Calculating high energy particle production
according to statistical theories
on an electronic computer

by R. Hagedorn

GENEVE
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ABSTRACT

A programme for an electronic computer is described, which evaluates the results obtained by another programme for phase space calculations *). By multiplying the spectra of the different particles by appropriate weight factors and superposing them in a suitable way, one obtains automatically printed:

(i) absolute probabilities for each single reaction
(ii) mean absolute numbers for each kind of particle
(iii) total energy spectra (CM-system) for each kind of particle
(iv) the spectra of decay particles emerging from two-particle decay of unstable products of the primary interactions; including \( \gamma \)-spectra

In addition to this, part of the results are punched out once more in a form, which is accepted by an automatical graph plotter namely:

(i) the spectra as described above
(ii) a least square fit to these spectra in order to smooth out the remaining statistical fluctuations.

*) Both programme tapes, together with a detailed description for their use on Ferranti Mercury computers,
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I. INTRODUCTION

In recent years, energies of protons up to 10 GeV became available, and next year 25 GeV will be reached. Shooting such particles on nucleons at rest leads to CM-energies of 4-8 nucleon masses roughly. Baryon conservation demands that at least 2 baryons come out, the remaining energy of 2-6 nucleon masses is sufficient to produce various particles (anti-nucleons, hyperons, k-mesons, \( \bar{\pi} \)-mesons, etc.) and to give them kinetic energy.

No rigorous theory for such events exists so far and perhaps it will never exist - even if the basis of field theory would be rebuilt. This comes from the fact that the number of degrees of freedom is too large. Therefore in this energy-range (and for higher energies even more) a situation arises, where a statistical theory seems adequate. In classical physics, nobody would try to solve the equations of motion for a system of many particles, though in principle this is possible. The same will probably remain true also in high energy quantum physics.

It seems worthwhile therefore to investigate the statis-
because it is at present the only one which allows predictions so urgently needed for the planning of experiments with the big machines, and secondly because this kind of theory may remain the only one at these and higher energies.

Recent research in the statistical model has shown that it is able to reproduce some of the experimental facts very well at some energies, if it is applied with the necessary caution and if the calculations are done rigorously \(^1\) \(^2\). The type of statistical theory studied here is that of E. Fermi. In his basic paper \(^3\) many things, which are nevertheless important, have been mentioned only very briefly and not worked out in detail. Very often the theory is applied in the form given there without adding all those refinements, which make it complicated, but which are absolutely necessary.

According to Fermi, the probability for a certain final state is given by an interaction volume, raised to a certain power and then multiplied by the total statistical weight of the final state, which is essentially a phase space volume times certain factors for the statistical weight of a single particle, counting spin and isospin multiplicities.

All this is described in detail in the above quoted report \(^1\).

It was realised very soon that the main difficulty was the exact calculation of the phase space integrals, where all approximations used by various authors become either unfeasible or wrong for higher numbers of particles produced. Regarding the many attempts (in particular of experimental physicists) to fit the observed data to Fermi’s theory, it seems necessary to state that:
exists no ready formula in which one has to insert, say, energy and particle numbers and masses, and which then gives in a short calculation the answer as to what Fermi's theory predicts. All formulas published so far are valid only in special cases or for very small numbers of particles.

This explains why it is stated so frequently, that Fermi's theory gives no agreement with the experimental facts. Indeed, we could show in 3) that in the phase space integrals, errors of a factor five are easily made, even with what one would tend to consider a good approximation. Frequently, factors as those given by Yevnin and de Shalit 5) and Barbašev and Barbašev 6) which count the number of final isospin states consistent with conservation of isospin and charge and factors as \((\prod N_i !)^{-1}\) compensating for the indistinguishability of particles, are simply omitted, which together with the errors in phase space calculation leads to figures that in fact have nothing to do with the spirit of Fermi's paper.

The "striking simplicity" of Fermi's theory does in fact not exist and this theory cannot be blamed for failing the experimental facts until it is carried through rigorously.

