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$SU(2)_k$ WZW AND Z_k PARA-FERMION MODELS
ON THE TORUS

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ABSTRACT

We study $SU(2)_k$ WZW and Z_k parafermion models using a Coulomb gas formulation related to that of Wakimoto and Zamolodchikov. We discuss the truncation of the Fock modules of the free bosons of the Coulomb gas system to obtain the parafermion and $SU(2)_k$ modules. Explicit expressions are given for the correlators of these models on the torus. A detailed discussion of their modular transformation and their monodromy is also given, the complications being due to the restrictions on the lattice summations.

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1. Introduction

Among the known two dimensional conformal field theories, the ones of particular interest are the Wess-Zumino-Witten (WZW) models [1,2,3], these being the conformal models in which the chiral algebra is generated by a current based on some Lie group G . At present, all the known rational conformal field theories are of the coset type G/H [4], these being the models in which the chiral algebra is constructed from that of G such that its fields commute with those of the subalgebra based on H ($H \subset G$). There is even a conjecture that all the rational conformal theories are of coset type. Therefore a full understanding of the WZW models is of importance.

Recently Witten [5] has shown a very interesting connection between WZW models and three dimensional Chern-Simons gauge theories. Subsequent works [6,7] have made the connection between the states of the Hilbert space of three-dimensional Chern-Simons theory and the conformal blocks of WZW theory more explicit. In particular, in Ref. [6,7] the known differential equations [2] satisfied by the conformal blocks for the case the Riemann surface is a two-sphere have been derived and the known characters for $G = SU(2)$ and arbitrary level k [8] have been obtained. In principle, in this approach one should be able to obtain expressions for the correlators on higher genus surfaces, however, this has not yet been done.

In this paper we will follow another approach, namely the Coulomb gas approach to $SU(2)_k$ and the related Z_k parafermion models (PF) [9,10] (i.e., $SU(2)_k/U(1)_k$). The Coulomb gas approach to minimal conformal models [11-14] and minimal superconformal models [15,16] has proven to be an efficient one. Here we will use this formulation to study the characters of $SU(2)_k$ WZW and Z_k parafermion models and write explicit expressions for some correlators on the torus in these models and study their properties.

The organization of this paper is as follows. In Section 2 we will first review the Coulomb gas formulation of Wakimoto-Zamolodchikov [17,18] and by introducing analogies similar to Felder's [12], will investigate the structure of the PF modules and give a derivation of the known PF character. In Section 3 we will give examples of correlation functions, both for PF and $SU(2)_k$ systems and study their properties. In particular we will discuss their monodromy which, due to the restrictions on the domain of lattice summations, is intricate. Section 4 contains the concluding remarks. In Appendix A we discuss some properties of the lattice sums for PF and $SU(2)_k$ systems which are useful for the discussions in section 3. Appendix B deals with the problem of the modular transformation for these models. Again the restrictions on the domain of lattice summations

make it more involved than the usual cases, and we will handle the problem by exploiting the connection with $N = 2$ minimal models whose modular transformation was worked out in Ref. [16].

2. Coulomb Gas Representation of PF and $SU(2)_k$ Systems and Their Characters

In this section, we will first review the Coulomb gas approach to $SU(2)_k$ WZW and the related Z_k parafermion models and then discuss a derivation of their characters. The $SU(2)_k$ currents $J^\pm(z)$, $J^3(z)$ and the energy momentum tensor $T(z)$ satisfy the operator product expansion (OPE):

$$\begin{aligned} J^a(z)J^b(z') &= \frac{\frac{k}{2}\delta^{ab}}{(z-z')^2} + \frac{f^{ab}J^c(z')}{(z-z')} + \text{regular} \\ T(z)J^a(z') &= \frac{J^a(z')}{(z-z')^2} + \frac{\partial J^a(z')}{(z-z')} + \text{regular} \\ T(z)T(z') &= \frac{c/2}{(z-z')^4} + \frac{2T(z')}{(z-z')^3} + \frac{\partial T(z')}{(z-z')^2} + \text{regular} \end{aligned} \quad (2.1)$$

where k is an integer, $c = \frac{3k}{k+2}$ and $f^{+-3} = f^{3+-} = f^{3+-} = 1$.

As shown by Wakimoto [17] and by Zamolodchikov [18] (see also Ref. [19]) these currents can be expressed in terms of one free boson ϕ (with a background charge) and a free (β, γ) system of weight $(1,0)$:

$$\begin{aligned} J^+(z) &= -\frac{1}{\sqrt{2}}\beta(z) \\ J^3(z) &= \beta(z)\gamma(z) - i\sqrt{\frac{k+2}{2}}\partial\phi(z) \\ J^-(z) &= \frac{1}{\sqrt{2}}(\beta(z)\gamma(z)^2 - i\sqrt{k+2}\partial\phi(z)\gamma(z) + k\partial\gamma(z)) \\ T(z) &= -\frac{1}{4}(\partial\phi(z)\partial\phi(z) + i\alpha_0\partial^2\phi(z) - \beta(z)\partial\gamma(z)) \end{aligned} \quad (2.2)$$

where $\alpha_0 = \frac{1}{2\sqrt{k+2}}$ and the normalizations are such that $\partial\phi(z)\partial\phi(z') = -\frac{2}{(z-z')^2}$ and $\beta(z)\gamma(z') = -\frac{1}{z-z'}$. Clearly the (β, γ) system contributes $c_{\beta\gamma} = 2$ to the central charge whereas the boson ϕ gives $c_\phi = 1 - 24\alpha_0^2$ and therefore the total central charge is $c = \frac{3k}{k+2}$.

One can bosonize the (β, γ) system in terms of two bosons (χ, η) in the usual way as:

$$\begin{aligned} \beta &= -\frac{1}{\sqrt{2}}\partial\chi e^{-\frac{1}{\sqrt{2}}(\chi+\eta)} \\ \gamma &= c\sqrt{2}(\chi+\eta) \end{aligned} \quad (2.3)$$

Note that η comes with imaginary momenta and contributes negatively to the dimensions. Now by making the following change of basis from (χ, η, ϕ) to (ϕ_0, ϕ_1, ϕ_2) :

$$\begin{aligned}\phi_0 &= -\sqrt{\frac{k+2}{k}}\phi + i\sqrt{\frac{k}{2}}\eta, \\ \phi_1 &= \phi - i\sqrt{\frac{k+2}{2}}\eta - \sqrt{\frac{k+2}{2}}\chi, \\ i\phi_2 &= -\sqrt{\frac{k+2}{k}}\phi + i\sqrt{\frac{k+2}{2}}\eta + \sqrt{\frac{k}{2}}\chi,\end{aligned}\quad (2.4)$$

one can rewrite the currents as:

$$\begin{aligned}J^+(z) &= \sqrt{\frac{k}{2}}\psi(z)c\frac{1}{\sqrt{k}}\phi_0(z), \\ J^3(z) &= i\sqrt{k}\partial\phi_0(z), \\ J^-(z) &= \sqrt{\frac{k}{2}}\psi^\dagger(z)c\frac{1}{\sqrt{k}}\phi_0(z),\end{aligned}\quad (2.5)$$

where $\psi(z)$ and $\psi^\dagger(z)$ are the Z_k parafermion currents,

$$\begin{aligned}\psi(z) &= -\frac{i}{2}\left(\sqrt{\frac{k+2}{k}}\partial\phi_1(z) + i\partial\phi_2(z)\right)c\frac{1}{\sqrt{k}}\phi_2(z), \\ \psi^\dagger(z) &= -\frac{i}{2}\left(\sqrt{\frac{k+2}{k}}\partial\phi_2(z) - i\partial\phi_1(z)\right)c\frac{1}{\sqrt{k}}\phi_2(z).\end{aligned}\quad (2.6)$$

These parafermions each carry a conformal weight $1 - \frac{1}{k}$. The energy momentum tensor is now:

$$T(z) = -\frac{1}{4}((\partial\phi_0)^2 + (\partial\phi_1)^2 + (\partial\phi_2)^2) + i\alpha_0\partial^2\phi_1. \quad (2.7)$$

One can construct screening operators which have dimension one and commute with the currents (i.e., their OPE with the currents are regular or if non-regular they are total derivative). In fact, there are three such operators:

$$\begin{aligned}V_1(z) &= \partial\phi_2(z)c^{2i\alpha_0\phi_1(z)}, \\ V_+(z) &= c^{-\frac{1}{2}}\sqrt{\frac{k+2}{k}}\phi_1(z)\pm\frac{1}{2}\sqrt{k}\phi_2(z).\end{aligned}\quad \begin{aligned}(2.8a) \\ (2.8b)\end{aligned}$$

One can easily verify the following OPE's of these screening operators with ψ and $\psi^\dagger$:

$$\begin{aligned}\psi(z)V_1(z') &= -\frac{k+2}{k\sqrt{2}}\partial_{z'}\left(\frac{1}{z-z'}c\frac{1}{\sqrt{k+2}}\phi_1(z')+\frac{1}{\sqrt{k}}\phi_2(z')\right) + \text{regular}, \\ \psi^\dagger(z)V_1(z') &= -\frac{k+2}{k\sqrt{2}}\partial_{z'}\left(\frac{1}{z-z'}c\frac{1}{\sqrt{k+2}}\phi_1(z')-\frac{1}{\sqrt{k}}\phi_2(z')\right) + \text{regular}, \\ \psi(z)V_+(z') &= \frac{1}{\sqrt{2}}\partial_{z'}\left(\frac{1}{z-z'}V_1(z')c\frac{1}{\sqrt{k}}\phi_2(z')\right) + \text{regular}, \\ \psi^\dagger(z)V_+(z') &= \text{regular}, \\ \psi(z)V_-(z') &= \text{regular}, \\ \psi^\dagger(z)V_-(z') &= \frac{1}{\sqrt{2}}\partial_{z'}\left(\frac{1}{z-z'}V_1(z')c\frac{1}{\sqrt{k}}\phi_2(z')\right) + \text{regular}.\end{aligned}$$

Therefore these screening charges commute with the currents. If we denote the momenta of $V_1(z)$ and $V_\pm(z)$ by $\vec{e}_1$ and $\vec{e}_\pm$ respectively, then they satisfy the relation $(k+2)\vec{e}_1 + \vec{e}_+ + \vec{e}_- = 0$. Thus these screening operators can be used to balance the charges on higher genus surfaces. In particular for genus-two surface one expects a background charge $+2\alpha_0$ on the surface which can be screened by the relation

$$(k+1)\vec{e}_1 + \vec{e}_+ + \vec{e}_- = (-2\alpha_0, 0, 0).$$

The parafermion system is described entirely by the two bosons ϕ_1 and ϕ_2 . The parafermion primary fields $\Phi_{\ell,m}$ are labelled by two integers ℓ and m ; $\ell = 0, \dots, k$ and $m = -k, \dots, k-1$. In terms of ϕ_1 and ϕ_2 they are expressed as:

$$\Phi_{\ell,m}(z) = C(\ell, m) \cdot e^{-\frac{i\ell}{2\sqrt{k+2}}\phi_1(z) + \frac{im}{2\sqrt{k}}\phi_2(z)}; \quad \text{for } -\ell \leq m \leq \ell \quad (2.9)$$

where $C(\ell, m) = \sqrt{\frac{\ell!}{(\frac{k+m}{2})!(\frac{k-m}{2})!}}$. For $m > \ell$ one can get the primary fields by the successive application of the parafermion field ψ on $\Phi_{\ell,m}$. The primary fields of $SU(2)_k$ of spin j and J_0^3 -eigenvalue m are then simply

$$\tilde{\Phi}_{j,m}(z) = \Phi_{2j,2m}(z)c^{\frac{im}{2\sqrt{k}}}\phi_0(z) \quad (2.10)$$

Now let us turn to the structure of parafermion modules. First note that the screening charges $Q_\pm = \int dz V_\pm(z)$ and $Q_1 = \int dz V_1(z)$ satisfy the relations:

$$Q_+^2 = Q_-^2 = 0, \quad (2.11)$$

$$Q_+Q_- + (-1)^k Q_-Q_+ = 0, \quad (2.12a)$$

$$Q_+Q_1 + Q_1Q_+ = 0. \quad (2.12b)$$

These relations can be verified by using the OPE of the screening operators. In the last relation, one obtains an integral of a total derivative which consequently vanishes.

