (67) and (72) under the condition $\rho + \nu = -1$ lead to

$$[\alpha, \alpha^*] = \frac{1}{q^{\tilde{K}}}. \quad (73)$$

Note that in the lefthand side of (73) there is the usual commutator. It is natural to use the following formula for the harmonic oscillator Hamiltonian

$$H = \alpha^+ M^{-1} \alpha = \frac{c}{2} [x q^{\rho \tilde{K} + 1/2} x + p q^{\nu \tilde{K}} p + q^{1/2} (p x - q^{-1} x p)], \quad (74)$$

One can find the first few eigenstates of this operator:

$$\psi_0, \alpha^+ \psi_0, \alpha^+ \alpha^+ \psi_0, \alpha \psi_0 = 0, \quad (75)$$

with the following eigenvalues

$$E_0 = 0, \quad E_1 = 1, \quad E_2 = c \left(\frac{1}{2} (q + 1/q) + 1 \right). \quad (76)$$

7 Multi-dimensional generalizations

Let us define q -deformed $2n$ -dimensional “classical” phase space to be formed by generators of position x^i and momentum p_j , subject to the relations

$$x^i x^j = \frac{1}{q} \hat{R}^{ij}_{kl} x^k x^l \quad (77)$$

$$p_i p_j = \frac{1}{q} \hat{R}^{lk}_{ji} p_k p_l \quad (78)$$

$$p_j x^i = q \hat{R}^{ik}_{jl} x^l p_k, \quad (79)$$

where the $\hat{R}$ -matrix is given by the usual formula

$$\begin{aligned} \hat{R}^{ij}_{kl} &= \delta_k^i \delta_l^j (1 + (q - 1) \delta^{ij}) + (q - 1/q) \delta_k^i \delta_l^j \theta(i - j) \\ \theta(i - j) &= 1, \quad i > j, \quad \theta(i - j) = 0, \quad i \leq j \end{aligned} \quad (80)$$

(compare with [11]).

It is natural to assume that under $\hbar$ -deformation, relations (77) and (78) do not change and the $\hbar$ -deformed version of (79) is

$$p_j x^i = -i q \hbar \delta_j^i + \hat{R}^{ik}_{jl} x^l p_k. \quad (81)$$

Note that to get from this relation the commutation relations (61) one should change q to $q^{1/2}$.

The algebraic relations (77), (78) and (81) with $\hat{R}$ as in (80) are the same as for the q -deformed oscillators [13, 14, 15],

$$\begin{aligned} A_i A_j &= 1/q A_j A_i, \quad A_i^+ A_j^+ = q A_j^+ A_i^+, \quad i < j. \\ A_i A_j^+ - q A_j^+ A_i &= \delta_j^i (1 + (q^2 - q) A_i^+ A_i + (q^2 - 1) \sum_{k>i} A_k^+ A_k). \end{aligned} \quad (82)$$

However the hermiticity properties are different. For the operators p and x , the usual hermiticity properties $(p_i)^+ = p_i$, $(x_i)^+ = x_i$ are assumed.

Note also that the algebra (77), (78) and (81) admits a generalization to the continuum case. Let us write down the corresponding generalization for creation and annihilation operators. We can assume that creation and annihilation operators are labelled by a momentum variable $k \in R^1$, the discretization of which gives a labelling by an index k , $k = 1, 2, \dots, n$, $A_k = \Delta^{1/2} a(k)$; A_k satisfies the algebra of q -deformed oscillators [13, 14, 15] and Δ is a spacing parameter. So the continuum version of the algebra (82) looks like

$$\begin{aligned} [a(k), a(l)]_q &= 0, \quad [a^+(k), a^+(l)]_q = 0, \quad k > l, \\ [a(k), a^+(l)]_q &= \delta(k - l) (1 + (q^{-2} - 1) \int_k^\infty dk' a^+(k') a(k')). \end{aligned} \quad (83)$$

8 Quantum groups and quantization of space-time

Non-commutative geometry and quantum groups are of relevance to the problem of the quantization of space-time. Although in recent physics papers a discussion of this possibility was avoided rather carefully, it seems that the mathematical formalism itself pushes us in this direction. In particular, Manin [6] discussed quantum vector spaces and his consideration looks rather appropriate in this context. Wess and Zumino [11] considered the deformed Schrödinger equation in quantum space; see also [16].

The idea of the quantization of space-time using non-commuting coordinates x_μ is an old one [17]. In this context relations like

$$x_\mu x_\nu - x_\nu x_\mu = i\hbar g_{\mu\nu} \quad (84)$$

or the anisotropic one

$$x_\mu x_\nu = q x_\nu x_\mu \quad (85)$$

were considered.

It is natural to attempt to relate the non-commutativity of coordinates to the uncertainty δx in the length measurement

$$\delta x > l_{Pl}, \quad (86)$$

where l_{Pl} is the Planck length. More recently the relation (86) was considered in string theory [18, 19]. However, trying to derive the uncertainty relation (86) from (84) or (85) by analogy with the Heisenberg uncertainty relation, we get a problem because (86) holds even in one dimension, i.e. on the line. In this context the definition of the quantum line as the reduction of the quantum plane (see Section 5) looks rather suitable. One can speculate that at the Planck scale there are fluctuations of the metric [20] and signature [1, 2], and the number field [5, 21] undergoes fluctuations; moreover, the space-time coordinates become non-commuting operators. Quantum cosmology [22] appears to be a more appropriate field for attempts to relate these ideas to reality.

In conclusion, let me say that of course there are a lot of problems in quantum group field theory but this quantum group challenge looks rather exciting.

Acknowledgements

I would like to thank the Theory Division at CERN for kind hospitality.

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