The first thing to do was to find a reliable method for calculating the phase space integrals, even for high particle numbers, and with the exact relativistic relation \(E = \sqrt{p^2 + m^2}\) between energy and momentum.

A Monte Carlo method was developed 4) which allows to calculate phase space integrals up to 16 particles with any prescribed statistical error.
that the reactions at 25 GeV for instance can be calculated with less than 2\% probable error in reasonable time.

This programme yields, not only the phase space integral proper, but also the spectra of the various particles in the form of histograms. On the other hand, there is always a large number of final states and the above information is punched out for each final state separately. An example shows to where this leads:

In a P-P collision at 25 GeV primary lab. energy, many hundred final states (counting only states with different particle numbers) are possible, the mean number of different kinds of particles being about 4. Then for each such final state, the MC-calculation yields the phase space density $C^*$ and in addition 4 spectra and mean energies, each being represented by 32 numbers. That is, for each final state, one obtains about 130 numbers - and that many hundred times. Obviously this large amount of data cannot be processed by desk computing.

In the above quoted paper 1), we had such a low energy, that the number of final states was small and the evaluation on a desk computer was still feasible, but the example shows clearly that at higher energies this would be impossible.

Therefore it was felt necessary to develop a programme which is able to read in the punched tapes containing the results of the MC-calculation, and to reduce this large amount of data to a small one containing only information which has a direct relation to measurable quantities.

Such a programme would reduce the time necessary for the

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1) Reference to the original paper is needed to complete the sentence.
results (which are that part of the theory which is invariant under change of philosophy) with different assumptions about how to represent that part of the matrix element squared, which remains after splitting off the phase space factor. The evaluation programme should therefore be as flexible as possible and allow to interpret the MC-data in different ways.

The present programme seems to meet these requests. It carries through simultaneously in one single calculation three different versions of the "interaction volume" and can combine the data once fed in several times differently without the need of reading them in again.

The programme may thus be used for different tasks, e.g., also for N-N-annihilation, or also for models which are more complicated than the Fermi model, as for instance the "excited nucleon model" 8, 9).

In the following, part II, the programme is described by saying what it does, without going into any detail.

The details of the programme as well as a prescription for users (preparation of interpreting data, running of the programme, layout of final results, etc.) will be given in part III, and will be published separately.

Part III will be rather technical and be useful only for Mercury owners. This part, as well as the description of the MC-programme will be available at CERN on request. The programme tapes of both programmes, also are available and the written descriptions should be sufficient to allow even non-specialists to use the programmes on any Ferranti-Mercury computer and to carry out calculations according to the Fermi
II. THE PROGRAMME

1) What the programme ought to do:

Suppose, we have obtained results from the Monte Carlo (MC) calculation of phase space integrals and spectra. They are given in a form, which has been described in detail in 7). After a small modification, these results are readable by Mercury and it is the present programme which evaluates them in physical terms.

What the programme has to do, is the following:

Consider a particle of kind \( i \) \((i = 1, \ldots, N = \) number of kinds of particles occurring in the final state\), and a particular reaction \( b \) (in the MC-programme, \( b \) is the number of the case). To be definite, let us take

\[
N + N \rightarrow N + Y + K + 5\pi
\]

This is only one of many possible reactions, and \( b \) numbers these reactions, whereas \( i \) numbers the kinds of particles. For instance \( i = 1 \) the nucleons, \( i = 2 \) hyperons, \( i = 3 \) pions etc.

Now each of these reactions has a certain probability \( P_b \), which is proportional to

(i) the phase space density \( \sigma_b^* \)

(ii) a certain weight factor:

\[
f = \prod_i (2S_i + 1) \sigma_j
\]
Here \( \mathcal{W}_{\alpha_1, \alpha_2}^{(T)} \) counts the number of isospin states of total isospin \( T \), which can be built up from \( \alpha \) particles of isospin \( \frac{1}{2} \), \( \beta \) particles of isospin 1 and \( \gamma \) particles of isospin 3/2.

