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Study of two-dimensional non-linear oscillations by means of an
electromechanical analogue model, applied to particle motion in
circular accelerators

by

M. Barbier and A. Schoch

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1. Introduction

For the study of non-linear effects on betatron oscillations in circular accelerators, carried out in connection with the design of the CERN alternating gradient proton synchrotron, it was found desirable to have an analogue model simulating these oscillations. Firstly, all analytical methods in use are only approximations whose reliability requires checking. For this an analogue model seemed advantageous as the problems involved are quite formidable and expensive for digital computers. Secondly there are important problems (e.g. in particular the behaviour of oscillations with parameters changing with time) so far more or less inaccessible to analytical methods. These problems can be studied experimentally by means of an analogue model. Experiments of this kind have also proved helpful in the interpretation of analytical results.

So an electromechanical model was devised, simulating the transverse motion of a particle in an accelerator by a two-dimensional pendulum. Non-linear forces and perturbations are simulated by electrostatic fields which, obeying Laplace's equation, make it easy to produce the particular spatial configurations of forces and perturbations occurring in the guide field accelerators. It was this feature which made a mechanical analogue easier to realize than a purely electronic one*.

A short description of the analogue model had been published in the Proceedings of the 1956 CERN Symposium (see Barbier [1956]) together with a first series of experiments carried out mainly to check some of the theoretical results on two-dimensional non-linear resonances. In the present

* It may be mentioned that another, purely electronic, analogue had been constructed in CERN for the simulation of (one-dimensional) phase oscillation. (see Schmelzer [1956] and Gabillard [1957 and 1958]).

report, details on the performance of the analogue are given as well as results of further experiments on non-linear resonances (with fixed parameters).

2. The mathematical problem to be simulated by the model

The mathematical problem to be simulated by the model is given by the equations of motion for the radial and vertical displacements x and y in a circular accelerator :

$$\begin{aligned} \frac{d^2 x}{d\vartheta^2} - nx &= - \frac{\partial V(x, y, \vartheta)}{\partial x} \\ \frac{d^2 y}{d\vartheta^2} + ny &= - \frac{\partial V(x, y, \vartheta)}{\partial y} \end{aligned} \quad (1)$$

ϑ being the angular position of the particle, n the field index (characterizing the field gradient) and $V(x, y, \vartheta)$ a perturbation potential taking account of deviations from the ideal field characterized by n . This potential satisfies Laplace's equation in two dimensions (if we consider the perturbations due to the guide and focusing field only, which dominate in practice). We write it in the form of a multipole expansion either in polar coordinates r, ϕ or in rectangular coordinates $x = r \cos \phi, y = r \sin \phi$:

$$\begin{aligned} V &= \sum_{k=1}^{\infty} V_k \left(\frac{r}{R} \right)^k \cos k (\phi - \epsilon_k) \\ &= \frac{V_1}{R} [x + \epsilon_1, y] + \frac{V_2}{R^2} [x^2 - y^2 + 2\epsilon_2, 2xy] \\ &\quad + \frac{V_3}{R^3} [x^3 - 3xy^2 + 3\epsilon_3, (3x^2y - y^3)] \\ &\quad + \frac{V_4}{R^4} [x^4 - 6x^2y^2 + y^4 + 4\epsilon_4, (4x^3y - 4xy^3)] + \dots \\ &= \left[r_0 \left(\frac{\delta B}{B_0} - \frac{\delta p}{p_0} \right) + n\zeta \right] x + \left[\epsilon r_0 - n\zeta \right] y \\ &\quad - \frac{1}{2} \left[\delta n (x^2 - y^2) + 2 \epsilon nxy \right] - \frac{1}{3!} \frac{dn}{dx} \left[x^3 - 3xy^2 + 3\epsilon (3x^2y - y^3) \right] \\ &\quad - \frac{1}{4!} \frac{d^2 n}{dx^2} \left[x^4 - 6x^2y^2 + y^4 + 4\epsilon (4x^3y - 4xy^3) \right] + \dots \end{aligned} \quad (2)$$

R is a reference distance from the centre to be chosen later. ϵ_k is the

angle by which the plane of symmetry of the k -th multipole ($2k$ - pole) is tilted against $y = 0$. The last of the equations expresses the multipole coefficient V_k in terms of accelerator parameters : r_0 is the radius of the equilibrium orbit $\delta B(\vartheta)$, δp , $\delta n(\vartheta)$ are deviations of field B , momentum p , and n -value from the ideal values; $\xi(\vartheta)$, $\zeta(\vartheta)$ are radial and vertical magnet displacements, $\epsilon(\vartheta)$ is the angle of tilt of the magnetic symmetry plane against $y = 0$. The radial derivatives $\frac{dn}{dx}$, $\frac{d^2n}{dx^2}$, .. (taken at $x = y = 0$) are responsible for non-linear terms in the equations of motion ($\frac{dn}{dx}$ producing a sextupole contribution V_3 , $\frac{d^2n}{dx^2}$ an octupole contribution V_4 etc.).

