CERN-TH-5882/90

TWO-DIMENSIONAL TOPOLOGICAL GRAVITY, TOPOLOGICAL MATTER AND INTERSECTION THEORY ON MODULI SPACE

T.P. Killingback

Theoretical Physics Division
CERN
CH-1211 Geneva 23

ABSTRACT

We propose a two-dimensional topological matter system associated to an arbitrary compact Lie group G . This topological quantum field theory is defined in terms of intersection theory on the space of based holomorphic maps from a Riemann surface to the based loop group ΩG of G , and corresponds physically to the topological sigma model with ΩG as the target space. This topological matter system may be naturally coupled to two-dimensional topological gravity and is also related to topological Yang-Milles theory in four dimensions. There appears also to be a connection between this theory and chiral models involving infinite-dimensional twistor space.

CERN-TH-5882/90
October 1990

String theory, in the formulation due to Polyakov [1], may be viewed as two-dimensional conformal field theory coupled to two-dimensional quantum gravity. It seems reasonable to expect, therefore, that a deeper understanding of conformal field theory coupled to quantum gravity will yield important insights into the structure of string theory. Conformal field theory coupled to two-dimensional gravity has been studied in the light-cone in [2] and in the Liouville theory approach in [3]. Significant progress on two-dimensional quantum gravity has resulted from studying matrix models [4-12]. An alternative approach to two-dimensional gravity is provided by topological quantum field theory. The theory of two-dimensional topological gravity defined in [13] reproduces the amplitudes of the one-matrix model for genus $g \leq 1$, and it has been conjectured that this is true for all g . This result has been generalized in [14] to the n -matrix models. Topological gravity is also related to Liouville theory by the argument of [15]. The relation between two-dimensional topological gravity and matrix models has also been discussed in [16] and [17], and the geometrical interpretation of the string equation has been considered in [18].

One conclusion that may be drawn from the work of [13, 14] is that, for central charge $c < 1$, two-dimensional quantum gravity coupled to conformal field theory appears to be equivalent to two-dimensional topological gravity coupled to topological matter. In this context a topological matter system refers to a topological quantum field theory which does not contain elementary gravitons. It is not clear that either the Liouville theory approach or the matrix model approach can be extended to yield significant results for conformal field theories with $c > 1$ coupled to two-dimensional gravity. In the physical region $c > 1$ a topological quantum field theory approach may prove to be more successful. We should consider, therefore, two-dimensional topological gravity coupled to interesting topological matter systems. One of the most natural topological matter systems is the topological sigma model [19], in which the dynamical field is a smooth map φ from a compact Riemann surface Σ to a compact Kähler manifold X . Topological sigma models coupled to two-dimensional topological gravity have been discussed in [13].

The fundamental example of a conformal field theory with $c > 1$ is the Wess-Zumino-Witten model for a compact Lie group [20]. In the topological approach, topological matter systems play the role of conformal field theories and so we might expect that particularly interesting topological matter systems (which may play the role of the Wess-Zumino-Witten

model) will be associated to a compact Lie group G . In this paper we will propose such a topological matter system in terms of a topological sigma model of maps from a compact Riemann surface Σ into the based loop group ΩG of G . This theory may be naturally coupled to two-dimensional topological gravity and is also related to topological Yang-Mills theory in four dimensions. There is also a connection with two-dimensional chiral models via an infinite dimensional twistor structure.

Let us first recall the definition of the topological sigma model defined on a compact Riemann surface Σ with a compact target manifold X [19]. In general, a topological sigma model of maps $\varphi : \Sigma \rightarrow X$ may be defined for X an arbitrary symplectic manifold. A symplectic structure on X determines an almost complex structure uniquely up to homotopy. The case of most practical importance, and the one that we will consider here, is that in which the almost complex structure on X is integrable; in this case, X is a Kähler manifold. The correlation functions of the topological sigma model, as defined by standard quantum field theoretic methods in [19], reduce to local calculations on the moduli space of sigma model instantons, i.e., the moduli space of holomorphic maps from Σ to X of topological type ν . The space of all sigma model instantons is $\mathcal{S} = \bigcup_{\nu} \mathcal{S}_{\nu}$. The correlation functions of the topological sigma model may now be defined in a model-independent way, which is closely related to the invariants of symplectic manifolds defined in [21]. The connection between the Lagrangian quantum field theoretic definition of the correlation functions and the model-independent definition to be used here is discussed in [19] (see also [22]).

The model-independent definition of the correlation functions of the topological sigma model is given in terms of integration of certain differential forms over the space $\mathcal{S}_{\nu}$ of sigma model instantons of topological type ν . For this integration to make sense it is certainly necessary that $\mathcal{S}_{\nu}$ should be finite dimensional. If X is a compact Kähler manifold then in general each component of the space of holomorphic maps from Σ to X will be finite dimensional. For example, if $X = \mathbb{CP}^n$ (n -dimensional complex projective space) then the topological type of a map from a Riemann surface Σ of genus g to X is specified by a single integer k and the complex dimension of the space $\mathcal{S}_k$ of holomorphic maps $\varphi : \Sigma \rightarrow \mathbb{CP}^n$ of degree k is given by

