TRANSVERSE VARIATION OF ACCELERATING FORCE

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We use a cartesian system of coordinates in the accelerating gap, with $z$ the direction of particle motion and $x$ for a transverse direction. Field quantities can be written as the product of their space-dependence function and the time-dependence $\exp(j\omega t)$. For a particle crossing the gap with velocity $\beta c$ and passing the plane $z = 0$ at phase $\psi$ we substitute for $\omega t$ the quantity:

$$\left( \frac{\omega z}{\beta c} + \psi \right) \tag{1}$$

The transverse variation of the longitudinal impulse in crossing the gap is then:

$$\frac{d\delta p_z}{dx} = \frac{\delta}{\delta x} \left[ \int \exp(j\left( \frac{\omega z}{\beta c} + \psi \right)) e^F_z \cdot \frac{dx}{\beta c} \right] \tag{2}$$

The integral is taken right through the gap, from and to regions where all A.F. fields vanish. It is assumed that any angle between the velocity and the $z$ axis is sufficiently small that we can take $x$ and $y$ constant in evaluating the integral, and that the amount of acceleration per gap-transit is small enough to allow $\beta c$ to be treated as constant.

Taking the differentiation inside the integral, using $H = \mu \text{Curl} \ E$ equation, and integrating by parts, we obtain:

$$\frac{d\delta p_z}{dx} = - \left( \frac{\mu \omega}{\beta c} \exp(j\left( \frac{\omega z}{\beta c} + \psi \right)) e^F_x \frac{dx}{\beta c} + \int \frac{j\omega}{c} \exp(j\left( \frac{\omega z}{\beta c} + \psi \right)) e^H_y \frac{dz}{\beta c} \right) \tag{3}$$
The transverse impulse in the $x$ direction, integrated through the gap, is:

$$
\delta p_x = \int \exp{j \left( \frac{\omega x}{\beta c} + \varphi \right)} e_x \frac{d\beta}{\beta c} - \int \beta \exp{j \left( \frac{\omega x}{\beta c} + \varphi \right)} e_y \frac{d\beta}{\beta c} \tag{4}
$$

To find the variation of this for particles differing in longitudinal position we differentiate with respect to $\varphi$ and multiply by $-\omega/\beta c$, and obtain:

$$
- \frac{\omega}{\beta c} \frac{\delta p_x}{\delta \varphi} = - \int \frac{d\omega}{\beta c} \exp{j \left( \frac{\omega x}{\beta c} + \varphi \right)} e_x \frac{d\beta}{\beta c} + \int \frac{d\omega}{\beta c} \exp{j \left( \frac{\omega x}{\beta c} + \varphi \right)} e_y \frac{d\beta}{\beta c} \tag{5}
$$

(it is not valid to differentiate with respect to $z$ for this purpose, because we have already made the substitution (1)).

The expressions (5) and (6) are the same. If we change our notation and use $z$ to indicate the longitudinal separation of a particle from some chosen reference particle, we can write:

$$
\frac{\delta p_x}{\delta z} - \frac{\delta p_z}{\delta x} = 0 \tag{6}
$$

The $\delta p$ quantities can equally well be regarded as changes, per gap, of the momenta referred to that of some reference particle.

Thus we have for motion in the $x$, $z$ plane around a reference particle a conservative force-field in which the work done on the particle round any closed path is zero and a potential exists, the total energy of oscillations being conserved.

**APPLICATIONS**

In a linear accelerator there is a phase-dependence of focusing force, and a radial variation of accelerating force. The former can have a substantial effect on the radial oscillations, while it is usual to take into account only the average of the latter, and neglect its effect on the phase oscillations.
The effect on the radial oscillations can be calculated by treating the phase oscillations as given, as for example in Bell (1954), Smith (1956), Hereward and Johnsen (1956).

The equation (6) shows that it is logically inconsistent to neglect the effect of radial motion on the phase oscillations and calculate the effect of given phase oscillations on the radial behaviour; for both effects arise from a conservative coupling and are due to a single term in the total Hamiltonian. On the other hand it provides a convenient check on the practical usefulness of this procedure; if typical amplitudes of phase oscillation contain much more energy than typical amplitudes of radial oscillation then the coupling can be calculated in this “one-way” manner without gross error.

There is possibly an application to cavities where one wants the simplest and cleanest possible dynamics, such as bunchers and debunchers for linear accelerators. Only the curl E equation of Maxwell’s four* is used to derive equation (6), and this is unaffected by the presence of charges or currents in the region concerned. Thus the result can equally be applied to accelerating gaps fitted with grids. A gridless cavity gives a radial increase of accelerating force and the associated phase dependence of focusing force, while the field between grids that form part of reasonably large plane surfaces is, near the axis, just that of a cylindrical $\delta_0$ resonator; with a radial decrease (like $J_0 (r/\pi_0)$) of accelerating force, and a focusing force (due to $H_\varphi$) with phase-dependence opposite to that of the gridless case. With grids slightly recessed from the plane faces of the cavity or its drift-tubes one must have some intermediate case, and the equation (6) shows that if they can be arranged to eliminate one of the unwanted effects the other will vanish too.