Whereas n is constant, independent of ϑ , in a weakly focusing accelerator, n changes with ϑ periodically in an A.G. synchrotron, covering several periods per revolution. In order to simplify the realization of a model, the alternation of n was not simulated, i.e. the model was designed to perform the "smooth" part of the motion only. To explain what this means we recall that the undisturbed motion ($V \equiv 0$) in the A.G. synchrotron can be represented by the Floquet form of solution :

$$x = \sqrt{a} \left[w(\vartheta) e^{i(Q\vartheta+\varphi)} + w^*(\vartheta) e^{-i(Q\vartheta+\varphi)} \right] \quad (3)$$

where $\sqrt{a}$ and φ are arbitrary constants defining amplitude and phase. The factor $w(\vartheta)$ is a function of ϑ having the same period as $n(\vartheta)$. Thus (3) is a "smooth" sine motion with frequency Q , modulated by wriggles with the period of $n(\vartheta)$. Q as well as the modulating factor $w(\vartheta)$ are determined by the shape of $n(\vartheta)$. Under the conditions applying in A.G. synchrotrons, the wriggle modulation is relatively small (about 15% of the sine wave part in the CERN PS). Properly defined "smooth motion" coordinates $\bar{x}$, $\bar{y}$ can then be shown to satisfy the following equations to a fair approximation (see Schoch [1957], in particular Appendix V).

$$\begin{aligned} \frac{d^2 \bar{x}}{d\vartheta^2} + Q^2 \bar{x} &= - \frac{\partial \bar{V}}{\partial \bar{x}} \\ \frac{d^2 \bar{y}}{d\vartheta^2} + Q^2 \bar{y} &= - \frac{\partial \bar{V}}{\partial \bar{y}} \end{aligned} \quad (4)$$

The unperturbed system can therefore be simulated by a simple two-dimensional harmonic oscillator. The perturbation potential in smooth motion coordinates becomes

$$\begin{aligned}\bar{V} = & \left[r_0 \left(\frac{\delta B}{B_0} - \frac{\delta p}{p_0} \right) - Q^2 \bar{\xi} \right] \bar{x} + \left[\epsilon r_0 - Q^2 \bar{\zeta} \right] \bar{y} \\ & - \frac{1}{2} \left[\bar{\delta n} (\bar{x}^2 - \bar{y}^2) + 2\bar{\epsilon} Q^2 \bar{x}\bar{y} \right] - \frac{1}{3!} \frac{d\bar{n}}{d\bar{x}} (\bar{x}^3 - 3\bar{x}\bar{y}^2) \\ & - \frac{1}{4!} \frac{d^2\bar{n}}{d\bar{x}^2} (\bar{x}^4 - 6\bar{x}^2\bar{y}^2 + \bar{y}^4) - \dots\end{aligned}$$

In this, the quantities $\bar{\xi}$, $\bar{\zeta}$, $\bar{\epsilon}$, $\bar{\delta n}$, $\frac{d\bar{n}}{d\bar{x}}$ etc. are modified with respect to those in (2) by the transformation to smooth motion coordinates (see above reference). In general these modifications are small enough to preserve the orders of magnitude of the multipole terms in (2).

The conversion from smooth motion back to the complete motion in the a.g. case can be obtained from

$$x = \sqrt{\frac{Q}{2}} (w + w^*) \bar{x} - \frac{i(w - w^*)}{\sqrt{2Q}} \frac{d\bar{x}}{d\vartheta}, \quad (5)$$

again to a fairly good approximation (formulae (8.8) and (V.2) of above reference).

3. Description of the analogue model

The model consists of a flexible quartz pendulum in the form of a thin fibre terminating in a ball. The fibre and the ball are made with a piece of fused quartz stretched in a flame. A thin layer of metal is then deposited on the quartz by sputtering in vacuum, thus enabling the ball to carry an electrical charge when the end of the fibre is connected to a voltage source.

The pendulum being elastic, has its own restoring force, linearly proportional to its displacement from the position of equilibrium. As the displacements occurring within the tube are small in comparison with the length of the pendulum, the linearity of the latter is very good.

The general lay-out is shown in fig. 1 (see p. 5). The ball and the fibre are held in the centre of a set of twelve cylindrical electrodes, arranged in a circle parallel to the fibre axis. These electrodes are made of glass covered with aquadag and can be connected to voltage sources. The whole is put in a glass tube, which is heated in a furnace and pumped until the vacuum reaches 10^{-6} mm Hg.

The natural damping of the pendulum is then so small that it takes several hours before the amplitude is reduced by a factor e.

Figs. 2 and 3 are photographs of the pendulum and the tube. Fig. 4 is a complete view of the set-up, showing how the illuminated pendulum can be observed in projection on a screen.