[23]

$$\dim_{\mathbb{C}} \mathcal{S}_k = (n+1)k - n(g-1), \quad (1)$$

for $k \geq 2g$. Now let us define the correlation functions for the topological sigma model of maps from Σ to X . First let Ψ denote the universal map

$$\Psi : \Sigma \times \text{Map}(\Sigma; X) \rightarrow X \quad (2)$$

defined by $\Psi(x, \varphi) = \varphi(x)$, for $x \in \Sigma$ and $\varphi \in \text{Map}(\Sigma; X)$. Now let Φ denote the universal instanton map obtained by restricting Ψ to $\mathcal{S}_\nu \subset \text{Map}(\Sigma; X)$, i.e.,

$$\Phi : \Sigma \times \mathcal{S}_\nu \rightarrow X. \quad (3)$$

For a point $x \in \Sigma$ we let $\Phi_x : \mathcal{S}_\nu \rightarrow X$ be the restriction of Φ to $x \times \mathcal{S}_\nu$. Consider now a cohomology class α in the real cohomology ring $H^*(X; \mathbb{R})$ of X . We obtain a corresponding cohomology class $W_\alpha = \Phi_x^*(\alpha)$ in $H^*(\mathcal{S}_\nu; \mathbb{R})$, which is independent of x . Given n cohomology classes $\alpha_1, \alpha_2, \dots, \alpha_n$ in $H^*(X; \mathbb{R})$ with corresponding cohomology classes $W_{\alpha_1}, W_{\alpha_2}, \dots, W_{\alpha_n}$ in $H^*(\mathcal{S}_\nu; \mathbb{R})$ we define the expectation value of these classes to be

$$(\mathcal{O}_{\alpha_1}(x_1) \mathcal{O}_{\alpha_2}(x_2) \dots \mathcal{O}_{\alpha_n}(x_n)) = \int_{\mathcal{S}_\nu} W_{\alpha_1} \wedge W_{\alpha_2} \wedge \dots \wedge W_{\alpha_n}. \quad (4)$$

In the quantum field theoretic approach the $\mathcal{O}_{\alpha_i}$ s are local operators. The expectation value (4) vanishes unless

$$\sum_{i=1}^n \dim \alpha_i = \dim \mathcal{S}_\nu. \quad (5)$$

The BRST invariance of the topological sigma model implies that the expectation values defined by (4) are independent of the complex structure on Σ [19].

Now let us consider extending this theory to the case in which the target space X is the based loop group ΩG of a compact Lie group G . The based loop group ΩG of G is the space

of all smooth based maps $f : S^1 \rightarrow G$ (where we regard $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ as the unit circle in the complex plane), such that $f(1) = 1$, with pointwise multiplication. An important fact about ΩG is that it is an infinite dimensional Kähler manifold (see [24], for example). This may be seen as follows. Let LG denote the free loop group of G , i.e., the group of all smooth maps $f : S^1 \rightarrow G$. The group ΩG may be identified with the quotient of LG by the group $G \subset LG$ given by the constant loops, i.e.,

$$\Omega G = LG/G. \quad (6)$$

Let $\mathfrak{g}$ be the Lie algebra of G and let Tr denote a positive definite invariant inner product on $\mathfrak{g}$. The Lie algebra $L\mathfrak{g}$ of LG is given by the space of smooth maps from S^1 to $\mathfrak{g}$. We can define a skew form $\omega : L\mathfrak{g} \times L\mathfrak{g} \rightarrow \mathbb{R}$ as follows:

$$\omega(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}(\xi(\theta)\eta'(\theta)) d\theta, \quad (7)$$

for $\xi, \eta \in L\mathfrak{g}$. This defines a closed two-form ω on LG/G , i.e., a symplectic structure on ΩG .

The space ΩG is also a complex manifold. The complexification $LG_{\mathbb{C}}$ of LG consists of the space of maps $f : S^1 \rightarrow G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is the complexification of G . Consider the subgroup $L^+G_{\mathbb{C}}$ of $LG_{\mathbb{C}}$ defined by those maps $f : S^1 \rightarrow G_{\mathbb{C}}$ which extend to holomorphic maps $\hat{f} : D_0 \rightarrow G_{\mathbb{C}}$, where $D_0 = \{z \in \mathbb{C} : |z| \leq 1\}$ is the unit disc. The loop group ΩG may be regarded as the quotient of $LG_{\mathbb{C}}$ by $L^+G_{\mathbb{C}}$:

$$\Omega G = LG_{\mathbb{C}}/L^+G_{\mathbb{C}} \quad (8)$$

and this endows ΩG with a natural complex structure. This complex structure and the symplectic structure defined by (7) are compatible and combine to yield a Kähler structure on ΩG .

In analogy with the case of the topological sigma model with a compact Kähler target space we expect that the correlation functions of the topological sigma model with ΩG as the

target space should be defined in terms of integration of certain differential forms over the space of instantons in the ΩG model. As ΩG is an infinite dimensional Kähler manifold the instantons in the ΩG model are the holomorphic maps from the Riemann surface Σ to ΩG . We will now impose a slight topological restriction on our theory and consider only based maps $\varphi : \Sigma \rightarrow \Omega G$, i.e., we introduce base points $x_0 \in \Sigma$ and $e \in \Omega G$ (given by the constant loop $e(\theta) = 1$) and require $\varphi(x_0) = e$. With this restriction the instantons in the ΩG model are the based holomorphic maps from Σ to ΩG . It is a remarkable fact that the moduli space of based holomorphic maps from Σ to ΩG , of a given topological type, is finite dimensional. It is possible, therefore, to make sense of intersection theory on the space of ΩG instantons.