Denote by $F_{\ell,m}$ the Fock module built on the vertex operators $:e^{-\frac{i\ell}{2\sqrt{k+2}}\phi_1 + \frac{im}{2\sqrt{k}}\phi_2}$. Then for $Q_\pm$ we have (see Fig. 1):

$$F_{\ell,m} \xrightarrow{Q_\pm} F_{\ell\pm(k+2),m\pm k},$$

and for Q_1

$$F_{2(k+2)-\ell-2,m} \xrightarrow{Q_1^{(k+1)}} F_{\ell,m} \xrightarrow{Q_1^{(k+1)}} F_{-\ell-2,m},$$

where $Q_1^{(i)}$ is defined as

$$Q_1^{(i)} = \int dz_1 V_1(z_1) \int dz_2 V_1(z_2) \dots \int dz_i V_1(z_i), \quad (2.13)$$

with the choice of contours being the same as Felder's [12]. The primary fields $\Phi_{l,m}$ listed above satisfy the following conditions:

$$Q_{\pm} \Phi_{l,m} = Q_1^{(l+1)} \Phi_{l,m} = 0 \quad (2.14a)$$

and

$$\Phi_{l,m} \notin \text{Im}(Q_1^{(k+l+1)}) \oplus \text{Im}(Q_+ Q_-). \quad (2.14b)$$

Consider the parafermion module $P_{l,m}$ which consists of the states obtained from the primary field $\Phi_{l,m}$ by applying equal numbers of Ψ and $\Psi^\dagger$ operators. The states obtained by applying unequal numbers of Ψ and $\Psi^\dagger$ differing by multiples of k will be discussed below. From Eq. (2.14) it is clear that the parafermion module $P_{l,m}$ is contained in $\frac{\text{Ker}(Q_+ Q_-) \cap \frac{\text{Ker}(Q_1^{(l+1)})}{\text{Im}(Q_+ Q_-)}}{\text{Im}(Q_+ Q_-)}$. Since $Q_{\pm}^2 = 0$ we have $\text{Im}(Q_+) \subset \text{Ker}(Q_{\pm})$. We now show that the cohomology of $Q_{\pm}$ (i.e., $\frac{\text{Ker}(Q_{\pm})}{\text{Im}(Q_{\pm})}$) is trivial on each Fock space. Indeed if there is a state ψ in some Fock module $F_{r,s}$ such that $Q_{\pm} \psi = 0$, then define the state ψ' as

$$\psi'(0) = \frac{1}{2\pi\alpha} \oint \frac{dz}{z} e^{\frac{i}{2}\sqrt{k+2}\phi_1(z) \mp \frac{1}{2}\sqrt{k}\phi_2(z)} \psi(0). \quad (2.15)$$

We then have

$$\begin{aligned} Q_{\pm} \psi'(0) &= \frac{1}{2\pi\alpha} \oint dw c^{-\frac{i}{2}\sqrt{k+2}\phi_1(w) \pm \frac{1}{2}\sqrt{k}\phi_2(w)} \psi'(0) \\ &= \frac{1}{2\pi\alpha} \oint \frac{dz}{z} \left(\frac{1}{2\pi\alpha} \oint_{C_z} dw + \frac{1}{2\pi\alpha} \oint_{C_w} dw \right) e^{-\frac{i}{2}\sqrt{k+2}\phi_1(w) \pm \frac{1}{2}\sqrt{k}\phi_2(w)} \\ &\quad \times e^{\frac{i}{2}\sqrt{k+2}\phi_1(z) \mp \frac{1}{2}\sqrt{k}\phi_2(z)} \psi(0) \end{aligned} \quad (2.16)$$

where the contour C_z is around z and not 0 whereas the contour C_w is around 0 and not z . The integral $\int_{C_w} dw$ is zero as $Q_{\pm} \psi(0) = 0$ and the integral $\int_{C_z} dw$ is just equal to one. Performing the z -integral we get $Q_{\pm} \psi'(0) = \psi(0)$. One can further ask whether the cohomology of $Q_{\pm}$ is trivial in $\text{Ker} Q_{\pm}$. In other words if $\psi \in \text{Ker} Q_{\pm} \cap \text{Ker} Q_{\pm}$ then does there exist a state ψ' such that $\psi = Q_{\pm} Q_{\mp} \psi'$? Indeed take the lowest level state $\Phi_{l,m}$ in $P_{l,m}$. We know that $\Phi_{l,m} \in \text{Ker}(Q_{\pm}) \cap \text{Ker}(Q_1)$. However if $\Phi_{l,m}$ is an image of $Q_{+} Q_{-}$ of some state ψ' then $\psi' \in F_{l-2(k+2),m}$. But the weight of the lowest level state in $F_{l-2(k+2),m}$ is greater than that of $\Phi_{l,m}$, therefore $\Phi_{l,m}$ cannot be in the image. On the

other hand for all other $F_{r,s}$, $(r,s) \neq (\ell, m)$, shown in Fig. 1, one can easily show that the lowest level states are in fact in the image of $Q_{+} Q_{-}$. Let us assume that this is true for the entire Fock module $F_{r,s}$ $((r,s) \neq (\ell, m))$, i.e.,

$$\begin{aligned} \frac{\text{Ker}(Q_+) \cap \text{Ker}(Q_-)}{\text{Im}(Q_+ Q_-)}(F_{r,s}) &= 0 \\ \text{for } (r,s) &= (\ell + n_1(k+2), m + n_2k) \quad n_1 > 0 \\ \text{or } (r,s) &= (-\ell - 2 + (n_1 + 2)(k+2), m + n_2k) \quad n_1 \geq 0 \end{aligned} \quad (2.17)$$

where $n_1 \geq |n_2|$ and $n_1 + n_2 \in 2\mathbb{Z}$. This implies that

$$\begin{aligned} \frac{\text{Ker}(Q_+ Q_-)}{\text{Ker}(Q_+) \oplus \text{Ker}(Q_-)}(F_{r,s}) &= 0 \\ \text{for } (r,s) &= (\ell - (n_1 + 2)(k+2), m + n_2k) \quad n_1 \geq 0 \\ \text{or } (r,s) &= (-\ell - 2 - n_1(k+2), m + n_2k) \quad n_1 > 0 \end{aligned} \quad (2.18)$$

where $n_1 \geq |n_2|$ and $n_1 + n_2 \in 2\mathbb{Z}$. This can be seen by introducing the following inner product between a state $\psi \in F_{r,s}$ at some level n above the primary field with a state $\psi' \in F_{-r-2,-s}$ at the level n above the primary field;

$$\langle \psi | \psi' \rangle = \text{the two point function} < \psi(0) \psi'(1) \rangle. \quad (2.19)$$

This is a non-degenerate inner product. In other words, for each ψ there is a ψ' such that $\langle \psi | \psi' \rangle > 0$. Now if $\psi \in \text{Ker} Q_+$, then due to the triviality of $\text{Coh}(Q_+)$, $\psi = Q_+ \psi''$ for some $\psi'' \in F_{r-(k+2),s-k}$. Since $\langle \psi' | \psi \rangle = \langle \psi' | Q_+ \psi'' \rangle = \langle Q_+ \psi' | \psi'' \rangle \neq 0$ then $Q_+ \psi' \neq 0$, where we have used the definition of the inner product (2.19), and deformed the contour of Q_+ so that it encircles the point 1. (Note that there is no contribution from infinity.) If we choose ψ' such that $\langle \psi' | \psi \rangle \neq 0$ and $\langle \psi' | \tilde{\psi} \rangle = 0$ for $\tilde{\psi} \in \text{Im}(Q_+ Q_-)$ then it follows by similar arguments that $\psi' \in \text{Ker}(Q_+ Q_-)$. Thus $\frac{\text{Ker}(Q_+ Q_-)}{\text{Ker} Q_+ \oplus \text{Ker} Q_-}(F_{-r-2,-s})$ is dual to $\frac{\text{Ker} Q_+ \cap \text{Ker} Q_-}{\text{Im}(Q_+ Q_-)}(F_{r,s})$ and therefore the triviality of the latter for the range of (r,s) given in Eq. (2.17) implies the triviality of the former. Finally, using the Z_2 symmetry $\phi_2 \leftrightarrow -\phi_2$, $Q_+ \leftrightarrow Q_-$ we can flip the sign of s to get Eq. (2.18).

Now we are in a position to calculate the character. Consider

$$\begin{aligned} \chi_{l,m} &= \text{Tr} \frac{\text{Ker}(Q_+ \cap \text{Ker}(Q_-))}{\text{Im}(Q_+ Q_-)}(F_{l,m}) (q^{l_{0,m}-c/24}) \\ &= \text{Tr}(\text{Ker}(Q_+) \cap \text{Ker}(Q_-))(F_{l,m}) (q^{l_{0,m}-c/24}) - \text{Tr} \frac{F_{l,m-2(k+2),m}}{\text{Ker}(Q_+ \cap \text{Ker}(Q_-))} (q^{l_{0,m}-c/24}) \end{aligned} \quad (2.20)$$

Now the first term is

$$\begin{aligned}
& \text{Tr}_{\text{Ker}(Q_+ \cap \text{Ker}(Q_-))}(t_{l,m}) (q^{l_{m-c/2+1}}) \\
&= \sum_{n=0}^{\infty} (-1)^n \text{Tr}_{\text{Ker}(Q_+ \cap \text{Ker}(Q_-))}(U_{l,m}(n, n+1)) (q^{l_{m-c/2+1}}) \\
&= \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} (-1)^{n+n'} \text{Tr}_{F_{l,m}(n, n+1)} \text{Tr}_{F_{l,m}(n', n'+1)} (q^{l_{m-c/2+1}})
\end{aligned} \quad (2.21)$$

and the second term

$$\begin{aligned}
& \text{Tr}_{\frac{F_{l,m}(l_{m-c/2+1})}{\text{Ker}(Q_+ \cap \text{Ker}(Q_-))}} (q^{l_{m-c/2+1}}) \\
&= \sum_{n=0}^{\infty} (-1)^n \text{Tr}_{F_{l,m}(n, n+1)} \text{Tr}_{\frac{F_{l,m}(l_{m-c/2+1})}{\text{Ker}(Q_+ \cap \text{Ker}(Q_-))}} (q^{l_{m-c/2+1}}) \\
&= \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} (-1)^{n+n'} \text{Tr}_{F_{l,m}(n, n+1)} \text{Tr}_{F_{l,m}(n', n'+1)} (q^{l_{m-c/2+1}})
\end{aligned} \quad (2.22)$$

Combining the above expressions we obtain

$$\chi_{l,m} = \frac{1}{\eta(\tau)^2} \sum_{\substack{n_1, n_2 \in \mathbb{Z}/2 \\ n_1 - n_2 \in \mathbb{Z} \\ n_1 \geq |n_2| \\ -n_1 > |n_2|}} (-1)^{2n_1} \text{sign}(n_1) q^{\frac{1}{4}(n_1 + 2n_2)^2 - \frac{1}{4}(n_1 - 2n_2)^2} \quad (2.23)$$

where $\text{sign}(0) = 1$. This is indeed the known expression for the parafermion character [8, 9]. Hence, if the assumption (2.17) is correct then the parafermion module can be identified with $\frac{\text{Ker}(Q_+ \cap \text{Ker}(Q_-))}{\text{Im}(Q_+ \cap \text{Ker}(Q_-))}$ and the latter is in $\text{Ker}(Q_+)$. The fact that none of the states in the latter is in $\text{Im}(Q_+)$ follows from the assumption (2.17) that $\frac{\text{Ker}(Q_+ \cap \text{Ker}(Q_-))}{\text{Im}(Q_+ \cap \text{Ker}(Q_-))}$ is trivial and the commutativity of Q_+ with Q_+ .