The electrodes are arranged so that the potentials they create are almost exactly the two dimensional solutions of Laplace's equation, i.e. they have the form

$$U(r, \varphi, z) = \sum_k U_k(t) \left(\frac{r}{R}\right)^k \cos k(\varphi - \epsilon_k)$$

where r , φ and z are the co-ordinates, R the radius of the cylinder through the electrodes and U_n the voltage harmonics (multipole components) of the field.

The equations of motion of the pendulum therefore become :

$$\frac{d^2 x}{dt^2} + \omega^2 x = - \frac{q}{m} \frac{\partial U}{\partial x} \quad (6)$$

$$\frac{d^2 y}{dt^2} + \omega^2 y = - \frac{q}{m} \frac{\partial U}{\partial y}$$

where q the charge, m the mass, and $\frac{\omega}{2\pi}$ the natural frequency of the pendulum. Comparison of (2) and (6) shows how the quantities have to be scaled to represent a given system to be simulated.

The pendulum, practically linear and undamped if left alone, can (a) be made non-linear by applying non-linear electrical forces, which are constant in time, and (b) be perturbed by time-dependent forces that can also be made non-linear with respect to displacement.

4. Measurements of the model's parameters.

For the use of the model as a computer, it is preferable to obtain the parameters of the model from measurements taken during its performance, involving only time (or frequencies) and lengths. For instance we calibrate the applied fixed non-linearities by the frequency shift they cause, and the

Fig. 1 General layout of the tube.

alternating perturbation voltages by the rate of rise of oscillation amplitude with time; in this manner we can treat any mathematical problem directly, i.e. eliminating model constants and scaling factors from a particle accelerator to a betatron analogue. This will allow us to use the model directly as a mathematical instrument for all non-linear oscillation problems.

4.1 The one-dimensional motion of the pendulum.

4.1.1 Behaviour of the pendulum if no tensions are applied to the electrodes. First the uncharged mechanical pendulum has to be examined. Satisfactory results have been obtained with a pendulum having the dimensions; length 1 cm, thickness 0.1 mm. and the volume of the ball 1 mm³. The fundamental frequency of the oscillation of the particular tube we used for all experiments described in the following was $f = 13.3$ c/s. The first higher mode of resonance (flexural oscillations) of the pendulum lies around 110 c/s, this being well outside the range we will use. Such a fiber cannot be made with exact rotational symmetry, so the pendulum has two preferred axes of oscillation, which are perpendicular to each other, and along which it will swing with two slightly different frequencies f_x and f_y .

The frequency difference can easily be found by measuring (with a stopwatch) the time T taken to describe a full cycle of change of the Lissajou pattern in the free oscillation. Then

$$\left| f_x - f_y \right| = \frac{1}{T},$$

where $T = 1.7$ sec. for the particular tube used here. Therefore the relative frequency difference is of the order of a few per cent.

$$\frac{f_x - f_y}{f_0} = 0.044.$$

Each of the eigenfrequencies $f_x + f_y$, can also be measured separately by comparison with the frequency f_g of some electronic generator. For this measurement it is convenient to derive an electrical signal which has the exact frequency of the oscillating pendulum or a multiple of it. This can be done by means of a photocell covered by a slit, which produces an electrical pulse each time the image of the illuminated ball passes over it, i.e. twice in a period.

Fig. 2 Photograph of the pendulum.

Fig. 4 Photograph of the analogue model.

Fig. 5 Photograph of the tube.

This pulse can be used directly for frequency comparison on a two-beam oscilloscope, when the other beam (from the electronic generator) shows pulses of known frequency; or it can be transformed into a sinusoidal current by means of a selective amplifier which singles out one of the harmonics. If the two pulse rows are used, and remain still with respect to each other, then the frequencies f_1 and f_2 are rational in a ratio $k_1 : k_2$. This value can be found from the number of pulses k_1 and k_2 in each row, between two coincidences, i.e. for the common period.

If the frequencies differ slightly there will be a drift of one row with respect to the other, and the deviation δf in frequency of the row which is drifting is given by

$$\delta f \cdot \tau = 1,$$

where τ is the time taken by one pulse to replace the preceding one with respect to a reference pulse of the other row.

In the case where sine waves are derived from the electronic generator, as well as from the pick-up system, and displayed as a Lissajou pattern on the oscilloscope, the interpretation in terms of frequency differences requires more care, both signals being displayed in such a manner as to be no longer separated. When the pattern is stationary both frequencies remain in an exact rational relationship recognizable by the Lissajou pattern :

$$\frac{f_1}{f_2} = \frac{k_1}{k_2}.$$

To a given frequency ratio still a continuous set of patterns exists, corresponding to the differences of phase. Therefore, if one of the frequencies, f_1 say, differs from the rational value by a small amount δf_1 , the pattern changes continuously and repeats after a time τ related to δf_1 by

$$\delta f_1 = \frac{1}{k_2 \tau} \text{ c/s.}$$

Similarly, if f_2 deviates,

$$\delta f_2 = \frac{1}{k_1 \tau}.$$

In the above formulae k_1 and k_2 are prime to each other.