Let us now consider in more detail the moduli space of based holomorphic maps from Σ to ΩG , which we will denote by $\mathcal{H}(\Sigma; \Omega G)$. First we note that the set of connected components of the space $\text{Map}(\Sigma; \Omega G)$ of all smooth maps from a Riemann surface Σ of genus g to ΩG is given by [23]:

$$\pi_0 \text{Map}(\Sigma; \Omega G) = [\pi_1(\Omega G)]^{2g} \oplus \pi_2(\Omega G). \quad (9)$$

It is a basic result in algebraic topology that the homotopy groups of ΩG satisfy

$$\pi_i(\Omega G) = \pi_{i+1}(G). \quad (10)$$

Using (9) and (10) together with the well-known fact that $\pi_2(G) = 0$, for any compact Lie group, yields

$$\pi_0 \text{Map}(\Sigma; \Omega G) = \pi_3(G). \quad (11)$$

If we assume that G is simple then $\pi_3(G) = \mathbb{Z}$, and we have that

$$\pi_0 \text{Map}(\Sigma; \Omega G) = \mathbb{Z}. \quad (12)$$

Thus the components of the space $\text{Map}(\Sigma; \Omega G)$ are determined by a single integer k . If $\varphi \in \text{Map}(\Sigma; \Omega G)$ then $w(\varphi) = \int_{\Sigma} \varphi^* \omega$, where ω is the Kähler form on ΩG defined by (7), depends only on the homotopy class of φ , and $w(\varphi)$ (suitably normalized) is the integer k corresponding

to φ under the isomorphism (12). The moduli space of based holomorphic maps from Σ to ΩG of degree k will be denoted by $\mathcal{H}_k(\Sigma; \Omega G)$, or sometimes simply by $\mathcal{H}_k$; we have that $\mathcal{H}(\Sigma; \Omega G) = \bigcup_{k \in \mathbb{Z}} \mathcal{H}_k(\Sigma; \Omega G)$.

A key result about $\mathcal{H}_k(\Sigma; \Omega G)$ is the following [25] (and also [24]). The moduli space of based holomorphic maps from Σ to ΩG may be identified with the space of isomorphism classes of holomorphic G_C -bundles on $\Sigma \times S^2$ which are trivial over $x_0 \times S^2$ and $\Sigma \times D_{\infty}$, where x_0 is a base point in Σ and $D_{\infty} = \{z \in \mathbb{C} : |z| \geq 1\}$. We may see this as follows. Let $\{U_i\}$ be some open covering of Σ . Relative to $\{U_i\}$ a holomorphic map $\varphi : \Sigma \rightarrow \Omega G$ is given by holomorphic maps $\varphi_i : U_i \rightarrow LG_C$, which agree modulo L^+G_C , i.e., $\varphi_i \varphi_j^{-1} : U_i \cap U_j \rightarrow L^+G_C$. Consider a holomorphic G_C -bundle over $\Sigma \times S^2$. We can assume that the covering $\{U_i\}$ is such that the bundle is trivial over each $U_i \times D_0$. Over $\Sigma \times D_{\infty}$ we have a canonical trivialization of the bundle. On each $U_i \times D_0$ we can pick any trivialization consistent with that over $x_0 \times S^2$, assuming that $x_0 \in U_i$. Such trivializations correspond to based holomorphic maps $U_i \times D_0 \rightarrow G_C$ and the holomorphic bundle is given by based holomorphic maps $\varphi_i : U_i \rightarrow LG_C$, which agree modulo L^+G_C . This defines a based holomorphic map $\varphi : \Sigma \rightarrow LG_C/L^+G_C = \Omega G$.

A holomorphic G_C -bundle P on $\Sigma \times S^2$ is classified topologically by a single integer k ; if $G = SU(n)$ then this integer is the second Chern class of P . The above result implies, therefore, that we can identify $\mathcal{H}_k(\Sigma; \Omega G)$ with the moduli space of holomorphic G_C -bundles on $\Sigma \times S^2$, trivial over $x_0 \times S^2$ and $\Sigma \times D_{\infty}$, and of characteristic class k . An immediate important consequence of this is that $\mathcal{H}_k(\Sigma; \Omega G)$ is finite dimensional. This follows because general results of algebraic geometry imply that the space of all holomorphic bundles of a given topological type on a compact complex manifold is a finite-dimensional algebraic variety (see [26]). We will give a formula for the dimension of $\mathcal{H}_k(\Sigma; \Omega G)$ when Σ is a surface of genus zero later in this paper.

We can now define the correlation functions of the ΩG model in terms of intersection theory on the space of instantons $\mathcal{H}_k(\Sigma; \Omega G)$. The definition is in complete analogy with the case of a compact target space. We introduce the universal map

$$\psi : \Sigma \times \text{Map}(\Sigma; \Omega G) \rightarrow \Omega G \quad (13)$$

defined by evaluation: $\Psi(x, \varphi) = \varphi(x)$, for $x \in \Sigma$ and $\varphi \in \text{Map}(\Sigma; \Omega G)$. The universal instanton Φ is obtained by restricting Ψ to $\mathcal{H}_k(\Sigma; \Omega G) \subset \text{Map}(\Sigma; \Omega G)$ to give

$$\Phi : \Sigma \times \mathcal{H}_k(\Sigma; \Omega G) \rightarrow \Omega G. \quad (14)$$

If x is a point in Σ , Φ_x is the restriction of Φ to $\Sigma \times \mathcal{H}_k$. Given a cohomology class $\alpha \in H^*(\Omega G; \mathbf{R})$ we obtain a cohomology class $W_\alpha = \Phi_x^*(\alpha) \in H^*(\mathcal{H}_k; \mathbf{R})$, which is clearly independent of the point x .