We have not been able to prove the assumption (2.17), however as shown above at least the lowest level states in $F_{r,s}$ (and hence the entire irreducible parafermion module built on them) are in the $\text{Im}(Q_+ \cap \text{Ker}(Q_-))$. It remains to be proven that this is so for all $\text{Ker}(Q_+ \cap \text{Ker}(Q_-))$ in $F_{r,s}$. For $k = 1$ one can in fact prove that this is so. Let $\psi \in F_{r,s}$ (with weight $\Delta \neq 0$) be in $\text{Ker}(Q_+ \cap \text{Ker}(Q_-))$. Define

$$\psi'(0) = \frac{1}{2i\pi\Delta} \oint dz z e^{\sqrt{3}\phi_+(z)} \psi(0)$$

Applying $Q_+ Q_-$ on ψ' and using the fact that $Q_+ \psi = 0$ we obtain

$$\begin{aligned}
Q_+ Q_- \psi'(0) &= \frac{1}{2i\pi\Delta} \oint dz z (Q_+ Q_- e^{\sqrt{3}\phi_+(z)}) \psi(0) \\
&= \frac{1}{2i\pi\Delta} \oint dz z T(z) \psi(0) \\
&= \psi'(0)
\end{aligned}$$

Thus every state in $\text{Ker}(Q_+ \cap \text{Ker}(Q_-))$ with $\Delta \neq 0$ is in $\text{Im}(Q_+ \cap \text{Ker}(Q_-))$. This is as expected for $k = 1$; the parafermion module consists of only one state, namely, the identity.

In case for $k \geq 2$ the statement (2.17) does not hold, one can proceed to obtain the character in the following way. Consider the modules $F_{-l-2+2p(k+2),m}$ ($p \geq 1$) and $F_{l+2p(k+2),m}$ ($p \geq 0$) which are connected to $F_{l,m}$ by the action of Q_+ . In Fig. 1, they are the modules on the horizontal line from $F_{l,m}$. For each of these, we compute the trace by summing (with alternating $\pm$ signs) over all the modules which are connected to it by the action of Q_+ and Q_- . This way one obtains a trace over the states in $\text{Ker}(Q_+ \cap \text{Ker}(Q_-))$ and $\frac{F_{l,m}}{\text{Ker}(Q_+ \cap \text{Ker}(Q_-))}$ which is:

$$\begin{aligned}
& \text{Tr}_{\frac{F_{l,m}}{\text{Ker}(Q_+ \cap \text{Ker}(Q_-))}} (q^{l_{m-c/2+1}}) = \text{Tr}_{\frac{F_{l,m}}{\text{Ker}(Q_+ \cap \text{Ker}(Q_-))}} (q^{l_{m-c/2+1}}) \\
&= \frac{1}{\eta(\tau)^2} \left[\sum_{j=0}^{\infty} (-1)^j q^{(k+2)(\frac{1}{2}j + \frac{k+1}{2(k+2)})^2 - k(\frac{1}{2}j + \frac{k}{2})^2} \right. \\
&\quad \left. + \sum_{j=1}^{\infty} (-1)^j q^{(k+2)(\frac{1}{2}j + \frac{k+1}{2(k+2)})^2 - k(\frac{1}{2}j - \frac{k}{2})^2} \right]. \quad (2.24)
\end{aligned}$$

Let us denote this space, which is obtained after the above summation as $\tilde{F}_{r,s}$. We now consider the following sequence:

$$\tilde{F}_{l,m} \xrightarrow{Q_+^{(k+1-r)}} \tilde{F}_{2(k+2)-l-2,m} \xrightarrow{Q_+^{(k+1-r)}} \tilde{F}_{4(k+2)-l-2,m} \xrightarrow{Q_+^{(k+1-r)}} \dots \quad (2.25)$$

Note that under the action of $Q_+ Q_-$ we have

$$\tilde{F}_{l,m} \xrightarrow{Q_+ Q_-} \tilde{F}_{2(k+2)+l,m} \xrightarrow{Q_+ Q_-} \tilde{F}_{4(k+2)+l,m} \xrightarrow{Q_+ Q_-} \dots$$

and

$$\tilde{F}_{2(k+2)-l-2,m} \xrightarrow{Q_+ Q_-} \tilde{F}_{4(k+2)-l-2,m} \xrightarrow{Q_+ Q_-} \tilde{F}_{6(k+2)-l-2,m} \xrightarrow{Q_+ Q_-} \dots$$

Therefore, if the trace is taken over the modules in (2.25) with alternating $\pm$ signs, then taking into account the fact that $\tilde{F}_{-l-2,m} \xrightarrow{Q_+ Q_-} \tilde{F}_{2(k+2)-l-2,m}$ and that $\tilde{F}_{-l-2}$ is dual to $\tilde{F}_{l,m}$, in effect we are taking the trace over the states in $\frac{\text{Ker}(Q_+ \cap \text{Ker}(Q_-))}{\text{Im}(Q_+ \cap \text{Ker}(Q_-))}$ and $\frac{\text{Ker}(Q_+ \cap \text{Ker}(Q_-))}{\text{Im}(Q_+ \cap \text{Ker}(Q_-))}$. But as we showed above, $\frac{\text{Ker}(Q_+ \cap \text{Ker}(Q_-))}{\text{Im}(Q_+ \cap \text{Ker}(Q_-))}$ is dual to $\frac{\text{Ker}(Q_+ \cap \text{Ker}(Q_-))}{\text{Im}(Q_+ \cap \text{Ker}(Q_-))}$. Therefore we can instead of the sequence in (2.25) consider the following sequence:

$$\begin{aligned}
& Q_+^{(k+1-r)} \tilde{F}_{l,m} \xrightarrow{Q_+^{(k+1-r)}} \tilde{F}_{2(k+2)-l-2,m} \xrightarrow{Q_+^{(k+1-r)}} \dots \\
& \dots \tilde{F}_{l,m} \xrightarrow{Q_+^{(k+1-r)}} \tilde{F}_{2(k+2)-l-2,m} \xrightarrow{Q_+^{(k+1-r)}} \dots \quad (2.26)
\end{aligned}$$

where $\hat{F}_{r,s} \equiv \frac{\text{Ker} Q_+ \cap \text{Ker} Q_-}{\text{Im}(Q_+ Q_-)}(F_{r,s})$. Now if $Q_1^{(t+1)} Q_1^{(k+1-l)} = Q_1^{(t+1-l)} Q_1^{(t+1)}$ and the cohomology of Q_1 is trivial on the modules in the sequence (2.26) except for $\hat{F}_{r,m}$ then, by taking the trace over the modules in (2.26) with alternating $\pm$ signs, we get a trace over $\frac{\text{Ker}(Q_1^{(t+1)})}{\text{Im}(Q_1^{(t+1-l)})}(\hat{F}_{r,m})$ which is the parafermion module, and by using (2.21) and making a trivial change of summation variables, we get:

$$\begin{aligned} \text{Tr} \frac{\text{Ker}(Q_1^{(t+1)})}{\text{Im}(Q_1^{(t+1-l)})}(\hat{F}_{r,m}) &= (q^{l_{m-r/24}}) \\ &= \frac{1}{\eta(\tau)^2} \sum_{\substack{n_1, n_2 \in \mathbb{Z}/2\mathbb{Z} \\ n_1 \geq |n_2| \\ -n_1 \geq |n_2|}} (-1)^{2n_1} \text{sign}(n_1) q^{\frac{1}{24}(l_{r+2n_1}(k+2)) - \frac{1}{24}(m+2n_2k)^2} \end{aligned} \quad (2.27)$$

We have so far discussed the spaces $F_{l,m}$. One may wonder what happens to the states obtained from $\Phi_{l,m}$ by applying unequal numbers of Ψ and $\Psi^\dagger$ differing by a multiple of k . These states could be contained in $F_{l,m+2nk}$, n being a non-zero integer. In Ref.[9], $(\Psi)^k$ and $(\Psi^\dagger)^k$ are identified with identity, so one would not expect to obtain any new states. But in the present Coulomb gas formalism, these states carry different momenta along the ϕ_2 direction. However, it turns out that there is an isomorphism between $F_{l,m}$ and $F_{l,m+2nk}$ for any $n \in \mathbb{Z}$. Indeed consider a nontrivial state $\eta \in F_{l,m}$. Then η is in $\text{Ker}(Q_-)$ and hence there exists a state η' such that $Q_- \eta' = \eta$. Furthermore $\bar{\eta} \equiv Q_+ \eta' \neq 0$, because otherwise due to the triviality of the cohomology of Q_+ there would exist a state η'' such that $\eta' = Q_+ \eta''$ which in turn would mean that $\eta \in \text{Im}(Q_+ Q_-)$ - a contradiction. Now we show that $\bar{\eta} \in F_{l,m+2k}$. Certainly $\bar{\eta} \in \text{Ker} Q_+$. It also belongs to $\text{Ker} Q_-$ as $Q_- \bar{\eta} = Q_- Q_+ \eta' = (-1)^{k+1} Q_+ \eta = 0$. Finally if $\bar{\eta} = Q_+ Q_- \eta'$, then $Q_- (\eta' - Q_- \eta') = \eta$ and $Q_+ (\eta' - Q_- \eta') = 0$ implying that $\eta \in \text{Im}(Q_+ Q_-)$, a contradiction. Therefore $\bar{\eta}$ is a nontrivial state in $F_{l,m+2k}$. It is easy to see that the map defined above between $F_{l,m}$ and $F_{l,m+2k}$ is an isomorphism, and that it can be extended to all $F_{l,m+2nk}$ successively. This isomorphism allows one to identify $(\Psi)^k = (\Psi^\dagger)^k = 1$ in our formalism and work with only $\hat{F}_{l,m}$ for example. This also means that all the physical quantities such as characters and correlation functions computed using $F_{l,m+2nk}$ would be independent of n . In particular, this implies the shift invariance of the domain of the lattice sums which plays a crucial role in proving monodromy and modular covariance of correlation functions and which we will discuss in next section.

Now let us discuss the computation of the n -point function on the sphere. For this we need to define a dual field at ∞ (say). The dual spaces have already been introduced

above; $(F_{l,m})^* \sim F_{-l-2,-m}$ and $\left(\frac{\text{Ker} Q_+ \cap \text{Ker} Q_-}{\text{Im}(Q_+ Q_-)}(F_{l,m}) \right)^* \sim \frac{\text{Ker}(Q_+ Q_-)}{\text{Ker}(Q_+ Q_-)}(F_{-l-2,-m})$. Thus we can consider a four-point function,

$$\langle \psi_1(0) \psi_2(z) \psi_3(1) \psi_4^*(\infty) \rangle >$$

where

$$\begin{aligned} \psi_i &\in \frac{\text{Ker} Q_+ \cap \text{Ker} Q_-}{\text{Im}(Q_+ Q_-)}(F_{l_i, m_i}) \quad i = 1, 2, 3 \\ \psi_4^* &\in \frac{\text{Ker}(Q_+ Q_-)}{\text{Ker}(Q_+) \oplus \text{Ker}(Q_-)}(F_{-l_4, -m_4}). \end{aligned} \quad (2.28)$$

To define this correlation function we must introduce a sufficient number of screening operators so that

$$\left(\frac{l_1 + l_2 + l_3 - l_4}{2\sqrt{k+2}} \right) + \sum (\text{screening charges}) = 0.$$

One can define the screening contours as in Doltsenko-Fateev [11] or in Felder [12]. The fact that the intermediate state is also a physical state can be seen in the usual way. That the dressed $\psi_1(0) \times \psi_2(z)$ is in $\text{Ker} Q_+ \cap \text{Ker} Q_-$ follows from the commutativity of $Q_\pm$ with Q_+ and Q_- . If on the other hand this product contains a state in $\text{Im}(Q_+ Q_-)$, then it decouples from the above correlation function as can be seen by deforming the Q_+ and Q_- contours and pulling them around ψ_3 and ψ_4^* and using Eq. (2.28).