The elastic restoring force of the fibre may not be exactly proportional to displacement, the effect becoming apparent when the oscillations reach a certain amplitude.

A non-linear restoring force makes the frequency of the pendulum depend on the amplitude of its oscillation. To a first order approximation the frequency deviation is proportional to the square of the amplitude, if the law of force (with displacement) contains terms of the second or third degree.

With the method of frequency measurement by comparison with an electronic oscillator, quadratic and cubic non-linear forces may be calibrated by measuring the curve of the frequency shift against the square of the amplitude of oscillation, for a time independent non-linearity and one-dimensional free oscillations.

We will examine now what happens, when we place a charge on the pendulum, by connecting its lead to a constant voltage source.

When the ball is thus electrically charged, as in normal operation, the image force, exerted on this charge by the walls of the chamber, i.e. the conducting electrode system, is a negative restoring force which is zero when the ball is in the centre and has linear and non-linear terms depending on displacement.

The linear term of the image force causes a shift of the oscillation frequency of the pendulum, which is independent of amplitude. The non-linear terms of the image force cause a further shift, which grows with amplitude. It will be seen later that the non-linearity resulting from the image force induced by the electrical charge on the ball is small compared with the non-linearity applied to the model by means of the electrode system in normal operation.

Fig. 5 (see p. 9) shows measured frequency shifts vs. square of free oscillation amplitude for the pendulum, in free motion, and for different voltages on the ball all electrodes being earthed. As is seen the pendulum has a small inherent non-linearity causing a frequency increase of 0.01 c/s when the amplitude in the tube has reached about 7 mm. (the maximum we shall use). As ball voltage is applied one sees that the oscillation frequency is lowered, and that as the oscillation amplitude is increased the frequency still decreases. The charge on the ball introduces a non-linearity which, for the pendulum used, is of opposite sign to the natural one. Both

effects cancel out for a voltage of about 1.3 kV on the ball.

Fig. 6 (below) shows the change in frequency for small oscillations vs. ball voltage, i.e. the term in the image force which is linear with displacement. The dependence on ball voltage appears to be quadratic, which is to be expected as image forces always depend on the square of the charge involved.

Fig. 7 (below) shows the difference in frequency for maximum (7mm) and nearly zero amplitudes vs. ball voltage, i.e. the term in the restoring force which is non-linear with displacement. The observed dependence is also quadratic with ball voltage.

Fig. 5 Frequency shift against the oscillation amplitude squared for various ball voltages.

Fig. 6 Frequency change of small oscillations against ball voltage.

4.1.2 Behaviour of the pendulum on application of time independent voltages to the electrodes, so as to form dipole, quadrupole sextupole and octupole fields.

The pendulum is arranged in the tube so that the principal axis of oscillation falls nearly in the middle of a gap between two electrodes, for symmetry and convenience in experiments on one dimensional oscillations. To produce an electrical dipole field with this system, which has twelve electrodes, four electrodes are connected to the voltage source and two to earth on each side. In this way, the elements in higher order field harmonics are very few, as even harmonics do not appear for symmetry reasons and as the third one vanishes in such an arrangement, where one third of the electrodes in a period are earthed.

Fig. 8 shows the displacements of the ball in a dipole field plotted against the voltage applied to one of the groups of four rods against the earth for various ball voltages against earth. The movement of the ball is seen to be linear with field strength except when

Fig. 7 Frequency shift at an amplitude of 7 mm against ball voltage.

Fig. 8 Displacement of the ball against dipole voltage for various voltages of the ball.

very high depole fields and small ball voltages are used. The effect seems to be due to the image force exerted by the charge resting on the electrode system, which is reflecting itself in the conducting surface of the ball.

A quadrupole field is equally well produced with a small content in higher field harmonics by earthing one and connecting two electrodes to the voltage appropriate for each quadrant

of the tube. The effect of a quadrupole field with axes of force in line with the principal oscillation axes is to introduce a new difference between the frequencies of oscillation along the two principal axes, one being increased and the other decreased by equal amounts.

Fig. 9a shows the measured changes in frequency of small oscillations dependent on electrode voltage along one of the principal axis for various voltages on the ball.

Fig. 9a Change of frequency of small oscillations against applied quadrupole voltage.

Fig. 9b shows how linear the frequency shift is with ball voltage per applied unit quadrupole voltage. A six pole field is produced by connecting the electrodes in

Fig. 9b Change of frequency per unit quadrupole voltage as a function of ball voltage.

pairs to the voltage sources of opposite polarity. Field components of higher order, owing to their high harmonic number, are probably irrelevant in the centre region of the tube, where the ball is oscillating.