In synchrotrons, it has been suggested (Crook and Hamermesh, 1954, Burrow et al, 1956) that the use of accelerating gaps which are so shaped as to give less acceleration to those particles crossing them with higher momentum would produce a useful increase in the damping of the synchrotron oscillations. Our result shows that the effect of such a tapered gap can only superpose a conservative force-field on the synchrotron oscillations, and the extra work done on

* H. Vogt-Hansen has pointed out that the generalisation of (6) to all three
them round a closed path in their phase plane can only be zero. Referring specifically to an ordinary a.c. synchrotron below transition energy; an additional forward force on the inner part of the gap and backward force on the outer part \( \frac{d}{dr} \) negative will cause the corresponding parts of the phase-plane trajectories to spiral inwards compared with the usual case, but a radial force with an equal \( \frac{d}{ds} \) will act on the left and right parts of these trajectories in such a way to produce an equal and opposite antidamping effect. The appropriate changes of sign must be used when applying this argument to machines above transition energy, or with negative momentum compaction factors, but the conclusion is the same. More complete analysis of the effect follows.

We are not concerned for the moment with normal adiabatic damping, so can ignore variation of parameters and write equations of motion for the synchrotron oscillations, fully linearised, in a fairly simple form. We use \( \Delta \omega / \omega \) and \( \Delta \phi \) as variables with \( \Delta R \) the radial displacement relative to synchronous particles.

Then, without the tapered gap field we have:

\[
\frac{d}{dt} \left( \frac{\Delta E}{\omega} \right) = \frac{eV}{2\pi} \cos \frac{\phi}{s} \cdot \Delta \phi
\]  

(7)

\[
\frac{d\Delta \phi}{dt} = -\frac{h}{m\gamma R^2} \frac{\Delta E}{\omega} - h \omega \frac{\Delta R}{R}
\]  

(8)

\[
\frac{\Delta R}{R} = -\frac{k}{m\gamma R^2} \frac{\Delta E}{\omega}
\]  

(9)

Here \( \gamma \) represents \( (1-\beta^2)^{-\frac{1}{2}} \) and \( k \) is the momentum compaction factor; otherwise the notation is that of equation (2) in Burron et al.

The tapered gap introduces a factor, say \( 1 - b \frac{\Delta R}{R} \), into the right hand side of (7); we are entitled to linearise and neglect the products of quantities.
\[ \frac{d}{dt} \left( \Delta \theta \right) = \frac{eV}{2\pi} \cos \varphi_s \cdot \Delta \varphi - \frac{beV}{2\pi} \cos \varphi_s \frac{\Delta R}{R} \] (10)

At this point we have taken into account the tapered gap field as such. If we were to substitute (9) into (8) and use this with (10) we should get equivalent equations and the same result of damping as Crotchie and Hemerzesh or Barron et al. We must now allow for the \( \varphi \)-dependent radial force implied by (6). This is done as follows: Equation (8) is unchanged, but (6) shows that the new term in (10) must be associated with a radial force which, averaged over a turn, has the value

\[ - \frac{beV}{2\pi \mu H} \cos \varphi_s \cdot \Delta \varphi \]

For a balance of centrifugal and magnetic forces, a radial force \( F \) say, requires a change of \( \frac{\Delta E}{\omega} \) of amount

\[ - F \frac{R}{\omega} \]

at constant radius. The force (11) therefore changes the radius-energy relation (9) to

\[ \frac{\Delta R}{R} = \frac{k}{\mu \omega R} \left( \frac{\Delta E}{\omega} - \frac{beV}{2\pi \mu H} \cos \varphi_s \cdot \Delta \varphi \right) \] (12)

Substituting in (8) we obtain:

\[ \frac{d\Delta \theta}{dt} = k_1 \frac{\Delta E}{\omega} + \frac{beV}{2\pi \alpha \mu H^2} \cos \varphi_s \cdot \Delta \varphi \] (13)
And substituting (12) into (10):

\[
\frac{d}{dt} \frac{\Delta \Phi}{\omega} = K_1 \Delta \phi - \frac{kbeV}{2m\gamma_0 r^2} \cos \psi_s \cdot \frac{\Delta \Phi}{\omega}
\]  

(14)

\(K_1\) is an expression equivalent to the \(\frac{h\omega_s^2}{E_s}\) of Browne et al., and \(K_2\) is the original \(\frac{eV}{2\pi} \cos \psi_s\), with the addition of a term in \(b^2\) which will change the synchrotron oscillation frequency. The last term in (14) is due to the tapered gap and would, alone, produce damping; but taking it together with the last term in (13) the whole effect of \(b\) on the oscillations is a distortion and frequency shift without any now damping or antidamping.

Betatron oscillations have been neglected in the above, and this justified if their frequency differs by a good factor from that of the synchrotron oscillations.

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