In the next section we describe how to compute two-point functions on the torus. The subtlety here is that one must make sure that the trace is taken over only the physical states and this will be ensured by making use of Eqs (2.20), (2.21) and (2.22) (or Eq. (2.27) in case the statement (2.17) does not hold).

Before ending this section we note that since $SU(2)_k$ is the product of $U(1)_k$ and $U(1)_k$, using the expression for the character of $U(1)$ and the character of $U(1)_k$ and following the steps sketched in Appendix A one obtains the well-known expression [8] for the character of $SU(2)_k$ given in Eq. (A.10) below.

3. Correlators in the PF and $SU(2)_k$ Systems

In this section we will use the Coulomb gas formulation of $SU(2)_k$ current algebra and parafermion system of the previous section to derive expressions for their correlation functions on the torus. For simplicity we will only give the correlators of the currents and

the correlators of the primary fields of type $j = \frac{1}{2}, m = 1 + \frac{1}{2}$. The former does not require screening, and thus there is no contour integral, whereas the latter requires one screening operator.

Consider first the correlation function of two PE currents: $\langle \Psi(z_1) \Psi^\dagger(z_2) \rangle$. Using the expression (2.6) for Ψ and $\Psi^\dagger$, we get

$$\begin{aligned} & \langle \Psi(z_1) \Psi^\dagger(z_2) \rangle_{l,m} \\ &= \frac{1}{k} \left(\frac{\theta_1(z_1 - z_2)}{\theta_1'(0)} \right)^{2/k} \left[-(k+1) \partial_{z_1}^2 \log \theta_1(z_1 - z_2) + (k+2) \partial_{w_1}^2 \right. \\ & \quad \left. - k^2 \partial_{w_2}^2 - (\partial_{z_1} \log \theta_1(z_1 - z_2))^2 - 2k \partial_{z_1} \log \theta_1(z_1 - z_2) \partial_{w_2} \right] \Gamma_{l,m}(0, w_2), \end{aligned} \quad (3.1)$$

where $w_2 = 2(z_1 - z_2)$ and

$$\begin{aligned} \Gamma_{l,m}(w_1, w_2) &= \frac{1}{\eta(\tau)^2} \sum_{\substack{n_1, n_2 \in \mathbb{Z}/2 \\ n_1 - n_2 \in \mathbb{Z} \\ n_1 \geq |w_2| \\ -n_1 \geq |w_2|}} (-1)^{n_1} \text{sign}(n_1) q^{\frac{n_1(n_1+1)}{2}} (t + 1 + 2n_1(k+2))^\tau - \frac{1}{4}(m+2n_2k)^\tau \\ & \quad \times e^{2ix(n_1 + \frac{k+1}{2(k+1)})w_1 - 2ix(n_2 + \frac{k}{2})w_2} \end{aligned} \quad (3.2)$$

In Appendix A we discuss some properties of $\Gamma_{l,m}$ which will be useful in what follows. In deriving (3.1), we have used the following expressions:

$$\langle \partial \phi_1(z_1) \partial \phi_1(z_2) \rangle = 2 \partial_{z_1}^2 \log \theta_1(z_1 - z_2) \Gamma(0, 0) = -(k+2) \partial_{w_1}^2 \Gamma(0, 0) \quad (3.3)$$

and

$$\begin{aligned} & \langle \partial \phi_2(z_1) e^{-\frac{1}{k} \phi_2(z_1)} \partial \phi_2(z_2) e^{-\frac{1}{k} \phi_2(z_2)} \rangle \\ &= \left(-\frac{\theta_1(z_1 - z_2)}{\theta_1'(0)} \right)^{2/k} \left[2 \partial_{z_1}^2 \log \theta_1(z_1 - z_2) \Gamma \right. \\ & \quad \left. + \frac{1}{k} (\partial_{z_1} \log \theta_1(z_1 - z_2))^2 \Gamma + 8 \partial_{z_1} \log \theta_1(z_1 - z_2) \partial_{w_2} \Gamma + 4k \partial_{w_2}^2 \Gamma \right]. \end{aligned} \quad (3.4)$$

Let us check the behaviour of the correlator (3.1) under $(z_1 - z_2) \rightarrow 0$, as well as its monodromy under $z_1 \rightarrow z_1 + \tau$. For $(z_1 - z_2) \rightarrow 0$ it has the expansion:

$$\langle \Psi(z_1) \Psi^\dagger(z_2) \rangle_{l,m} \rightarrow (z_1 - z_2)^{-2+2/k} \chi_{l,m}(\tau) + \frac{k+2}{k} (z_1 - z_2)^{2/k} (2i\pi) \partial_{z_1} \chi_{l,m}(\tau) + \dots \quad (3.5)$$

where $\chi_{l,m}(\tau)$ is the character of the h.w.r. $\Phi_{l,m}$. This expansion is expected from the following OPE:

$$\Psi(z_1) \Psi^\dagger(z_2) = (z_1 - z_2)^{-2+2/k} \left[1 + \frac{k+2}{k} (z_1 - z_2)^{2/k} T(z_2) + \dots \right] \quad (3.6)$$

and the fact that $\langle T(z) \rangle = 2i\pi \partial_z \chi_{l,m}(\tau)$. Under $z_1 \rightarrow z_1 + \tau$, using

$$\Gamma_{l,m}(0, w_2) \rightarrow q^{1/k} e^{2ixw_2/k} \Gamma_{l,m+2}(0, w_2) \quad (3.7)$$

we get

$$\langle \Psi(z_1 + \tau) \Psi^\dagger(z_2) \rangle_{l,m} = e^{-2ix/\tau} \langle \Psi(z_1) \Psi^\dagger(z_2) \rangle_{l,m+2}. \quad (3.8)$$

This is expected from the fusion rule

$$\Psi \times \Phi_{l,m} = \Phi_{l,m+2}. \quad (3.9)$$

Since $0 \leq m < 2k$, the change in $\Gamma_{l,m}$ given above is correct as long as $m+2$ is also in this range. However if $m+2$ is outside this range, then one has to shift the lattice summation. For instance, if $m = 2k-1$ then in the lattice sum (3.2) one needs to write $n_2 + \frac{m+2}{2} = (n_2+1) + \frac{1}{2}$ which means in turn that the cone $|w_2| < |n_1|$ is shifted. The difference between the contribution of these two cones is simply the contribution of the lines $n_1 = -n_2$ and $n_1 = n_2 + 1$ to the lattice sum (3.2). As we discuss in Appendix A, this difference, i.e., the contribution of these two lines is of the form $F^{(+)}(w_1, w_2) \theta_1(\frac{w_1 + w_2}{2}) + F^{(-)}(w_1, w_2) \theta_1(\frac{-w_1 + w_2}{2})$ where $F^{(\pm)}(w_1, w_2)$ satisfy $k \partial_{w_2} F^{(\pm)} = \pm(k+2) \partial_{w_1} F^{(\pm)}$. However when this difference is plugged into Eq. (3.1), its contribution vanishes and one has the shift invariance of the cone, and therefore the correct monodromy.

Now let us consider the correlator of $SU(2)_k$ current:

$$\langle J^a(z_1) J^b(z_2) \rangle = \delta^{ab} \epsilon^i(z_1, z_2). \quad (3.10)$$

We evaluate this by using the expression (2.5) for $J^a(z)$. Since the expression for $J^3(z)$ is simpler we first consider the correlator $\langle J^3(z_1) J^3(z_2) \rangle$. Using $J^3(z) = \frac{ik}{2} \partial \phi_0(z)$ we simply get:

$$\langle J^3(z_1) J^3(z_2) \rangle = -k/2 \partial_{z_1}^2 \log \theta(z_1 - z_2) \Gamma_l(0, 0, 0) + k^2 \partial_{w_3}^2 \Gamma_l(0, 0, 0), \quad (3.11)$$

where

$$\Gamma_l(w_1, w_2, w_3) = \frac{1}{\eta(\tau)} \sum_m \sum_{n_3} \Gamma_{l,m}(w_1, w_2) q^{k(n_3 + \frac{m}{2})} e^{2ix(n_3 + \frac{m}{2})w_3}.$$

As we discuss in Appendix A, it is possible to bring this expression for Γ_l to the more useful expression (A.6). Using (A.6) and setting $w_1 = w_2 = w_3 = 0$, we get

$$\begin{aligned} \langle J^3(z_1) J^3(z_2) \rangle &= -\frac{k}{2} (\partial_{z_1}^2 \log \theta_1(z_1 - z_2) - 4i\pi \partial_z \log \eta(\tau)) \chi_l(\tau) \\ & \quad + \frac{2ix}{3} (k+2) \partial_z \chi_l(\tau) \end{aligned} \quad (3.12)$$

where $\chi_\ell(\tau)$ is the character of $SU(2)_k$ in the rep $j = \ell/2$:

$$\chi_\ell(\tau) = \frac{1}{\eta(\tau)^3} \sum_n (2(k+2)n + \ell + 1) q^{(k+2)(n + \frac{\ell+\ell+1}{2(k+2)})^2} \quad (3.13)$$

It is easy to verify that this correlator has the correct $(z_1 - z_2) \rightarrow 0$ expansion and the correct monodromy under $z_1 \rightarrow z_1 + \tau$ †. It is clearly invariant under $z_1 \rightarrow z_1 + \tau$, which is expected. Under $(z_1 - z_2) \rightarrow 0$, since

$$\partial_{z_1}^2 \log \theta_1(z_1 - z_2) - 4i\pi \partial_\tau \log \eta(\tau) = -\frac{1}{(z_1 - z_2)^2} + O(z_1 - z_2) \quad (3.14)$$

we get:

$$< J^3(z_1) J^3(z_2) > \rightarrow \frac{k/2}{(z_1 - z_2)^2} + \frac{2i\pi}{3} (k+2) \partial_\tau \chi_\ell(\tau) + O(z_1 - z_2) \quad (3.15)$$

This is in accordance with the following OPE:

$$\sum_n J^a(z_1) J^a(z_2) = \frac{3k/2}{(z_1 - z_2)^2} + (k+2) I(z_2) + \dots \quad (3.16)$$

and the fact that $< I(z) > = 2i\pi \partial_\tau \chi_\ell(\tau)$.