The effect of a six-pole field is to introduce an oscillation frequency dependent on the square of the amplitude,

which is demonstrated in fig. 10 for various six-pole and ball voltages.

Fig. 11 shows the frequency shift obtained with an amplitude of 7 mm. for a six-pole versus the product of the ball and the six-pole voltages.

The curve is nearly parabolic save for the effect of the natural non-linearity

which is only noticeable near the origin. The parabolic dependence of frequency shift on six-pole strength must be expected from theoretical considerations whereas in the eight-pole case the dependence will be linear, as will be confirmed by experimental work.

An eight-pole field may be produced in the following way. Take a system, connect one electrode to a voltage source and two electrodes to another source of opposite polarity and of half the magnitude.

As in the six-pole case, the eight-pole introduces a linear dependence of oscillation frequency on the square of the amplitude as is shown in fig. 12 for various ball and eight-pole voltages. The frequency differences obtained are proportional to the applied eight-pole force as shown in fig. 13 where the frequency differences for maximum amplitude are drawn as a function

Fig. 10 Frequency shift against the oscillation amplitude squared for various six-pole and ball voltages.

Fig. 11 Frequency shift at an amplitude of 7 mm for a six-pole against the product of six-pole and ball voltages.

Fig. 12 Frequency shift against the oscillation amplitude squared for various eight-pole and ball voltages.

Fig. 13 Frequency shift at an amplitude of 7 mm for an eight-pole against the product of eight-pole and ball voltages.

of the product of the eight-pole voltage and the ball voltage.

4.2 The two-dimensional motion of the pendulum.

4.2.1 Natural two-dimensional oscillations of the uncharged pendulum.

If the oscillation frequencies of the ball, along each principal axis were the same, the motion of the ball would describe an ellipse. As both frequencies differ slightly, the phase between the oscillations along the x and y axes changes slowly, thus altering the shape of the ellipse. The ellipse, however, should remain always inscribed in a rectangle, when the pendulum is perfectly linear. If the pendulum is subject to non-linear forces the figure in which the ellipses are described takes a non-rectangular form.

Fig. 14a is a photograph showing a free two-dimensional oscillation of the ball, with no voltages applied whatsoever. Owing to the non-rectangular shape of the envelope, one must conclude that there is a structural non-linearity in the pendulum.

Fig. 14b shows the same free two-dimensional oscillation, this time with an applied ball voltage of 1.3 kV. For this ball voltage the oscillation frequency was found independent of the amplitude, the frequency drift of the image force non-linearity compensating that of the natural non-linearity. As the image force does not differ basically from the preceding one, it is seen that the image force non-linearity cannot compensate the structural one for a two-dimensional motion.

Fig. 14c shows another free two-dimensional motion, with a ball voltage of .3 kV, which makes the image force several times larger. As is seen, the form of the pattern is not substantially altered.

As the image force is a pure radial one, and independent of azimuth, it follows that the structural non-linearity must be an azimuth dependent force.

4.2.2 Correction of the inherent non-linearity by a static eight-pole field

An attempt was made to correct the pendulum's non-linearity by applying an external eight-pole. Fig. 15 shows the result for an eight-pole DC voltage of - 0.5, + 1 kV and a ball voltage of 1.5 kV. The pendulum became linear, although its amplitude was very large. The eight-pole field is somewhat larger than expected judging from the one-dimensional frequency shift curves. It has however the correct sign.

4.2.3 Motion of a linear conical pendulum in six and 8-pole fields.

It was tempting after having produced an ideally two-dimensional pendulum, to

Fig. 14 a

Fig. 15

b

Fig. 16 a

c

b

Fig. 14 Free two-dimensional oscillation of pendulum (ball voltage a) 0, b) 1.3, c) 3 kV optical magnification 5)

examine the form of the oscillation in pure six-pole and eight-pole fields. The corrected eight-pole voltage was maintained, to which one added a strong six-pole or eight-pole DC voltage. The results are shown in figs 16a and 16b.

In the six-pole case, the envelope has only one axis of symmetry and in the other case it has two perpendicular ones, which is in accordance with the symmetry properties of the applied non-linear fields (i.e. a six-pole and an eight-pole).

5. Behaviour of the model under perturbations of constant frequency.

One will now study the effect of multipole perturbations of fixed frequency on one and two dimensional oscillations : first without and then with constant applied non-linear restoring forces (so called stabilizing non-linearity).

5.1 Beating of amplitudes in one dimension.

The experiments we will describe now have been carried out using mainly a one-dimensional oscillation. When the exciting multipole fields are arranged to be symmetrical with respect to the main oscillation axis of the pendulum and when the latter performs a one-dimensional oscillation along its axis, (which lies in the direction of the field at the moment when the perturbation is switched on) then the motion remains one-dimensional, even when excited by multipole fields of a higher order than the dipole.