The real cohomology ring of the loop group ΩG of a compact simple simply connected Lie group was determined in [27]. The result may be stated as follows. Let ε be the evaluation map

$$\varepsilon : S^1 \times \Omega G \rightarrow G, \quad (15)$$

defined by $\varepsilon(\theta, f) = f(\theta)$, for $\theta \in S^1$ and $f \in \Omega G$. The transgression map

$$\tau : H^n(G; \mathbf{R}) \rightarrow H^{n-1}(\Omega G; \mathbf{R}) \quad (16)$$

is defined as follows: let β be a closed n -form on G , define $\tau(\beta)$ to be the closed $(n-1)$ -form on ΩG obtained by pulling β back to $S^1 \times \Omega G$ by ε and then integrating over the circle S^1 , i.e.,

$$\tau(\beta) = \int_{S^1} \varepsilon^*(\beta). \quad (17)$$

The real cohomology ring of ΩG is then a polynomial algebra on the even dimensional cohomology classes obtained by transgressing the generators of $H^*(G; \mathbf{R})$ using the map τ .

Thus, for example, if $G = SU(2)$ then the real cohomology ring of $\Omega(SU(2))$ is given by

$$H^i(\Omega(SU(2)); \mathbf{R}) = \begin{cases} \mathbf{R}, & i \text{ even} \\ 0, & \text{otherwise.} \end{cases} \quad (18)$$

Cohomology classes $\alpha_1, \alpha_2, \dots, \alpha_n \in H^*(\Omega G; \mathbf{R})$ determine cohomology classes $W_{\alpha_1}, W_{\alpha_2}, \dots, W_{\alpha_n} \in$

$H^*(\mathcal{H}_k; \mathbf{R})$ and we define the expectation value of these classes to be

$$\langle \mathcal{O}_{\alpha_1}(x_1) \mathcal{O}_{\alpha_2}(x_2) \dots \mathcal{O}_{\alpha_n}(x_n) \rangle = \int_{\mathcal{H}_k} W_{\alpha_1} \wedge W_{\alpha_2} \wedge \dots \wedge W_{\alpha_n}. \quad (19)$$

Here the x_i 's are arbitrary points on Σ and $W_{\alpha_i} = \Phi_x^*(\alpha_i)$.

The notation used in (19) indicates that with respect to a suitable extension of the Lagrangian formulation of the topological sigma model to the case in which the target space is ΩG the cohomology classes W_{α_i} should correspond to local operators $\mathcal{O}_{\alpha_i}$. It seems likely that such a Lagrangian formulation results from appropriately fixing the gauge in the purely topological action:

$$S_{\text{top}}[\varphi] = \int_{\Sigma} \varphi^* \omega \quad (20)$$

where $\varphi : \Sigma \rightarrow \Omega G$ and ω is the Kähler form on ΩG defined by (7). This is the analogue for the ΩG model of the gauge fixing procedure discussed in [22] for the topological sigma model with compact target space. The BRST invariance of such a Lagrangian formulation would imply that the expectation values defined by (19) are independent of the complex structure on Σ .

We can generalize (19) by introducing an r -dimensional homology cycle γ on Σ . Let Φ_γ be the restriction of the universal instanton Φ , defined by (14), to $\gamma \times \mathcal{H}_k$. Given a p -dimensional cohomology class $\alpha \in H^p(\Omega G; \mathbf{R})$ we obtain a $(p-r)$ -dimensional cohomology class on $\mathcal{H}_k$ defined by

$$W_{\alpha, \gamma} = \int_{\gamma} \Phi_\gamma^*(\alpha). \quad (21)$$

Given n cohomology classes $\alpha_1 \in H^{p_1}(\Omega G; \mathbf{R})$, $\alpha_2 \in H^{p_2}(\Omega G; \mathbf{R})$, $\dots$, $\alpha_n \in H^{p_n}(\Omega G; \mathbf{R})$ and n homology cycles on Σ , $\gamma_1, \gamma_2, \dots, \gamma_n$ of dimensions $r_1, r_2, \dots, r_n$ respectively, we obtain n cohomology classes on $\mathcal{H}_k$; $W_{\alpha_1, \gamma_1} \in H^{p_1-r_1}(\mathcal{H}_k; \mathbf{R})$, $W_{\alpha_2, \gamma_2} \in H^{p_2-r_2}(\mathcal{H}_k; \mathbf{R})$, $\dots$, $W_{\alpha_n, \gamma_n} \in H^{p_n-r_n}(\mathcal{H}_k; \mathbf{R})$. The expectation value of these classes may then be defined as:

$$\langle \mathcal{O}_{\alpha_1, \gamma_1} \mathcal{O}_{\alpha_2, \gamma_2} \dots \mathcal{O}_{\alpha_n, \gamma_n} \rangle = \int_{\mathcal{H}_k} W_{\alpha_1, \gamma_1} \wedge W_{\alpha_2, \gamma_2} \wedge \dots \wedge W_{\alpha_n, \gamma_n}. \quad (22)$$

The expectation value (22) vanishes unless

$$\sum_{i=1}^n p_i - r_i = \dim \mathcal{H}_k. \quad (23)$$

We will not discuss here the subtleties which arise in the integration in (22) over $\mathcal{H}_k$ except to note that the spaces $\mathcal{H}_k$ are non-compact so there will be infrared problems with the integration. One way to overcome this is to consider a suitable compactification of the $\mathcal{H}_k$ s. A very similar problem occurs in four-dimensional Donaldson theory which requires detailed analysis of the moduli space of instantons near its boundary.