These factors have been computed and published by several authors \( 5 \), \( 6 \). They account for isospin and charge conservation.

The other factor \( \frac{\Pi(2S_1+1)}{\Pi N_1} \) is for the multiplicity of ordinary spin states and for the indistinguishability of equal particles, thus \( \sigma_j \) is the number of particles of spin \( j = \frac{1}{2}, 1, \ldots, 5/2 \).

\( N_1 \) is the number of equal particles (here particles of one isospin multiplet are considered as equal, e.g., if there is one \( \pi^+ \), one \( \pi^0 \), one \( \pi^- \), then \( N_1 = 3 \)).

(iii) A so-called "interaction volume", which represents a mean value of the transition matrix element squared. It is not the place here to discuss its physical implications. We shall use it in a form, which gives more freedom, namely

\[
\Omega_1 \cdot \Omega_2 \cdot \Omega_3
\]

which allows one to assign different coupling strengths to different processes. One can then try out various combinations.
If for a given energy all final states are considered, i.e. if \( b \) runs through all possible reactions, then in the theory underlying this programme, it is assumed that the probability for the final state \( b \) is given by:

\[
P_b = \frac{\left\{ \mathcal{Q}^* \cdot \frac{W_{\alpha \beta y}(T)}{N_{b'i}} \frac{(2S_{i}+1)\sigma^j\gamma}{N_{i}!} \right\}_b}{\sum_b \left\{ \mathcal{Q}^* \cdot \frac{W_{\alpha \beta y}(T)}{N_{b'i}} \frac{(2S_{i}+1)\sigma^j\gamma}{N_{i}!} \right\}_b} = \frac{P'_b}{\sum_b P'_b}
\]

(1)

where \( P'_b \) is the numerator of the above expression.

This quantity, \( P'_b \) is easily calculated, once \( \mathcal{Q}^* \), \( Q_1^* \cdot Q_3^* \) and the other factors are given. The \( W_{\alpha \beta y} \) can be read into the magnetic store and are there available as a table.

The procedure for a particle \( i \) in a reaction \( b \) would then be:

(i) Read in the \( \mathcal{Q}^* \) and spectrum of that particle as they resulted from the MC-calculation;

(ii) Multiply the whole spectrum by \( P'_b \) and by the number \( \ell_i \) of particles it represents (the MC-spectra are all normalized to one, whatever the number of particles represented may be).

(iii) Add the weighted spectrum to the other weighted spectra of the particle \( i \).
(iv) Print out $P'_{b}$ and add $P'_{b}$ to $\sum_{b} P'_{b}$.

When all cases $b$ have been read into the machine, we have the following situation:

A sector of the magnetic drum can contain 32 floating point numbers, that is just one whole spectrum. Let all the particle - $i$ - spectra be added up in a sector $S_{i}$:

Sector $S_{i}$ contains the weighted total spectrum of particle $i$

$$W_{i}(\mathcal{E}) = \sum_{b} \ell_{i}^{(b)}, P'_{b}, w_{i}^{(b)}(\mathcal{E})$$

(2)

Here $\ell_{i}^{(b)}$ and $P'_{b}$ are explained above. $W_{i}^{(b)}(\mathcal{E})$ is the normalized MC-spectrum of particle $i$ in the reaction $b$. Furthermore somewhere all $P'_{b}$ are added and their sum $\sum P'_{b}$ is available.