For the correlator $< J^+(z_1) J^-(z_2) >$ of course we expect to get the same expression as (3.12). Let us see how this comes about. Using the expressions (2.5) for $J^+(z)$ and $J^-(z)$ we get:

$$\begin{aligned} < J^+(z_1) J^-(z_2) > \\ = & \left(-(k+1) \partial_{z_1}^2 \log \theta_1(z_1 - z_2) + (k+2)^2 \partial_{w_1}^2 - k^2 \partial_{w_2}^2 \right. \\ & \left. - (\partial_{z_1} \log \theta_1(z_1 - z_2))^2 - 2k \partial_{z_1} \log \theta_1(z_1 - z_2) \partial_{w_2} \right) \Gamma_\tau(0, w_2, w_3) \Big|_{w_3 \rightarrow w_2} \end{aligned} \quad (3.17)$$

where $w_2 = 2(z_1 - z_2)$ and the limit $w_3 \rightarrow w_2$ should be taken. Now using expression (A.6) for $\Gamma_\tau(w_1, w_2, w_3)$ and taking the limit $(w_3 - w_2) \rightarrow 0$ we get:

$$\begin{aligned} < J^+(z_1) J^-(z_2) > \\ = & \left(-(k+1) \partial_{z_1}^2 \log \theta_1(z_1 - z_2) + \frac{4i\pi}{3} (k+2) \partial_\tau + 4i\pi (k+2) \partial_\tau \log \eta(\tau) \right. \\ & \left. 4i\pi \partial_{z_1} f(z_1 - z_2) - (\partial_{z_1} \log \theta_1(z_1 - z_2))^2 - 4i\pi \partial_{z_1} \log \theta_1(z_1 - z_2) f(z_1 - z_2) \right. \\ & \left. - \frac{\pi^2}{3} + 4\pi^2 f(z_1 - z_2) - 8\pi^2 g(z_1 - z_2) \right) \chi_\ell(\tau) \end{aligned} \quad (3.18)$$

† In Ref. [20] this correlator was first written by these requirements, since they uniquely fix it.

where

$$f(z) = \sum_{r \neq 0} (-1)^r \frac{q^{\frac{1}{2}r(r+1)} e^{2i\pi r z}}{1 - q^r} \quad (3.19)$$

and

$$g(z) = \sum_{r \neq 0} (-1)^r \frac{q^{\frac{1}{2}r(r+1)} e^{2i\pi r z}}{(1 - q^r)^2}. \quad (3.20)$$

Comparing this expression with (3.12) we get the following identity:

$$\begin{aligned} 4i\pi \partial_z f(z) - (\partial_z \log \theta_1(z))^2 - 4i\pi \partial_z \log \theta_1(z) f(z) - \frac{\pi^2}{3} \\ + 4\pi^2 f(z) - 8\pi^2 g(z) = \partial_z^2 \log \theta_1(z) - 8i\pi \partial_\tau \log \eta(\tau). \end{aligned} \quad (3.21)$$

We have checked that both sides of this identity are invariant under $z \rightarrow z + \tau$ and they both have the expansion $-\frac{1}{z^2} - \frac{1}{3} \left(\frac{\partial_z''(0)}{\theta_1'(0)} \right) + O(z^2)$. From this it follows that this identity holds. This identity will also be useful when we discuss the correlation function of primary fields of $SU(2)_k$ to simplify the expression.

We now consider the correlation function of primary fields. For simplicity we restrict ourselves to examples which involve only one screening operator. For this we consider the parafermion primary fields $\Phi_{1,\pm 1}(z) = e^{\pm \frac{1}{2\sqrt{k+2}} \phi_1(z) \pm \frac{1}{2\sqrt{k}} \phi_2(z)}$ which correspond to $\ell = 1, m = \pm 1$ in expression (2.9). The screening operator $V_1(z) = \partial \phi_2(z) e^{\frac{1}{2\sqrt{k+2}} \phi_1(z)}$ should be used to screen the charge. We therefore need the following expression:

$$\begin{aligned} < \partial \phi_2(z) e^{\frac{1}{2\sqrt{k}} \phi_2(z)} e^{\frac{1}{2\sqrt{k+2}} \phi_1(z)} > \\ = & \left(\frac{\theta_1(z_1 - z_2)}{\theta_1'(0)} \right)^{\frac{1}{2k}} \left(-\frac{1}{\sqrt{k}} \partial_z \log \left(\frac{\theta_1(z_1 - z)}{\theta_1(z - z_2)} \right) + 2\sqrt{k} \partial_{w_2} \right) \Gamma_{1,m}(0, w_2) \end{aligned} \quad (3.22)$$

where $w_2 = 2(z_1 - z_2)$. Using this we then simply get

$$\begin{aligned} < \Phi_{1,1}(z_1) \Phi_{1,-1}(z_2) >_{1,m} \\ = & -\frac{1}{\sqrt{k}} \left(\frac{\theta_1(z_1 - z_2)}{\theta_1'(0)} \right)^{\frac{k+1}{2(k+2)}} \oint_C dz \left(\frac{\theta_1(z_1 - z)}{\theta_1'(0)} \right)^{-\frac{1}{k+2}} \left(\frac{\theta_1(z - z_2)}{\theta_1'(0)} \right)^{-\frac{1}{k+2}} \\ & \times [\partial_z \log \left(\frac{\theta_1(z_1 - z)}{\theta_1(z - z_2)} \right) - 2k \partial_{w_2}] \Gamma_{1,m}(w_1, w_2) \end{aligned} \quad (3.23)$$

where $w_1 = z_1 + z_2 - 2z$ and $w_2 = z_1 - z_2$. The s-channel contours $C_1^{(s)}$ and $C_2^{(s)}$ and the t-channel ones $C_1^{(t)}$ and $C_2^{(t)}$ are as in Ref. [13] and shown in Fig. 2. We have checked that this correlator has the correct monodromy and modular transformation properties and in the identity channel the limit $(z_1 - z_2) \rightarrow 0$ reproduces the character. The check is

basically the same as the one given for the minimal models in Refs. [13, 16], except for the restriction of the range $0 \leq m < 2k$ and $0 \leq \ell \leq k$. Under $z_1 \rightarrow z_1 + \tau$ we have

$$\tilde{V}_{\ell,m}(w_1, w_2) \longrightarrow q^{\frac{m}{2k} - \frac{\ell}{2k}} e^{-i\pi(\frac{m}{2k} - \frac{\ell}{2k})} \tilde{V}_{\ell+1, m+1}(w_1, w_2) \quad (3.24)$$

First note that the expression (3.23) vanishes for $\ell = -1$ and $\ell = k+1$. To see this consider $\ell = -1$ and in the expression (3.23) replace $\tilde{V}_{\ell,m}$ by $\tilde{V}_{\ell,m}$ where $\tilde{V}_{\ell,m}(w_1, w_2)$ is defined with the domain of summation appearing symmetric for $n_1 > 0$ and $n_1 < 0$ (i.e., $-n_2 \leq |n_1|$ for $n_2 < 0$ and $n_2 > |n_1|$ for $n_2 > 0$). Thus the difference between $\tilde{V}_{\ell,m}$ and $\tilde{V}_{\ell,m}$ is just the contribution of the line $n_2 = n_1$ which is equal to $F_0^{(-)}(w_1, w_2) \theta_1(\frac{w_1 - w_2}{2})$ where $F_0^{(-)}$ is given in Appendix A (Eq. (A.3) with $\beta = 0$) and satisfies $2k\partial_{w_2} F_0^{(-)}(w_1, w_2) = -2(k+2)\partial_{w_1} F_0^{(-)}(w_1, w_2)$. By plugging this difference in (3.23) one finds that the change in the integral is a total derivative and therefore it is irrelevant meaning that one can replace $\tilde{V}_{\ell,m}(w_1, w_2)$ by $\tilde{V}_{\ell,m}(w_1, w_2)$ without any change. Now since $\tilde{V}_{\ell,m}(-w_1, w_2) = -\tilde{V}_{\ell,m}(w_1, w_2)$, by a change of integration variable $z \rightarrow z_1 + z_2 - z$ one can see that for both contours $C_1^{(0)}$ and $C_2^{(0)}$ the integral is minus of itself and therefore it vanishes. The same argument also applies to $\ell = k+1$. Therefore as far as ℓ is concerned the transformation (3.24) is the expected one. For $m = 2k-1$ since $\frac{m'}{2k} = \frac{m+1}{2k} = 1$ we need to shift $n_2 \rightarrow n_2 - 1$. We saw in the example of correlators of currents that this shift introduced terms proportional to $\theta_1(\frac{w_1 - w_2}{2})$ and $\theta_1(\frac{w_1 + w_2}{2})$ which did not contribute. For the present example their contribution is a total derivative term to the integrand and therefore do not contribute to the correlator either. We conclude that as a result of this shift invariance of the cone we get the correct monodromy also for $m = 0$ and $m = 2k-1$. Moreover the transformation (3.24) is in accordance with the fusion rule [3, 10]:

$$\Phi_{\ell,m} \times \Phi_{\ell',m'} = \sum_{n=\ell-\ell'}^{\min(\ell+\ell', k-\ell-\ell')} \Phi_{n, m+m'} \quad (3.25)$$

The modular transformation property of (3.23) can be easily obtained by using the modular transformation of $\tilde{V}_{\ell,m}(w_1, w_2)$ derived in Appendix B.

We now consider our example of $Sl(2)_k$ which is closely related to the previous example of parafermion, namely the correlator of the primary field $\tilde{\Phi}_{j,m}$ with $j = \frac{1}{2}$ and $m = \pm \frac{1}{2}$. Using expression (2.10) for this primary field we get

$$\begin{aligned} & < \tilde{\Phi}_{1/2, 1/2}(z_1) \tilde{\Phi}_{1/2, -1/2}(z_2) > \\ &= -\frac{1}{\sqrt{k}} \left(\frac{\theta_1(z_1 - z_2)}{\theta_1'(0)} \right)^{\frac{1}{\pi k + 1}} \oint_C dz \left(\frac{\theta_1(z_1 - z)}{\theta_1'(0)} \right)^{-\frac{1}{\pi k + 1}} \left(\frac{\theta_1(z - z_2)}{\theta_1'(0)} \right)^{-\frac{1}{\pi k + 1}} \\ & \quad \times \left[\partial_z \log \left(\frac{\theta_1(z_1 - z)}{\theta_1(z - z_2)} \right) - 2k\partial_{w_2} \right] \tilde{V}_\ell(w_1, w_2, w_3) \end{aligned} \quad (3.26)$$

where $w_1 = z_1 + z_2 - 2z$, $w_2 = z_1 - z_2$ and the limit $w_3 \rightarrow w_2$ should be taken. In fact it is possible to write this correlator in a much simpler form in the following way. Using expression (A.6) for $\tilde{V}_\ell$ we have

$$\tilde{V}_\ell(w_1, w_2, w_3) \xrightarrow{w_3 \rightarrow w_2} \frac{k+2}{2} \partial_{w_1} \tilde{V}_\ell(w_1) + \left[f\left(\frac{w_1 + w_2}{2}\right) - f\left(\frac{w_1 - w_2}{2}\right) \right] \tilde{V}_\ell(w_1)$$

where

$$\tilde{V}_\ell(w_1) = \sum_n q^{\frac{(k+2)(n + \frac{k+1}{2})}{\pi(k+1)}} e^{2i\pi(n + \frac{k+1}{2})w_1}$$

and $f(z)$ was defined in (3.19). Also we have

$$\begin{aligned} 2k\partial_{w_2} \tilde{V}_\ell(w_1, w_2, w_3) & \xrightarrow{w_3 \rightarrow w_2} 2k\partial_{w_2} \left[f\left(\frac{w_1 + w_2}{2}\right) - f\left(\frac{w_1 - w_2}{2}\right) \right] \tilde{V}_\ell(w_1) \\ & + 2(k+2) \left[f\left(\frac{w_1 + w_2}{2}\right) + f\left(\frac{w_1 - w_2}{2}\right) \right] \partial_{w_1} \tilde{V}_\ell(w_1) \\ & + 2i\pi \left[f\left(\frac{w_1 + w_2}{2}\right) - f\left(\frac{w_1 - w_2}{2}\right) \right] \tilde{V}_\ell(w_1) \\ & - 4i\pi \left[g\left(\frac{w_1 + w_2}{2}\right) - g\left(\frac{w_1 - w_2}{2}\right) \right] \tilde{V}_\ell(w_1) \end{aligned}$$

where $g(z)$ is as in (3.20). Now upon using the identity (3.21) the expression (3.26) reduces to

$$\begin{aligned} & < \tilde{\Phi}_{1/2, 1/2}(z_1) \tilde{\Phi}_{1/2, -1/2}(z_2) > \\ & \sim \left(\frac{\theta_1(z_1 - z_2)}{\theta_1'(0)} \right)^{\frac{1}{\pi k + 1}} \oint_C dz \left(\frac{\theta_1(z_1 - z)}{\theta_1'(0)} \right)^{-\frac{1}{\pi k + 1}} \left(\frac{\theta_1(z - z_2)}{\theta_1'(0)} \right)^{-\frac{1}{\pi k + 1}} \\ & \quad \times \partial_z^2 \log \left(\frac{\theta_1(z_1 - z)}{\theta_1(z - z_2)} \right) \tilde{V}_\ell(w_1) \end{aligned}$$

Again we have checked that this expression has the correct monodromy, modular transformation properties as well as the correct $(z_1 - z_2) \rightarrow 0$ limit. Moreover it reproduces the correct four point function on sphere upon degeneration $q \rightarrow 0$. For instance, for the two contours $C_1^{(0)}$ and $C_2^{(0)}$ shown in Fig. 2, in the limit $q \rightarrow 0$ after making a change of coordinate $x = e^{2i\pi z}$ and choosing $x_1 = 1$ we get

$$(1 - x_2)^{-\frac{1}{\pi k + 1}} x_2^{\frac{k+1}{2}} F\left(\frac{2j}{k+2}, -\frac{1}{k+2}, -\frac{2}{k+2}; 1 - x_2\right)$$

and

$$(1 - x_2)^{1 + \frac{1}{\pi k + 1}} x_2^{\frac{k+1}{2}} F\left(1 + \frac{1}{k+2}, 1 + \frac{2j+2}{k+2}, 2 + \frac{2}{k+2}; 1 - x_2\right)$$

which are the known expressions [21] for the two blocks in the t -channel when the external fields $j_1 = \frac{1}{2}$ and $j_2 = \frac{1}{2}$ at the points 1 and x_2 are in the total $j_2 = 0$ representation. These two blocks are respectively the identity block ($j = 0$) and $j = 1$ block.