Now let us consider this theory coupled to two-dimensional topological gravity. First we will recall the definition of two-dimensional topological gravity [13]. Let $\mathcal{M}_{g,n}$ be the moduli space of Riemann surfaces of genus g with n marked points (or punctures) and let $\overline{\mathcal{M}}_{g,n}$ be the Mumford compactification of $\mathcal{M}_{g,n}$ consisting of stable curves. Consider now a stable curve $\Sigma \in \overline{\mathcal{M}}_{g,n}$ with n marked points $x_1, x_2, \dots, x_n$. For each x_i the complex cotangent space $T_{x_i}^* \Sigma$ is a complex line, and as Σ varies over $\overline{\mathcal{M}}_{g,n}$ these complex lines vary holomorphically to give n holomorphic line bundles $\mathcal{L}_{(i)}$ over $\overline{\mathcal{M}}_{g,n}$. If $d_1, d_2, \dots, d_n$ are n non-negative integers then the correlation function of two-dimensional topological gravity is defined by

$$\langle \sigma_{d_1} \sigma_{d_2} \dots \sigma_{d_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} c_1(\mathcal{L}_{(1)})^{d_1} \wedge c_1(\mathcal{L}_{(2)})^{d_2} \wedge \dots \wedge c_1(\mathcal{L}_{(n)})^{d_n}. \quad (24)$$

Here $c_1(\mathcal{L}_{(i)})^{d_i}$ denotes the d_i -fold wedge product of the first Chern class of $\mathcal{L}_{(i)}$. The moduli space $\overline{\mathcal{M}}_{g,n}$ has real dimension $2(3g-3+n)$ so the expectation value defined by (24) will vanish unless

$$\sum_{i=1}^n d_i = 3g - 3 + n. \quad (25)$$

The ΩG model may be coupled to two-dimensional topological gravity in analogy with the case of the topological sigma model with compact target space, which has been discussed in [13] and also in [28]. We consider the moduli space $\mathcal{N}_{g,n}$ of pairs (Σ, φ) , where Σ is a stable curve of

genus g with n marked points $x_1, x_2, \dots, x_n$ and $\varphi : \Sigma \rightarrow \Omega G$ is a based holomorphic map. Now for each marked point $x_i \in \Sigma$, the complex cotangent space $T_{x_i}^* \Sigma$ at x_i is a complex line, and as Σ varies over $\mathcal{N}_{g,n}$ these lines fit together to give n holomorphic line bundles $\widehat{\mathcal{L}}_{(i)} \rightarrow \mathcal{N}_{g,n}$. We can also consider the universal fibration $\mathcal{CN}_{g,n}$ over $\mathcal{N}_{g,n}$ defined by requiring that the fibre of $\mathcal{CN}_{g,n}$ over a point $\Pi \in \mathcal{N}_{g,n}$ is a pair (Σ, φ) in the isomorphism class Π . The marked point x_i can then be viewed as a section $x_i : \mathcal{N}_{g,n} \rightarrow \mathcal{CN}_{g,n}$ of the universal fibration. If we also introduce the universal map $\widehat{\Phi} : \mathcal{CN}_{g,n} \rightarrow \Omega G$, where $\widehat{\Phi}(\Sigma, \varphi)$ is the map $\varphi : \Sigma \rightarrow \Omega G$, then we obtain n maps $\widehat{\Phi} \circ x_i : \mathcal{N}_{g,n} \rightarrow \Omega G$. To each cohomology class $\alpha \in H^*(\Omega G; \mathbb{R})$ there corresponds the cohomology class $W_{\alpha,i} = (\widehat{\Phi} \circ x_i)^*(\alpha)$ in $H^*(\mathcal{N}_{g,n}; \mathbb{R})$. Thus for each marked point $x_i \in \Sigma$ and non-negative integer d we can define a cohomology class $c_1(\widehat{\mathcal{L}}_{(i)})^d \wedge W_{\alpha,i}$ in $H^*(\mathcal{N}_{g,n}; \mathbb{R})$. The expectation values of the ΩG model coupled to two-dimensional topological gravity may now be defined by

$$\langle \sigma_{d_1} \sigma_{d_2} \dots \sigma_{d_n} \rangle = \int_{\mathcal{N}_{g,n}} c_1(\widehat{\mathcal{L}}_{(1)})^{d_1} \wedge W_{\alpha,1} \wedge c_1(\widehat{\mathcal{L}}_{(2)})^{d_2} \wedge W_{\alpha,2} \wedge \dots \wedge c_1(\widehat{\mathcal{L}}_{(n)})^{d_n} \wedge W_{\alpha,n}. \quad (26)$$

In this expression, $d_1, d_2, \dots, d_n$ are n non-negative integers and the $\sigma_{d_i, \alpha, i}$ s represent local quantum field theory operators.

We now wish to discuss a connection between the ΩG model considered above and four-dimensional topological Yang-Mills theory [29]. Here we will describe topological Yang-Mills theory in a model independent formulation which is closely related to Donaldson theory [30]. The correspondence between this formulation and the Lagrangian formulation of [29] is discussed in [31].

Let M be a compact smooth Riemannian four-manifold and let $P \rightarrow M$ be a principal G -bundle over M . In addition let $\mathcal{A}$ be the space of connections on P and $\mathcal{G}$ be the group of gauge transformations of P . It is conventional to work with the space of irreducible connections on P , and the group $\mathcal{G}$ of all gauge transformations, here, however, we will consider a slight variant and work with the space $\mathcal{A}$ of all connections and the group of based gauge transformations $\mathcal{G}_*$, which consists of those gauge transformations $g \in \mathcal{G}$ which leave a point in P fixed.