The purpose of the work is to examine the behaviour of an anharmonic pendulum, submitted to a periodic force, which is acting in the direction of its line of oscillation and which depends on a power of the displacement. Mathematical treatment of this problem, with special reference to particle accelerators, has been carried out by Judd [1950], Sturrock [1958], Moser [1955], Hagedorn [1957], Hagedorn and Schoch [1957] and Courant [1956].

It was desired to check the above work experimentally in order to justify its application to the designing of the machines.

The pendulum was made anharmonic by the application of a static eight-pole (the so-called stabilizing non-linearity) well lined up along the main axis of oscillation.

Fig. 17 is a photograph showing how well the pendulum can be made to oscillate along a straight line (even at large amplitudes). The scale is

represented by the electrodes system.

Excitation voltages were applied in the forms of a dipole, a quadrupole and a six-pole respectively, in order to have the cases where the perturbation force was independent of displacement, proportional to it, and the square of it. For these experiments we had an electrical pick up system, which provided a current proportional to the displacement of the pendulum. It measured the variation of the capacity between the pendulum and a pair of adjacent electrodes of the model. This capacity was put in parallel with a tank circuit in the screen grid of a tetrode, while a quartz was driving the control grid. The tube was operated on its dynatron characteristic, in order to provide a negative resistance. With this device it was possible to display the motion of a particle in the phase plane, by feeding the output to the x-plates of an oscillograph, the derivative of it to the y-plates, and by adjusting the deflection voltage so that a circle would be described in the case of an undamped oscillation. By triggering the oscillograph spot intensely at intervals equal to the period of the excitation voltage, the succession of points appearing on the screen plotted out a phase space diagram.

5.1.1 Dipole excitation with stabilizing eight-pole. In this case the excitation field is homogenous i.e. the time varying force is independent of the position of the pendulum. When the excitation frequency is in the neighbourhood of the eigenfrequency of the pendulum, it corresponds to an integer resonance line in the machine. This case deserves to be examined in detail, because it shows all the main features of the behaviour of the anharmonic oscillator of a main resonance. It will

not be basically different at higher order resonance modes, when the excitation force depends on the position of the pendulum as with a quadrupole or a six-pole.

Fig. 18 shows resonance curves taken point by point. They are obtained by measuring the maximum amplitude of oscillation reached by the pendulum, starting from zero, at different frequencies and for different excitation voltages.

Fig. 19 shows a series of patterns described by the representative

Fig. 18 Resonance curves for one-dimensional oscillation with dipole excitation and eight-pole stabilizing non-linearity.

Fig. 17

c

Fig. 19 a

d

b

e

Fig. 17 View of the tube taken along the axis showing pendulum performing one-dimensional motion.

Fig. 20 a

d

b

e

c

f

g. 20 Maps of the phase plane with dipole excitation and stabilizing eight-pole at several distances from eigen-frequency, a) $\Delta f = -0.015$ e/s, b) $\Delta f = -0.012$, c) $\Delta f = 0$, d) $\Delta f = 0.03$, e) $\Delta f = 0.0385$, f) $\Delta f = 0.0485$.

point, taken at equal periodic intervals in the phase plane, starting at zero oscillation amplitude and with constant perturbation voltage at different frequencies.

The phase space curves have been taken with the same ball and eight-pole voltages as for the previous figure but with an excitation voltage of 16 volts, i.e. they are referring to a situation almost identical with the one of the upper of the three resonance curves.

It was interesting to examine the motion when the starting perturbation for the initial oscillation amplitude differed from zero, but using the same exciting voltage at the same frequency, in order to get a general view of the patterns described over the phase plane by the representative point.

Fig. 20 shows several such cases taken at various distances from the resonance and shows rather well the way in which these curves divide when passing through resonance.

The voltages used were 1.3 kV on the ball, (-1, + 2 kV) on the eight-pole and 7.7 volts for excitation i.e. near the second resonance curve in fig. 18. The static displacements of the ball, in a dipole field, could have been taken to calibrate the dipole exciting force; however, it was possible to measure the rate of growth of oscillation amplitude.

Fig. 21 shows the rate of growth of oscillation amplitude at exact resonance frequency for several dipole alternating voltages and ball voltages.

The non-linearity induced by the eight-pole has already been calibrated, by the shift of the oscillation frequency with its amplitude.

The eight-pole can be calibrated otherwise, when the dipole excitation and distance from resonance are known, by using the velocity along its trajectory at the representative point in phase space. This can be measured on some of the photographs by counting the number of dots along the trace, the dots appearing in synchronism with the excitation voltage.

5.1.2 Quadrupole excitation of double eigen-frequency with stabilizing eight-pole. The perturbation force is now proportional to the displacement of the ball and its frequency is twice f_x . This corresponds to being on a half-integer resonance line, for the betatron oscillation in a synchrotron. The motion was again kept one-dimensional.