4. Concluding Remarks

In this paper, we have shown how one can construct two-point functions for Z_k -parafermion and $SU(2)_k$ WZW models on the torus, and have shown that they satisfy the correct monodromy and modular transformation. This construction rests on the Coulomb gas description of these models, and the truncation of extra states in the Fock modules of the free bosons that are not contained in the irreducible modules of the respective chiral algebras, via screening operators. We have argued that these irreducible modules can be identified with the certain cohomology associated with certain BRST operator, and have obtained the known characters. The basic assumption in this has been on the triviality of this cohomology in the Fock modules other than the central one.

It would be interesting to prove this assumption directly using the properties of the screening operators. If one can do this then one may try to construct rational conformal field theories directly by specifying screening charges as in Ref. [22]. In [22] an approach to construct finite dimensional representation of modular and monodromy groups was given but an understanding of the positivity condition is still lacking. If one can prove the triviality of the cohomologies then one might be able to get conditions on the screening charges to ensure the positivity.

One may also try to extend this analysis to genus-2 surfaces. There one expects an extra charge $+2\alpha_0$ which can be balanced by introducing $(k+2)Q_1$ and one Q_+ each. However an analysis of the monodromy and modular transformation of such multicontour problems is technically much more complicated.

While completing this work, we received an interesting paper by D. Bernard and G. Felder [23] in which a detailed investigation of $SU(2)_k$ has been carried out in the formulation in terms of (β, γ) system and the boson ϕ . Two other related works have also appeared, one by D. Neumeschansky [24] and the other by J. Distler and Z. Qiu [25]. Two of the screening operators introduced in [24] are total derivative, and therefore one expects every state to be in the kernel of the Q 's constructed from them, i.e., they can not give any non trivial information. We do not understand the Q constructed in Ref. [25] since it contains $Q^{(k+\gamma)}$ and one expects it also not to give any non-trivial information.

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Appendix A

In this appendix we discuss some properties of the lattice sums, $\Gamma_{\ell,m}(w_1, w_2)$ of the PF system and $\Gamma_\ell(w_1, w_2, w_3)$ of $SU(2)_k$. First consider the lattice sum for the PF system:

$$\Gamma_{\ell,m}(w_1, w_2) = \frac{1}{\eta(\tau)^2} \sum_{\substack{n_1, n_2 \in \mathbb{Z}/2\mathbb{Z} \\ n_1 - n_2 \in \mathbb{Z} \\ n_1 \geq |n_2| \\ -n_1 > |n_2|}} (-1)^{2n_1} \text{sign}(n_1) q^{\frac{(k+2)(n_1 + \frac{k+1}{2(k+1)})^2 - k(n_2 + \frac{m}{2})^2}{2}} \times e^{2i\pi(n_1 + \frac{k+1}{2(k+1)})w_1 - 2i\pi(n_2 + \frac{m}{2})w_2} \quad (\text{A.1})$$

As we saw in section 3, under $z_1 \rightarrow z_1 + \tau$ we may get a situation in which the cone $|n_1| > |n_2|$ is shifted. Thus let us see what is the change in $\Gamma_{\ell,m}(w_1, w_2)$ when it is defined over the cone $|n_1| > |n_2| - \alpha$ for some non-zero integer α . It is not difficult to see that the difference between the two is the sum over the points $n_1 = n_2 - \beta$ and $n_1 = -n_2 + (\beta - 1)$ where $1 \leq \beta \leq \alpha$ for $\alpha \geq 1$ and $0 \geq \beta \geq \alpha + 1$ for $\alpha \leq -1$. It is clear from the expression for $\Gamma_{\ell,m}(w_1, w_2)$ that the lines $n_1 = -n_2 + \beta - 1$ and $n_1 = n_2 - \beta$ should give respectively terms proportional to $\theta_1(\frac{w_1 + w_2}{2})$ and $\theta_1(\frac{w_1 - w_2}{2})$. Explicitly, the difference

$$\sum_{\substack{n_1, n_2 \in \mathbb{Z}/2\mathbb{Z} \\ n_1 - n_2 \in \mathbb{Z} \\ n_1 \geq |n_2 - \alpha| \\ -n_1 > |n_2 - \alpha|}} - \sum_{\substack{n_1, n_2 \in \mathbb{Z}/2\mathbb{Z} \\ n_1 - n_2 \in \mathbb{Z} \\ n_1 \geq |n_2| \\ -n_1 > |n_2|}}$$

is equal to

$$\sum_{\beta} \left[F_{\beta}^{(+)}(w_1, w_2) \theta_1\left(\frac{w_1 + w_2}{2}\right) + F_{\beta}^{(-)}(w_1, w_2) \theta_1\left(\frac{-w_1 + w_2}{2}\right) \right]$$

where

$$F_{\beta}^{(+)}(w_1, w_2) = -(-1)^{\frac{k+1+m+2k(\beta-1)}{2}} q^{\frac{(k+1)^2 - (m+2k(\beta-1))^2}{4k}} \times e^{2i\pi[\frac{k+1}{2(k+1)}w_1 - \frac{m+2k(\beta-1)}{2}w_2 - \frac{k+1+m+2k(\beta-1)}{4}(w_1 + w_2)]} \quad (\text{A.2})$$

and

$$F_{\beta}^{(-)}(w_1, w_2) = \{F_{\beta+1}^{(+)} \text{ with } (\ell+1) \rightarrow -(\ell+1) \text{ and } w_1 \rightarrow -w_1\} \quad (\text{A.3})$$

In section 3, what was useful for our discussion of monodromy of the correlation function was the fact that $2kQ_{w_2} F_{\beta}^{(\pm)}(w_1, w_2) = \pm 2(k+2)Q_{w_1} F_{\beta}^{(\pm)}(w_1, w_2)$. We made use of this when showing that the additional terms do not contribute. Note that for the character $\chi_{\ell,m}(\tau) = \Gamma_{\ell,m}(0, 0)$ since $\theta_1(0) = 0$ there are no additional terms and therefore it is independent of the shift of the cone.

For what follows it will be more convenient to write the summation appearing symmetric for $n_1 > 0$ and $n_1 < 0$, i.e., to have the lines, say, $|n_1| = -n_2 > 0$ included but $|n_1| = +n_2 \geq 0$ excluded. Denote this by $\tilde{\Gamma}_{l,m}(w_1, w_2)$. Then the difference between $\tilde{\Gamma}_{l,m}(w_1, w_2)$ and $\Gamma_{l,m}(w_1, w_2)$ is the contribution of the points $n_1 = n_2$ which is

$$\tilde{\Gamma}_{l,m}(w_1, w_2) - \Gamma_{l,m}(w_1, w_2) = F_n^{(-)}(w_1, w_2) \theta_1 \left(\frac{w_1 - w_2}{2} \right). \quad (\text{A.4})$$

Now consider the lattice sum for $SU(2)_k$:

$$\Gamma_l(w_1, w_2, w_3) = \frac{1}{\eta(\tau)} \sum_m \sum_{n_3} \Gamma_{l,m}(w_1, w_2) q^{\frac{k(n_3 + \frac{m}{2k})^2}{2}} e^{2i\pi(n_3 + \frac{m}{2k})w_3}. \quad (\text{A.5})$$

It is possible to make a change of summation and rewrite this sum in a form which is simpler. The expression that we are aiming at is the following:

$$\sum_{n=-\infty}^{\infty} q^{\frac{(k+2)(n + \frac{k+1}{2(k+1)})^2}{2}} e^{2i\pi(n + \frac{k+1}{2(k+1)})w_1} \sum_{r=-\infty}^{\infty} (-1)^r \frac{q^{\frac{1}{2}r(r+1)}}{1 - q^r} e^{2i\pi r \zeta} \times \left[e^{-2i\pi((k+2)n + \frac{k+1}{2})\zeta + 2i\pi r(\frac{w_1 + w_2}{2} - e^{2i\pi((k+2)n + \frac{k+1}{2})\zeta + 2i\pi r(\frac{w_1 + w_2}{2} - \frac{w_1 + w_2}{2})} \right] \quad (\text{A.6})$$

where $\zeta = \frac{w_2 - w_3}{k}$. To obtain this, consider $\tilde{\Gamma}_l(w_1, w_2, w_3)$ which is defined as in (A.5) but with $\tilde{\Gamma}_{l,m}(w_1, w_2)$ replacing $\Gamma_{l,m}(w_1, w_2)$ on the right hand side. Make the change of summation $n_3 \rightarrow n_3 - n_2$, then since the quadratic term in n_2 and n_3 changes to a linear term in n_3 , it is possible to do its summation. Moreover since the dependence on m now is only in the form $n_2 + \frac{m}{2k}$ we can also do the summation over m . The range of m is $0 \leq m \leq 2k - 2$ for $\ell \in 2Z$ and $1 \leq m \leq 2k - 1$ for $\ell \in 2Z + 1$. Define $p = kn_2 + \frac{m}{2}$ where $\nu = 0$ for $\ell \in 2Z$ and $\nu = 1$ for $\ell \in 2Z + 1$, then summing over m we get $-kn_1 \leq p \leq kn_1 - 1$ for $n_1 > 0$ and $kn_1 \leq p \leq -kn_1 - 1$ for $n_1 < 0$. Now the summation over p can be explicitly done;

$$\sum_p q^{xp} = \text{sign}(n_1) \frac{q^{-kn_1x} - q^{kn_1x}}{1 - q^x} \quad (\text{A.7})$$

where $x = 2n_3 + \frac{\zeta}{k}$. Collecting all the terms, we have the following sum

$$\sum_{\substack{n_1, n_3 \in Z/2 \\ n_1 - n_3 \in Z}} (-1)^{2n_3} q^{\frac{(k+2)(n_1 + \frac{k+1}{2(k+1)})^2}{2} + kn_3^2 - 2kn_1n_3 + \nu\zeta/2 + n_1w_3} \times e^{2i\pi[(n_1 + \frac{k+1}{2(k+1)})w_1 - kn_1\zeta + \nu\zeta/2 + n_1w_3]} - [(\ell + 1) \rightarrow -(\ell + 1), \quad w_1 \rightarrow -w_1]$$

Now we make the following change of summation variable: $n = n_1 - n_3$ and $r = 2n_3$. Obviously $n, r \in Z$. Then we get:

$$\sum_{n=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} q^{\frac{(k+2)(n + \frac{k+1}{2(k+1)})^2}{2}} (-1)^r \frac{q^{\frac{1}{2}r(r+1) + \nu\zeta/2}}{1 - q^r} e^{2i\pi[(n + \frac{k+1}{2(k+1)})w_1 - kn\zeta + r(\frac{w_1 + w_2}{2} + \nu\zeta/2]} - [(\ell + 1) \rightarrow -(\ell + 1), \quad w_1 \rightarrow -w_1] \quad (\text{A.8})$$

Upon using

$$\frac{(q^r e^{2i\pi\zeta})^{2n + \frac{k+1}{2}}}{1 - q^r e^{2i\pi\zeta}} = \sum_{p=0}^{2n + \frac{k+1}{2} - 1} q^{rp} e^{2i\pi\zeta p} + \frac{1}{1 - q^r e^{2i\pi\zeta}},$$

it is easy to see that the difference between expression (A.8) and (A.6) is a term of the form $F^{(+)}(w_1, w_2) \theta_1(\frac{w_1 + w_2}{2}) + F^{(-)}(w_1, w_2) \theta_1(\frac{-w_1 + w_2}{2})$. As in the case of PF, for $\chi_\ell(\tau) = \Gamma_\ell(0, 0, 0)$ this additional term vanishes and as we have seen in the examples considered in section 3 this difference does not contribute to the correlation functions.