It is now possible to define a universal bundle as follows [32]. The group $\mathcal{G}_*$ acts on $P \times \mathcal{A}$

by $(p, A) \mapsto (g(p), A^g)$, for $g \in \mathcal{G}_*$. This action is free and defines a principal $\mathcal{G}_*$ -bundle

$$\mathcal{G}_* \rightarrow P \times \mathcal{A} \rightarrow \mathcal{Q}, \quad (27)$$

where $\mathcal{Q} = (P \times \mathcal{A})/\mathcal{G}_*$. The compact group G also acts freely on $\mathcal{Q}$ to give a principal G -bundle

$$G \rightarrow \mathcal{Q} \rightarrow \mathcal{Q}/G, \quad (28)$$

where $\mathcal{Q}/G = M \times \mathcal{A}/\mathcal{G}_*$. The G -bundle $\mathcal{Q} \rightarrow M \times \mathcal{A}/\mathcal{G}_*$ has a natural connection w which depends on the Riemannian metric on M (see [32]). The connection w is universal in the following sense [32]. If $[A] \in \mathcal{A}/\mathcal{G}_*$ denotes a gauge equivalence class of connections on P , $\mathcal{Q}|_{[A]}$ denotes the restriction of $\mathcal{Q}$ to $M \times [A]$ and the restriction of w to $\mathcal{Q}|_{[A]}$ is denoted by $w|_{[A]}$, then $w|_{[A]}$ is in the equivalence class $[A]$.

The topological type of the gauge bundle $P \rightarrow M$ is determined by the four-dimensional characteristic number

$$k = \frac{1}{8\pi^2} \int_M \text{Tr}(F \wedge F), \quad (29)$$

where F is the curvature of a connection A on P . If $G = SU(n)$ then k is the second Chern number of P . Let $\mathcal{B}_k$ be the space of anti-self-dual connections $\{A \in \mathcal{A} : F = - * F\}$ on the G -bundle labelled by the integer k . The space $\mathcal{R}_k = \mathcal{B}_k/G$ is the moduli space of instantons and $\mathcal{R}_k^* = \mathcal{B}_k/G_*$ is the moduli space of based instantons.

The universal instanton bundle

$$P \rightarrow M \times \mathcal{R}_k^* \quad (30)$$

is obtained by restricting the universal bundle $\mathcal{Q} \rightarrow M \times \mathcal{A}/\mathcal{G}_*$ to $M \times \mathcal{R}_k^*$. Given a point $p \in M$ we let $\mathcal{P}_p$ be the restriction of $\mathcal{P}$ to $p \times \mathcal{P}_k^*$. The Chern classes $c_i(\mathcal{P}_p) \in H^{2i}(\mathcal{R}_k^*; \mathbb{R})$ are then determined by $\mathcal{P}$. More generally, we can consider a q -dimensional homology cycle Γ in M . We let $\mathcal{P}_\Gamma$ be the restriction of $\mathcal{P}$ to $\Gamma \times \mathcal{R}_k^*$. Integrating $c_i(\mathcal{P}_\Gamma)$ over Γ gives a $(2i - q)$ -dimensional

cohomology class on $\mathcal{R}_k^*$

$$\lambda_{i,\Gamma} = \int_\Gamma c_i(\mathcal{P}_\Gamma) \in H^{2i-q}(\mathcal{R}_k^*; \mathbb{R}). \quad (31)$$

Given n homology cycles $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ of dimensions $q_1, q_2, \dots, q_n$ respectively we may define the expectation values of topological Yang-Mills theory by

$$\langle \mathcal{O}_{i_1, \Gamma_1} \mathcal{O}_{i_2, \Gamma_2} \dots \mathcal{O}_{i_n, \Gamma_n} \rangle = \int_{\mathcal{R}_k^*} \lambda_{i_1, \Gamma_1} \wedge \lambda_{i_2, \Gamma_2} \wedge \dots \wedge \lambda_{i_n, \Gamma_n}. \quad (32)$$

This expectation value vanishes unless

$$\sum_{\ell=1}^n (2i_\ell - q_\ell) = \dim \mathcal{R}_k^*. \quad (33)$$

The expectation values of topological Yang-Mills theory are given therefore by intersection theory of natural homology classes on the moduli space of based instantons. That the expectation values defined by (32) are independent of the metric on M follows physically from the BRST invariance of the Lagrangian formulation given in [29].

The connection with the ΩG model is seen by considering a Riemann surface Σ of genus zero. The correlation functions of the ΩG model are now defined by (22) in terms of intersection theory on the space $\mathcal{H}_k$ of based holomorphic maps from S^2 to ΩG . There is a remarkable connection, due to Atiyah, between $\mathcal{H}_k(S^2; \Omega G)$ and the moduli space of Yang-Mills instantons on the four-sphere S^4 . Let P be a fixed principal G -bundle over S^4 . Such a bundle is classified topologically by an element of $\pi_3(G) = \mathbb{Z}$, where the integer k corresponding to P is given by (29). The moduli space of based Yang-Mills instantons on $P \rightarrow S^4$, with respect to the standard metric on S^4 , is $\mathcal{R}_k^*(S^4; G)$. The theorem of Atiyah [25] states that there is a natural diffeomorphism between $\mathcal{H}_k(S^2; \Omega G)$ and $\mathcal{R}_k^*(S^4; G)$:

$$\mathcal{H}_k(S^2; \Omega G) \cong \mathcal{R}_k^*(S^4; G). \quad (34)$$

Thus, for arbitrary G , both the ΩG model on a surface of genus zero and topological Yang-Mills theory on S^4 are defined in terms of intersection theory on the same moduli space of instantons.