Fig. 21 Rate of growth of oscillation amplitude against exciting dipole voltage.

Figure 22 shows several diagrams of the traces in the phase plane. The ball voltage was 1.3 kV and the quadrupole excitation voltage, 250 volt. Figures 22a and 22b show the case where the eight-pole is not yet applied; b is taken very near resonance and shows the hyperbolae along which the particle escapes in the phase plane. Figure 22a, however, is taken rather far from resonance, so that the particle performs a beating.

The other diagrams are taken with an applied static eight-pole (-1, + 2 kV). One sees how the elliptical but upright beating curves deform to the right, giving rise to a central and two lateral islands of stability.

5.1.3 Six-pole excitation of triple eigen-frequency with stabilizing eight-pole. The perturbation force is now proportional to the square of the displacement of the ball and its frequency is about three times f_x , corresponding to a third order subresonance line in a synchrotron. The motion is one dimensional as before.

Fig. 23 shows several braces in the phase plane : a, b, c without eight-pole, d, e, f with eight-pole. Voltages were 1.3 kV on the ball, (+ 1, - 0.5 kV) on the eight-pole and 250 V for excitation.

The phase space diagrams of the three cases resemble closely those of Dr. Schoch in his report. Thus a satisfactory agreement is established between the theoretical and the experimental work.

5.2 Beating of amplitudes in two-dimensions.

In this paragraph we will examine the effect of multipole perturbations of fixed frequency on two dimensional oscillations. Constant non-linear restoring forces were not applied, except in one particular case, where we wanted to show the form of the oscillation pattern when the resonance was quenched.

5.2.1 The resonance lines of the linear conical pendulum in the f_x, f_y plane when subject to multipole perturbations. The perturbing multipole field has a frequency f_g , given by the electronic generator. It has been found that the pendulum has two different frequencies along its axis of oscillation. These frequencies, which are already different in the natural pendulum, can be made more so by applying a quadrupole force with the electrodes, which is constant with respect to time.

As is known, the effect of such a quadrupole is only to change the difference between the two natural frequencies. Let f_x and f_y denote the natural frequencies, which will have been adjusted within this range, and

Fig. 22 a

d

b

e

c

f

Fig. 22 Maps of the phase plane in the case of quadrupole excitation with and without stabilizing eight-pole.

Without eight-pole, a) $\Delta f = f_x - \frac{1}{2} f_g = -0.055$ c/s, b) $\Delta f \approx 0$. With eight-pole, c) $\Delta f = -0.07$ c/s,

Fig. 23 a

d

b

e

c

f

Fig. 23 Maps of the phase plane in the case of six-pole excitation with and without static eight-pole. Without static eight-pole, a) $\Delta f = f_x - \frac{1}{3} f_g = -0.04$, b) $\Delta f \geq 0$, c) $\Delta f = 0.0555$ o/s. With eight-pole, d) $\Delta f = -0.0238$,

which can be measured directly with a photocell. We are now looking for the perturbation frequency f_g , for which the resonances of the pendulum occur with a given couple of pendulum frequencies f_x, f_y .

It is convenient to plot the resonance points in a two-dimensional coordinate system, as a function of the quantities $\frac{f_x}{f_g}, \frac{f_y}{f_g}$. Such a graph (see fig. 24) gives all the required information for the resonance states.

Using our type of multipole excitation, the resonances are expected to occur (according to Hagedorn) along the bunches of lines which have been drawn in the diagram going through the points $\frac{f_x}{f_g} = \frac{f_y}{f_g} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ for the two, four, six and eight-pole cases respectively. The dotted lines are those not expected to appear when the axis of perturbation ($\varphi = 0$ in $\cos k \varphi$) coincides with the x-axis of the pendulum. As it is impractical to make f_x very different from f_y , each bunch of resonance lines, will be examined experimentally only in the region where all the lines of the bunch have a point in common.

An accurate measurement can be made of a set of frequencies in the region of a nodal point, with the help of the beating between the oscillation of the pendulum, along one of its axis and the perturbation generator frequency.

In the region of the knot of the resonance lines of order n , the generator frequency is approximately

$$f_g = n f_x + \delta f_g$$

where δf_g is the small deviation from the measured value of $n f_x$. Dividing by $n f_g$:

$$\frac{f_x}{f_g} - \frac{1}{n} = - \frac{\delta f_g}{n f_g}$$

giving the coordinate of the point in $\frac{f_x}{f_g}$ direction. The quantity $\frac{f_y}{f_g} - \frac{1}{n}$ can be found in a similar way, or f_y can be measured directly from f_x with absence of perturbations, by the period of the Lissajou pattern f_x, f_y of the pendulum itself

$$\left| \frac{f_x}{f_g} - \frac{f_y}{f_g} \right| = \frac{1}{T}$$

Fig. 24 The frequency plane of the model, with the sum resonance lines.