Also note that the $U(1)$ valued character defined by

$$\chi_\ell(\tau, \zeta) = \text{Tr}(q^{L_{m-\ell/24} - 2i\pi\zeta J_0^3}), \quad (\text{A.9})$$

is obtained from $\Gamma_\ell(w_1, w_2, w_3)$ by setting $w_1 = w_2 = 0$. Using expression (A.6) we get the well-known expression [8]

$$\chi_\ell(\tau, \zeta) = \left(\frac{\theta_{\ell+1, k+2}(\zeta) - \theta_{\ell+1, k+2}(-\zeta)}{\theta_1(\zeta)} \right) \left(\frac{\theta_1'(0)}{2i\pi e^{i\pi\zeta}} \right) \quad (\text{A.10})$$

where $\theta_{r,s}(\zeta)$ is defined as

$$\theta_{r,s}(\zeta) = \sum_n q^{(n + \frac{r}{2})^2} e^{i\pi(2n + r)\zeta} \quad (\text{A.11})$$

In deriving (A.10) we made use of the following identity:

$$\frac{\theta_1'(0)}{\theta_1(\zeta)} = -2i\pi e^{i\pi\zeta} \sum_{r=-\infty}^{\infty} \frac{q^{\frac{1}{2}r(r+1)}}{1 - q^r e^{2i\pi\zeta}}$$

This identity can be easily obtained by writing $\theta_1(\zeta)$ in product form and then doing a partial fraction.

Appendix B

The main difficulty in the modular transformation of $\Gamma_{\ell,m}$ is that the lattice sum is over a restricted domain. This was also the problem in showing the monodromy behaviour of $\Gamma_{\ell,m}$, however, as shown in Appendix A and in section 3 the extra terms that appear upon shifting the domain either vanish (for zero screening charge case) or are total derivative (for simple screening contour case). We will use this shift invariance to embed $\Gamma_{\ell,m}$ in a three dimensional lattice $\bar{\Gamma}_{\ell,m}$ which essentially corresponds to correlators for $N = 2$ minimal series. Even though $\bar{\Gamma}_{\ell,m}$ is also a restricted lattice sum, the existence of a null vector in the lattice allows one to carry out the modular transformation explicitly, as described in Ref.[16]. However, we will first describe the case $w_1 = w_2 = 0$, i.e., the character $\chi_{\ell,m}(\tau)$. In this case one can directly study the modular transformation of $\chi_{\ell,m}$ upon using the following identity

$$\begin{aligned} \frac{1}{2\tau} \int_{|x|>|y|} dx dy e^{-\frac{ix}{2\tau}(x^2-y^2)} e^{2i\pi(xp_1-yp_2)} \text{sign}(x) \\ = \begin{cases} \text{sign}(p_1) q^{p_1^2-p_2^2} & \text{if } |p_1| > |p_2| \\ 0 & \text{if } |p_1| < |p_2| \end{cases} \end{aligned} \quad (B.1)$$

where $p_1 \neq p_2$. Denote the positive light cone $(p_1 > |p_2|)$ by L_+ and the negative one $(-p_1 > |p_2|)$ by L_- . Note that the integration in (B.1) is $\int_{L_+} - \int_{L_-}$. However p_1 and p_2 for our problem are $\sqrt{k+2}(n_1 + \frac{k+1}{2(k+2)})$ and $\sqrt{k}(n_2 + \frac{m}{2k})$ with $|n_1| > |n_2|$. It is easy to see that $\vec{p} = (p_1, p_2)$ lies in $(L_+ \cup L_-)/G_0$ where G_0 is a discrete subgroup of the proper Lorentz group generated by the element

$$g_0 = \begin{pmatrix} k+1 & \sqrt{k(k+2)} \\ \sqrt{k(k+2)} & k+1 \end{pmatrix}.$$

Now we can bring the integration domains L_+ and L_- to $L_+/(G_0)$ and $L_-/(G_0)$ by applying various elements of G_0 on (x, y) and reabsorbing it by applying it on $\vec{p}$. The effect of applying G_0 on $\vec{p}$ is to expand the domain of $\vec{p}$ from $(L_+ \cup L_-)/(G_0)$ to $L_+ \cup L_-$. In fact, since the integral in Eq. (B.1) vanishes when $\vec{p}$ is outside $L_+ \cup L_-$, we can extend the domain of $\vec{p}$ to the entire $\mathbb{R}^2$. Thus we have

$$\eta(\tau)^2 \chi_{\ell,m}(\tau) = \frac{1}{2\tau} \int_{\frac{L_+ \cup L_-}{G_0}} dx dy \text{sign}(x) e^{-\frac{ix}{2\tau}(x^2-y^2)} \sum_{\substack{n_1, n_2 \in \mathbb{Z}/2\mathbb{Z} \\ n_1 - n_2 \in \mathbb{Z}}} (-1)^{2n_1} e^{2i\pi(xp_1-yp_2)} \quad (B.2)$$

Now the summation

$$\sum_{\substack{n_1, n_2 \in \mathbb{Z}/2\mathbb{Z} \\ n_1 - n_2 \in \mathbb{Z}}} (-1)^{2n_1} e^{2i\pi(xp_1-yp_2)}$$

can be replaced by

$$e^{i\pi(x-\frac{k+1}{2(k+2)}, y-\frac{m}{2k})} \sum_{n_1, n_2 \in \mathbb{Z}} \delta(x\sqrt{k+2} - n_1) \delta(y\sqrt{k} - n_2) (1 - e^{i\pi(x\sqrt{k+2} - y\sqrt{k})})$$

and we can do the integrals to get:

$$\eta(\tau)^2 \chi_{\ell,m}(\tau) = \frac{1}{2\tau\sqrt{k(k+2)}} \sum_{n_1, n_2} e^{-\frac{i\pi}{2\tau}(\frac{n_1^2}{k+2} - \frac{n_2^2}{k})} \text{sign}(n_1) e^{i\pi(n_1\frac{k+1}{k+2} - n_2\frac{m}{k})} (1 - e^{i\pi(n_1 - n_2)}). \quad (B.3)$$

Now write $n_1 = 2(k+2)\tilde{n}_1 + (\ell' + 1)$ and $n_2 = 2k\tilde{n}_2 + m'$ where $\ell' = 0, \dots, 2k+3$ and $m' = 0, \dots, 2k-1$. Note that because of the factor $(1 - e^{i\pi(n_1 - n_2)})$, there is a non-zero contribution only if $(\ell' - m') \in 2\mathbb{Z}$. Thus

$$\begin{aligned} \eta(\tau)^2 \chi_{\ell,m}(\tau) &= \frac{1}{\tau\sqrt{k(k+2)}} \sum_{\substack{\ell', m' \in \mathbb{Z} \\ \ell' - m' \in 2\mathbb{Z} \\ n_1 \geq |n_2| \\ -n_1 > |n_2|}} \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n_1 \geq |n_2| \\ -n_1 > |n_2|}} \text{sign}(n_1) \times e^{-\frac{2i\pi}{\tau}((k+2)(n_1 + \frac{\ell'+1}{2(k+2)})^2 - k(n_2 + \frac{m'}{2k})^2)} \\ &\quad \times e^{i\pi(\frac{(k+1)(\ell'+1)}{k+2} - \frac{mm'}{k})}. \end{aligned} \quad (B.4)$$

To bring the range of ℓ' to the standard one, we transform $\ell' \rightarrow k+2 + \ell'$ and $m' \rightarrow k + m' \bmod 2k$ when $\ell' \geq k+2$. Finally, making use of the transformation $(\ell' + 1) \rightarrow -(\ell' + 1), \tilde{n}_1 \rightarrow -\tilde{n}_1$ and the fact that for $\ell' = k+1$ the sum vanishes, we obtain:

$$\chi_{\ell,m}(\tau) = \frac{1}{\sqrt{k(k+2)}} \sum_{\ell'=0}^k \sum_{\substack{m'=-k \\ m'=k}}^k \sin\left(\pi \frac{(\ell'+1)(\ell'+1)}{k+2}\right) e^{i\pi mm'/k} \chi_{\ell',m'}\left(-\frac{1}{\tau}\right). \quad (B.5)$$

Here we have used $\eta(-\frac{1}{\tau}) = (-i\tau)^{1/2} \eta(\tau)$.

Unfortunately this technique does not work when w_1 and w_2 are not zero; the G_0 symmetry, used to enlarge the domain of $\vec{p}$ to $L_+ \cup L_-$ is only a symmetry of the spectrum and not of lattice states. Therefore we will proceed by embedding $\Gamma_{\ell,m}$ in a three dimensional lattice corresponding to $N = 2$ minimal series whose modular transformation was worked out in Ref. [16]. First for $0 \leq \ell \leq k$ and $-(k+2) \leq m \leq (k+2)$, define $\bar{\Gamma}_{\ell,m}(\tau; w_1, w_2, \zeta)$ as follows:

$$\bar{\Gamma}_{\ell,m}(\tau; w_1, w_2, \zeta) = \frac{1}{\eta(\tau)} \Gamma'_{\ell,m}(\tau; w_1, \tilde{w}_2) \theta_3(\tau; \frac{-w_2 + \zeta}{k})$$

where θ_3 is a Jacobi θ -function

$$\theta_3(\tau; z) = \sum_{n_3=-\infty}^{\infty} q^{\frac{1}{2}n_3^2} e^{2\pi i n_3 z},$$

and

$$\begin{aligned} \Gamma'_{L,m}(\tau; w_1, \tilde{w}_2) &= \frac{1}{\eta(\tau)^2} \sum_{n_1} (-1)^{2n_1} \text{sign}(n_1) q^{\frac{(k+2)}{2}(n_1 + \frac{k+1}{2(k+2)})^2 - (n_2 + \frac{m}{2(k+2)})^2} \\ &\quad \times e^{2\pi i (n_1 + \frac{k+1}{2(k+2)})w_1 - 2\pi i (n_2 + \frac{m}{2(k+2)})w_2} \end{aligned} \quad (\text{B.7})$$