It is not hard to see that the expectation values defined by the ΩG model and by four-dimensional topological Yang-Mills theory are directly related. We will discuss elsewhere in greater detail the connection between the ΩG model on compact surfaces and four-dimensional topological Yang-Mills theory [33].

One other consequence of (34) is that it enables us to derive a formula for the dimension of $\mathcal{H}_k(S^2; \Omega G)$. The dimension of the moduli space $\mathcal{R}_k(S^4; G)$ of unbased instantons on S^4 has been calculated in [34]. The spaces $\mathcal{R}_k(S^4; G)$ and $\mathcal{R}_k^*(S^4; G)$ are related by $\mathcal{R}_k(S^4; G) = \mathcal{R}_k^*(S^4; G)/G$. Thus,

$$\dim \mathcal{R}_k^*(S^4; G) = \dim \mathcal{R}_k(S^4; G) + \dim G. \quad (35)$$

Combining (34), (35) and the results of [34] we obtain a formula for the dimension of $\mathcal{H}_k(S^2; \Omega G)$:

$$\dim \mathcal{H}_k(S^2; \Omega G) = 4n_G k, \quad (36)$$

where n_G is the dual Coxeter number of G . For example, if $G = SU(n)$ then $n_G = n$ and $\dim \mathcal{H}_k(S^2; \Omega G) = 4nk$. It also follows from (34) together with the results of [34] that $\mathcal{H}_k(S^2; \Omega G)$ is a smooth manifold for general G .

To conclude this paper we shall consider a connection between ideas related to the ΩG model, the two-dimensional chiral model and twistor theory. It may be helpful to first recall briefly how twistor theory enters the subject of non-linear sigma models (see [35-38] for reviews).

Let Σ be a compact Riemann surface with metric $g_{\mu\nu}$ and let N be a compact Riemannian manifold with metric h_{ab} . We are interested in smooth maps $\varphi : \Sigma \rightarrow N$. The sigma model action is

$$S[\varphi] = \frac{1}{2} \int_{\Sigma} |d\varphi(x)|^2 d\mu. \quad (37)$$

Maps φ which extremize this action are called harmonic. The action (37) is invariant under a conformal deformation of the metric $g_{\mu\nu}$ on Σ , so the sigma model defined by (37) depends only on Σ as a Riemann surface.

In the context of sigma models, twistor space arises in the following way. A twistor fibration $\pi : Z \rightarrow N$ is a fibration of an almost complex manifold Z over N with the property that if $\varphi : \Sigma \rightarrow Z$ is a holomorphic map then $\pi \circ \varphi : \Sigma \rightarrow N$ is harmonic. The space Z is called the twistor space of N . For example, if N is a $2n$ dimensional Riemannian manifold we may define a twistor fibration $\pi : J(N) \rightarrow N$, where $J(N)$ is the bundle of almost complex structures on N , i.e., for $x \in N$,

$$J_x(N) = \{J \in \text{End}(T_x N) : J^2 = -1\}. \quad (38)$$

The space $J(N)$ has a natural almost complex structure with respect to which $J(N) \rightarrow N$ is a twistor fibration [38]. When $n = 2$, $J(N)$ is the standard Penrose twistor space of N .

In favourable circumstances the twistor space Z is such that every harmonic map $\psi : \Sigma \rightarrow N$ lifts to a holomorphic map $\varphi : \Sigma \rightarrow Z$. In such cases the study of harmonic maps reduces to a study of holomorphic maps into twistor space (see [35-38]).

Now let us take the surface Σ to be the two-sphere S^2 and the target space N to be a compact Lie group G with h_{ab} the metric on G defined by the Killing form on the Lie algebra $\mathfrak{g}$ of G . The action (37) then defines the chiral model associated to G . Let Θ be the Maurer-Cartan form of G . Consider a harmonic map $\psi : S^2 \rightarrow G$ and let $\beta = \psi^* \Theta$. The harmonic map equation for ψ implies

$$d^* \beta = 0. \quad (39)$$

and the pull-back of the Maurer-Cartan equations imply that

$$d\beta + \frac{1}{2}[\beta, \beta] = 0. \quad (40)$$

On the trivial G -bundle $S^2 \times G \rightarrow S^2$ we can define a family of connections

$$\nabla_{\theta} = (\partial + \frac{1}{2}(1 - \theta^{-1})\beta^{1,0}, \quad \bar{\partial} + \frac{1}{2}(1 - \theta)\beta^{0,1}) \quad (41)$$

for $\theta \in S^1$. Equations (39) and (40) are equivalent to ∇_{θ} being flat. As S^2 is simply connected, each ∇_{θ} is gauge equivalent to the trivial flat connection, thus we obtain a family of maps $\psi_{\theta} : S^2 \rightarrow G$ such that $\psi_1 = e$ and $\psi_{-1} = \psi$. Hence we may define a map $\varphi : S^2 \rightarrow \Omega G$ by $\varphi(x)(\theta) = \psi_{\theta}(x)$, for $x \in S^2$, $\theta \in S^1$. It may be shown [35] that $\varphi : S^2 \rightarrow \Omega G$ is holomorphic.