We can then calculate the following quantities:

a) the absolute probability of any given set $B$ of reactions:

$$P(B) = \frac{\sum_{b \in B} P'_{b}}{\sum_{b} P'_{b}}$$

(3)
This can be done using the printed values $P'_b$ and $\sum_b P'_b$ by means of hand-computing. This is simple and should not be contained in the general programme, since that is far more complicated than to do it on a sheet of paper.

b) The absolute mean number of particles $i$ per event:

$$\langle l_i \rangle = \frac{\sum_b l_i^{(b)} P'_b}{\sum_b P'_b} = \frac{1}{\sum_b P'_b} \int w_i(\varepsilon) d\varepsilon$$

(4)

This is done by the programme (remember $\int w_i^{(b)}(\varepsilon) d\varepsilon = 1$)

c) The mean kinetic energy of the particle $i$

$$\langle \varepsilon_i \rangle = \frac{\int \varepsilon w_i(\varepsilon) d\varepsilon}{\int w_i(\varepsilon) d\varepsilon}$$

This is also done by the programme.

d) The normalized kinetic energy spectrum of particle $i$

$$w_i(\varepsilon) = \frac{W_i(\varepsilon)}{\int w_i(\varepsilon) d\varepsilon}$$

This is also done by the programme.
e) The quantities (b), (c), (d) can be calculated also for the decay products of unstable particles, if these decay into two other particles, which then may again decay and so on.

By a suitable grouping of spectra during their reading in, one may be able also to calculate the spectra and probabilities of certain final states, which are not specified in terms of particles, but rather in terms of numbers of prongs— which might be useful for application to nuclear emulsion data. One also might treat some types of final state interactions, e.g., the annihilation of pairs formed in the primary interaction.

It seems that the quantities calculated above cover essentially all measurable ones. There is of course no angular distribution, since the physical model assumes isotropy. The programme can, however, be used for the treatment of models in which moving excited nucleons emit particles isotropically in their rest system. This would then lead to predictions of angular distribution according to the details of the model considered.

This is the principle— it remains to add a few refinements:

(i) One will often be interested in trying different assumptions on the interaction volumes (matrix elements). Since the time for reading in MC-results and punching out final results is by far longer than the time needed for the calculations proper, it seemed reasonable to do all calculations a few times at once; we choose 1200 times. That was practical, on the condition that...
of interaction volumes:

\[ \Omega_1 ; \Omega_2 ; \Omega_3 \]
\[ \Omega' ; \Omega'_2 ; \Omega'_3 \]
\[ \Omega'' ; \Omega''_2 ; \Omega''_3 \]

and all results (a) ... (e) are printed accordingly in three versions. If one wishes more versions, the whole calculation must be repeated with new reading in etc. This then yields three more versions and so on.

(ii) Although the final spectra are weighted sums over many contributing MC-spectra, there will always be some to which not too many MC-spectra contribute - e.g., for anti-nucleons. They will then still show some statistical fluctuations due to the MC-process, whereas those spectra to which many MC-spectra contribute, will be smooth - e.g., for \( \pi \) and \( \eta \).

Therefore the programme contains a part in which the non-vanishing (lower energy -) part of the spectra is smoothed out by a least square Fourier analysis.

\[ \omega(\xi) \]

\[ \text{Fig. 1} \]
The forms of the spectra vary very strongly. A spectrum as (1) in figure 1 is hard to represent by a few sin-waves, whereas (3) is quite easy. We first transform the MC-calculated curve by multiplying it by \( \varepsilon^\alpha \), where \( \alpha \) is found in the following way:

We find for each spectrum separately the position \( \chi_0 \) of the maximum and the position \( n \) of the lowest energy cell above which the spectrum vanishes practically. We put then for all \( \chi_0/n < 0.5 \) (> 0.5 does practically not occur)

\[
\alpha = (1 - 2 \frac{\chi_0}{n})^\beta
\]

(see Fig. 2)

and first multiply the spectrum by \( \varepsilon^\alpha \), then approximate it by

\[
F(\varepsilon) = \sum_{\mu=1}^{N} a_{\mu} \sin \frac{\mu \pi}{n} \varepsilon
\]

between 0 and \( n \) and finally transform back by multiplying \( F(\varepsilon) \) by \( \varepsilon^{-\alpha} \). By trial, the optimal \( \beta \) was found to be \( \approx 2 \).

\[\text{Fig. 2}\]
By this transformation, all spectra are first brought into a form which is easy to represent by a few sin-terms. One should not go to high orders in interpolation, since then the fluctuations would be represented too.