Fig. 25 The observed second order sub-resonances in the frequency plane.

It is generally known which of the values f_x or f_y is the greater or it may easily be determined by tuning a selective amplifier to these two frequencies. The measured resonance points for subresonances of order $n = 2, 3, 4$ as seen in figs. 25, 26 and 27 were found to fit exactly with Hagedorn's theoretical lines and so a few measurements were considered sufficient. It was also made sure that no supplementary lines could be found in between these lines.

The dotted resonance lines did not appear when the x axis of the pendulum was the same as the axis of the multipole perturbation i.e. when it went through the point of the periphery having the maximum voltage distribution [$\cos n\varphi=1$]. These lines could be excited only when an angle between the axes was introduced, by shifting the potential distribution of the perturbation around the azimuth. This too agreed with Hagedorn's conclusions.

5.2.2 The quadratic invariant of the two-dimensional oscillator subject to multipole perturbations of fixed frequency. The behaviour of the particle itself at subresonances is shown in the following photographs taken of the actual motion of the ball.

Fig. 28a shows the growth of oscillation amplitude at a second order sub-resonance with a quadrupole excitation for $f_g = 2 f_x$. This is the quasi one-dimensional case where the amplitude increases in the x direction only.

Fig. 28b shows the motion when $f_g = f_x + f_y$ i.e. using the 45° sub-resonance line. In this case the amplitudes increase in both the x and y

Fig. 27 Observed fourth order sub-resonances

Fig. 28 a

Fig. 29 a

b

b

c

Fig. 28 Growth of oscillation amplitude at second order sub-resonance with quadrupole a) $2f_y = f_g$, b) $f_x + f_y = f_g$, c) $f_x + f_y = f_g$ with static eight-pole.

Fig. 30 a

Fig. 32 b

b

c

Fig. 32 a

d

Fig. 30 Growth of oscillation amplitude at fourth order sub-resonance a) $2f_x + 2f_y = f_g$, b) $3f_x + f_y = f_g$.

Fig. 32 Behaviour of pendulum on difference resonance lines

a) with quadrupole $f_x = f_y = f_g$

directions. As the central amplitudes were nearly equal, the corners of the envelopes of the successive Lissajou pattern are seen to lie on two perpendicular straight lines, which are in fact the quadratic invariant for this case where $n_1 = n_2$, $r_1 = r_2$.

Fig. 28c shows what happens when a constant eight-pole non-linearity as well as the perturbation is applied to the pendulum. The amplitudes are limited and the oscillations increase and decrease in a cyclical manner.

Fig. 29 shows the motion of the particle at a 3rd order sub-resonance with a six-pole perturbation on a sloped resonance line $f_g = f_x + 2 f_y$. In fig. 29a the initial oscillation is directed along the horizontal axis and in fig. 29b it is directed along the vertical axis. The corners of successive envelopes move along hyperbolae, having half axes in the ratio $\sqrt{2} : 1$ (the quadratic invariant for the case where $n_1 = 1$, $n_2 = 2$).

Fig. 30 shows the growth of amplitude with an eight-pole perturbation of frequencies (a) $f_g = 2f_x + 2f_y$ and (b) $f_g = f_x + 3f_y$. The quadratic invariants are for (a) an equilateral hyperbola (b) a hyperbola with axes in the ratio $\sqrt{3} : 1$ and this is fairly well shown by the photographs.

The cases where the quadratic invariant is an ellipse are shown next. We wished to ascertain in the above cases that the effect of the so-called difference resonance lines is not causing particle loss for not too large amplitudes.

Fig. 31 shows frequency diagrams with some possible difference resonance lines drawn.

Fig. 32 shows photographs taken at certain points indicated in fig. 31. In fig. 32 the motion of the pendulum is shown when excited

- (a) with a quadrupole and a frequency $f_g = f_x - f_y$, for $f_x \approx f_y$
- (b) with a six-pole and a frequency $f_g = 2f_x - f_y$, for $f_x \approx f_y$
- (c) with an eight-pole and a frequency $f_g = 3f_x - f_y$, for $f_x \approx f_y$
- (d) with a six-pole and a frequency $f_g = 2f_y - f_x$, for $f_y/f_x = 4/3$.

The envelope of all the oscillations is seen to be in (a) a circle (b) an ellipse with the ratio of the axes equal to $\sqrt{2}$ and in (c) an ellipse with the ratio of the axes equal to $\sqrt{3}$, as was expected. The ratio for (d) is 1.2 instead $\sqrt{2}$.

It is probable that the oscillations do not cover the whole circle or ellipse for the following reasons : inadequacy of exciting voltage, inexact timing or drift, presence of natural or electric non-linear forces, which exert a limiting action on the amplitude.

Fig. 31 The difference resonance lines in the frequency plane showing the points a, b, c and d where photographs of figure 32 were taken.

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