This may be reexpressed as follows. Let $\varphi : \Omega G \rightarrow G$ be the projection defined by evaluation at -1 , i.e., $\varphi(f) = f(-1)$, for $f \in \Omega G$. Given a harmonic map $\psi : S^2 \rightarrow G$ there is a holomorphic map $\varphi : S^2 \rightarrow \Omega G$ such that $\varphi \circ \varphi = \psi$ [39]. Conversely, if $\varphi : S^2 \rightarrow \Omega G$ is holomorphic (and satisfies a suitable equation [39]) then $\varphi \circ \varphi : S^2 \rightarrow G$ is harmonic. Thus the loop space ΩG may be regarded as the infinite dimensional twistor space of G in the sense described earlier. It follows from this that ΩG instantons are related to solutions of the chiral model with group G . It would be interesting to consider whether such a twistorial framework can provide a link with the Wess-Zumino-Witten model.

REFERENCES

1. A.M. Polyakov, *Phys.Lett.* **103** (1981), 207.
2. V. Knizhnik, A. Polyakov and A. Zamolodchikov, *Mod.Phys.Lett.* **A3** (1988), 819.
3. F. David, *Mod.Phys.Lett.* **A3** (1988), 1651;
J. Distler and H. Kawai, *Nucl.Phys.* **B321** (1989), 509.
4. E. Brézin and V. Kazakov, *Nucl.Phys.* **B236** (1990), 144.
5. M. Douglas and S. Shenker, *Nucl.Phys.* **B335** (1990), 635.
6. D.J. Gross and A. Migdal, *Phys.Rev.Lett.* **64** (1990), 127.
7. T. Banks, M. Douglas, N. Seiberg and S. Shenker, *Phys.Lett.* **238B** (1990), 279.
8. D.J. Gross and A. Migdal, *Nucl.Phys.* **B340** (1990), 333.
9. E. Brézin, M. Douglas, V. Kazakov and S. Shenker, *Phys.Lett.* **237B** (1990), 43.
10. D.J. Gross and A. Migdal, *Phys.Rev.Lett.* **64** (1990), 717.
11. C. Crnković, P. Ginsparg and G. Moore, "The Ising Model, the Yang-Lee edge singularity and 2D quantum gravity", Yale Preprint YCTP-P20-S9 (1989).
12. M. Douglas, *Phys.Lett.* **238B** (1990), 176.
13. E. Witten, *Nucl.Phys.* **B340** (1990), 281.
14. R. Dijkgraaf and E. Witten, *Nucl.Phys.* **B342** (1990), 486.

15. J. Distler, *Nucl.Phys.* **B342** (1990), 523.
16. E. Verlinde and H. Verlinde, "A solution of 2D topological quantum gravity", IAS Preprint IASSNS-HEP-90/40 (1990).
17. R. Dijkgraaf, E. Verlinde and H. Verlinde, "Loop equations and Virasoro constraints in non-perturbative 2D quantum gravity", IAS/Princeton Preprint IASSNS-HEP-90/48, PUPT-1184 (1990).
18. G. Moore, "Geometry of string equations", Yale Preprint YCTP-PA-90 (1990).
19. E. Witten, *Commun.Math.Phys.* **118** (1988), 411.
20. E. Witten, *Commun.Math.Phys.* **92** (1984), 455.
21. M. Gromov, *Invent.Math.* **82** (1985), 307.
22. L. Baulieu and I. Singer, *Commun.Math.Phys.* **125** (1989), 227.
23. T.P. Killingback, *Commun.Math.Phys.* **100** (1985), 481.
24. A. Pressley and G. Segal, Loop Groups. Oxford University Press (1986).
25. M.F. Atiyah, *Commun.Math.Phys.* **93** (1984), 437.
26. D. Mumford and J. Fogarty, Geometric Invariant Theory, (2nd edition). Springer-Verlag, New York (1982).
27. R. Bott, *Michigan Math.J.* **5** (1958), 35.
28. E. Witten, "Two-dimensional gravity and intersection theory on moduli space", IAS Preprint IASSNS-HEP-90/45 (1990).
29. E. Witten, *Commun.Math.Phys.* **117** (1988), 353.
30. S. Donaldson, "Polynomial invariants of smooth four manifolds", to appear in Topology.
31. L. Baulieu and I. Singer, *Nucl.Phys. (Proc.Suppl.)* **B5B** (1988), 12.
32. M.F. Atiyah and I. Singer, *Proc.Nat.Acad.Sci.* **81** (1984), 2597.
33. T.P. Killingback, CERN Preprint, to appear.
34. M.F. Atiyah, N. Hitchin and I. Singer, *Proc.R.Soc.London* **A362** (1978), 425.

35. F. Burstall, "Recent developments in twistor methods for harmonic maps", In: Harmonic Mappings, Twistors and σ -Models. P. Gauduchon (Ed.), World Scientific (1988).
36. J. Eells and S. Salamon, *Ann.Scuola Norm.Sup.Pisa* **12** (1985), 589.
37. J. Rawnsley, "f structures, f twistor spaces and harmonic maps", In: Geometry Seminar "Luigi Bianchi", Lecture Notes Math. 1164. Springer 1985.
38. S. Salamon, "Harmonic and holomorphic maps", in: Geometry Seminar "Luigi Bianchi", Lecture Notes Math. Springer (1985).
39. K. Uhlenbeck, "Harmonic maps into Lie Groups", *J.Diff.Geom.* **30** (1989), 1.