Here $\tilde{w}_2 \equiv \frac{(k+2)w_2 - 2\zeta}{k}$, and the summation is over $n_1, n_2 \in \frac{1}{2}Z$ with $n_1 - n_2 \in Z$ and $|n_1 + \frac{k+1}{2(k+2)} + \frac{\text{Im} w_1}{2(k+2)} + \frac{\text{Im} \tilde{w}_2}{2\pi\tau} + \epsilon| > |n_2 + \frac{m}{2(k+2)} + \frac{\text{Im} w_2}{2\pi\tau}|$ where $\epsilon \rightarrow 0_+$. Note that the domains are shifted by $\frac{\text{Im} w_1}{2\pi\tau}$ and $\frac{\text{Im} \tilde{w}_2}{2\pi\tau}$ so that the lattice sums are absolutely convergent. Hence $\bar{\Gamma}_{L,m}$ is given by a three-dimensional lattice sum. By making the change of summation variable $n_3 = k\tilde{n}_3 + m' - 2n_2$, where $m' = 0, \dots, k-1$, and $n_2 = \tilde{n}_2 - \tilde{n}_3$, the sum becomes

$$\begin{aligned} \bar{\Gamma}_{L,m}(\tau; w_1, \tilde{w}_2, \zeta) &= \frac{1}{\eta(\tau)^3} \sum_{m'=0}^{k-1} \sum_{\substack{n_1, \tilde{n}_2 \in Z/2 \\ n_1 - \tilde{n}_2 \in Z \\ |n_1 + \epsilon| \geq |n_2|}} (-1)^{2n_1} \text{sign}(n_1) q^{\frac{(k+2)}{2}(n_1 + \frac{k+1}{2(k+2)})^2 - k(\tilde{n}_3 + \frac{m+k\tilde{n}_3}{2})^2} \\ &\quad \times e^{2\pi i (n_1 + \frac{k+1}{2(k+2)})w_1 - 2\pi i (\tilde{n}_2 + \frac{m+k\tilde{n}_3}{2})w_2} \\ &\quad \times \theta_{m+k(k+2)m', k(k+2)/2}(\frac{1}{2}k(k+2)\zeta) \end{aligned} \quad (\text{B.8})$$

where $\theta_{s,l}(z)$ was defined in (A.11). In obtaining this, we have used the fact that the $n_1, \tilde{n}_2$ sum is exactly the one appearing in the parafermion lattice sum $\Gamma_{L,m}(\tau; w_1, w_2)$ and consequently using the shift invariance of the domain to bring the latter into the standard form $|n_1 + \epsilon| \geq |n_2|$. Thus

$$\bar{\Gamma}_{L,m} = \frac{1}{\eta(\tau)} \sum_{m'=0}^{k-1} \Gamma_{L,m}(\tau; w_1, w_2) \theta_{m+k(k+2)m', k(k+2)/2}(\frac{1}{2}k(k+2)\zeta). \quad (\text{B.9})$$

$\bar{\Gamma}_{L,m}(\tau, w_1, w_2, \zeta)$ is essentially the lattice part of the correlator for the $N=2$ minimal series in NS sector. In fact, by setting $\zeta = 0$ (ζ just keeps track of the $U(1)$ quantum number) we obtain the expression for the lattice sum given in Ref. [16], and whose modular transformation was also worked out. For the sake of completeness we briefly outline the steps involved in evaluating the modular transformation of $\Gamma'(\tau; w_1, \tilde{w}_2)$. Because of the existence of light-like vectors in the lattice, the sum over the direction parallel to light

ray becomes linear and can be done explicitly. Thus splitting the region into four parts

$$\begin{aligned} L_{++} : \quad n_1 + \frac{k+1}{2(k+2)} + \frac{\text{Im}(w_1 - \tilde{w}_2)}{2(k+2)\tau_2} &\geq 0 \quad \text{and} \quad n_2 + \frac{m}{2(k+2)} \geq 0 \\ L_{+-} : \quad n_1 + \frac{k+1}{2(k+2)} + \frac{\text{Im}(w_1 - \tilde{w}_2)}{2(k+2)\tau_2} &\geq 0 \quad \text{and} \quad n_2 + \frac{m}{2(k+2)} < 0 \\ L_{-+} : \quad n_1 + \frac{k+1}{2(k+2)} + \frac{\text{Im}(w_1 - \tilde{w}_2)}{2(k+2)\tau_2} &< 0 \quad \text{and} \quad n_2 + \frac{m}{2(k+2)} \geq 0 \\ L_{--} : \quad n_1 + \frac{k+1}{2(k+2)} + \frac{\text{Im}(w_1 - \tilde{w}_2)}{2(k+2)\tau_2} &< 0 \quad \text{and} \quad n_2 + \frac{m}{2(k+2)} < 0 \end{aligned}$$

the sum over L_{++} and L_{--} can be done by changing the summation variable $n_1 = n'_1 + n_2$. The quadratic term in n_2 disappears and the sum over n_2 just gives a denominator. Putting together the contribution from L_{++} and L_{--} , one gets a sum over n'_1 from $-\infty$ to $+\infty$. Similarly by changing $n_1 = n'_1 - n_2$ in L_{+-} and L_{-+} and summing over n_2 one gets another denominator with unrestricted sum over n'_1 . The result is

$$\begin{aligned} \Gamma'_{L,m}(\tau; w_1, \tilde{w}_2, x) &= q^{-\frac{m^2}{4(k+2)}} e^{-2\pi i \frac{m}{2(k+2)} \tilde{w}_2} \sum_{n=-\infty}^{\infty} q^{\frac{(k+2)}{2}(n + \frac{k+1}{2(k+2)} + x)^2} e^{2\pi i (n + \frac{k+1}{2(k+2)} + x)w_1} \\ &\quad \times \left[\frac{1}{1 + q^{\frac{(k+2)}{2}(n + \frac{k+1}{2(k+2)} + x) - m/2} e^{-i\pi(\tilde{w}_2 - w_1)}} - \frac{1}{1 + q^{-(k+2)(n + \frac{k+1}{2(k+2)} + x) - m/2} e^{-i\pi(\tilde{w}_2 + w_1)}} \right] \end{aligned} \quad (\text{B.10})$$

where we have introduced x to carry out the Poisson summation. By making Fourier expansion, one gets an integral over x on a real line of the following form

$$\int_{-\infty}^{\infty} dx q^{\frac{(k+2)}{2}x^2} \left[\frac{1}{1 + q^{\frac{(k+2)}{2}x - m/2} e^{-i\pi\tilde{w}_2 + ixp}} - \frac{1}{1 + q^{-(k+2)x - m/2} e^{-i\pi\tilde{w}_2 + ixp}} \right]$$

where $y = x + \frac{w_1 - p}{2(k+2)}$ and p is an integer labelling the Fourier mode. Now by the residue theorem

$$\int_{-\infty}^{\infty} dx = \int_{-\infty - i\text{Im}(\frac{w_1 - p}{2(k+2)})}^{\infty - i\text{Im}(\frac{w_1 - p}{2(k+2)})} dx + 2i\pi \sum (\text{Residues}).$$

But the latter integral is zero as the integrand is odd under $y \rightarrow -y$. Taking together all the residues one gets

$$\begin{aligned} \Gamma'_{L,m}(\tau; w_1, \tilde{w}_2) &= -\frac{i}{k+2} \sum_{p=0}^{k+1} \sum_{\substack{n=-\infty-(k+2) \\ p-n \in Z}}^{k+1} \frac{\sin(\frac{\pi(k+1)(p+1)}{k+2})}{\sin(\frac{\pi(k+1)(p+1)}{k+2})} e^{-i\pi \frac{m p^2}{k+2}} \\ &\quad \times e^{-\frac{2\pi i}{k(k+2)}(w_1^2 - \tilde{w}_2^2)} \Gamma'_{L,m}(-\frac{1}{\tau}, \frac{1}{\tau}, \frac{\tilde{w}_2}{\tau}). \end{aligned} \quad (\text{B.11})$$

Using Eq. (B.6), modular transformation of θ_3 and Eq. (B.9) one gets:

$$\begin{aligned} \bar{\Gamma}_{\ell,m}(\tau; w_1, w_2, \zeta) &\sim e^{-\frac{1}{2}\pi\left(\frac{w_1^2}{k+2} + \frac{w_2^2}{k}\right)} \\ &\times \sum_{\substack{k-1 \\ \tilde{m}'=0}}^{k+1} \sum_{\substack{k+1 \\ \tilde{m}=0}}^{\tilde{m}'-(k+2)} \sum_{\substack{k+1 \\ \tilde{m}'=0}}^{\tilde{m}'-(k+2)} \sin\left(\frac{\pi(\ell+1)(\ell'+1)}{k+2}\right) e^{-i\pi\tilde{m}\tilde{m}'/(k+2)} \\ &\times \Gamma_{\ell',m+2\tilde{m}'}\left(-\frac{1}{\tau}, \frac{w_1}{\tau}, \frac{w_2}{\tau}\right) \theta_{\frac{1}{2}(k+2)\tilde{m}', k(k+2)/2}\left(\frac{1}{2}k(k+2)\zeta, -\frac{1}{\tau}\right). \end{aligned} \quad (\text{B.12})$$

One can re-express this by transforming θ appearing in the right-hand side:

$$\begin{aligned} \bar{\Gamma}_{\ell,m}(\tau; w_1, w_2, \zeta) &\sim e^{-\frac{1}{2}\pi\left(\frac{w_1^2}{k+2} + \frac{w_2^2}{k}\right)} \\ &\times \sum_{\substack{k-1 \\ \tilde{m}'=0}}^{k+1} \sum_{\substack{k+1 \\ \tilde{m}=0}}^{\tilde{m}'-(k+2)} \sum_{\substack{k+1 \\ \tilde{m}'=0}}^{\tilde{m}'-(k+2)} \sin\left(\frac{\pi(\ell+1)(\ell'+1)}{k+2}\right) e^{-i\pi\tilde{m}\tilde{m}'/(k+2)} \Gamma_{\ell',m+2\tilde{m}'}\left(-\frac{1}{\tau}, \frac{w_1}{\tau}, \frac{w_2}{\tau}\right) \\ &\times \sum_{\substack{k+1 \\ s=0}}^{\tilde{m}'-(k+2)} \sum_{\substack{k-1 \\ s'=0}}^{\tilde{m}'-(k+2)} e^{-2i\pi\frac{(\tilde{m}+1)(k+2)\tilde{m}'(\ell'+1)(k+2)\ell'}{k(k+2)}} \theta_{s+1(k+2)s', k(k+2)/2}\left(\frac{1}{2}k(k+2)\zeta\right) \end{aligned} \quad (\text{B.13})$$

As $\Gamma_{\ell,m}$ is defined for $m \bmod 2k$, let $m + 2m' = \tau \bmod 2k$ and $\tilde{m} + 2\tilde{m}' = \tilde{\tau} \bmod 2k$, one can sum over $\tilde{m}'$ and $\tilde{m}$ fixing $\tilde{\tau}$. This sum vanishes for $s \neq m \bmod (k+2)$ and for $s = m \bmod (k+2)$ one gets a factor of $(k+2)$. Thus, comparing Eq. (B.13) with Eq. (B.9) and equating various powers of $e^{2i\pi\zeta}$ one gets

$$\Gamma_{\ell,m}(\tau; w_1, w_2) = \frac{1}{\sqrt{k(k+2)}} e^{-\frac{1}{2}\pi\left(\frac{w_1^2}{k+2} + \frac{w_2^2}{k}\right)} \sum_{\substack{k+1 \\ \ell'=0}}^{\ell+1} \sum_{\substack{k-1 \\ m'=0}}^{\ell'-1} \sin\left(\frac{\pi(\ell+1)(\ell'+1)}{k+2}\right) e^{-i\pi\tilde{m}\tilde{m}'/(k+2)} \Gamma_{\ell',m'}\left(-\frac{1}{\tau}, \frac{w_1}{\tau}, \frac{w_2}{\tau}\right) \quad (\text{B.14})$$

modulo irrelevant terms that give a total derivative with respect to the screening contour, when inserted in the full correlation function.

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Fig. 1

Fig. 2