In fact, the order of the interpolating sin-series is chosen accordingly to how many points of the spectrum really contribute. The order $N$ is taken to be half the number of contributing points, but not larger than 7.

One can also interpolate everything with a 5th order series, simply by setting a handswitch. This will be done if one knows that the statistics are bad and a 7th order interpolation would show waves in the spectrum, which are of statistical origin and should be suppressed.

This method showed sufficiently good reproduction when tried on given smooth curves of the types of Fig. 1. It breaks down for every shape which strongly deviates from the above type. In any case there are also plotted the original MC-spectra and one can see by visual inspection whether the smooth curve can be accepted or not.

2) How the calculation is carried out:

There are two types of information given to the computer. One is the output of the MC-calculation. The other one contains the necessary data which tell the computer how to interpret the results. These data are read in three steps. Three different
data tapes are prepared:

a) The main tape, containing three sets of interaction volumes and the total isospin. These data remain the same for the whole calculation.

b) Next the MC-results are read in. But since the different final states contain different kinds and numbers of particles, each case has its particular number of spectra and the machine cannot possibly know neither how many spectra will come next, nor how to interpret them. Therefore each single case of the MC-results will be preceded by a tape, which contains all the necessary information for the machine, namely:

- the number of the case (to check against the number of the MC-case).
- the numbers \(\lambda, \mu, \nu\) of particles with isospins \(\frac{1}{2}, 1, 3/2\).
- the numbers of particles with spin \(\frac{1}{2} \ldots 5/2\).
- the numbers of equal particles.
- the numbers \(n_1, n_2, n_3\), giving the powers of \(\Omega_1^{n_1} \Omega_2^{n_2} \Omega_3^{n_3}\)

(e.g. \(n_1\) for nucleons and pions, \(n_2\) for hyperons and \(n_3\) for K-mesons)

This is enough in order to calculate the relative probability \(P_b\) up to \(\delta^*\), which will be read in.

Next the number of spectra to follow is specified and this enables the machine to read them into the common store. They are headed by \(b\) (the case
an alarm if the b given above and that read in are different. \( \mathcal{S}^* \) is multiplied by the other factors and thus \( P_b' \) is complete.

Furthermore there are specified for each spectrum which one wishes to include three numbers, namely:

\[ p, \mathcal{S}, \text{ and } z, \]

where \( p \) denotes the location of the spectrum in question, \( \mathcal{S} \) denotes the sector number where it has to be added and \( z \) is a number by which it will be multiplied before being added. Automatically, \( z \) will be multiplied by \( P_b' \) such that the quoted spectrum will be added with its correct weight. In general \( z \) will be the number of particles represented by the spectrum, but it need not be integer. For instance, if one wishes to carry out a charge analysis, one will multiply a spectrum with the appropriate charge probability (see ref. 1). The same spectrum may be interpreted as often as desirable.

Then the next WC-case is read in and so on. In this way, one adds up all spectra of particle \( i \) coming from the different final states in one (or if desired, e.g. for charge analysis in several different) sector \( \mathcal{S}_i \) of the magnetic drum store. After all information has been read in, the spectra of all kinds of particles are present there properly weighted.

c) Now the third stage begins. A new tape for the final calculation is read in. On this tape, one specifies
are six numbers given to the machine. It calculates then the decay spectra for $\mu^* \rightarrow M + \mu$ assuming that the decaying spectrum is on sector $\mathcal{J}^*$ and the resultant spectra are to be placed on sectors $\mathcal{J}_M$ and $\mathcal{J}_\mu$.

This may be repeated as often as necessary. In particular, any spectrum resultant from a decay may decay itself again, e.g.,

$$N^* \rightarrow N + \pi^0 \rightarrow N + \gamma + \gamma.$$

After all decays have taken place, one may still superpose the resultant spectra in any arbitrary way:

$$\mathcal{J}_1, \ z_1, \ \mathcal{J}_2, \ z_2, \ \mathcal{J}_3$$

are five numbers whose reading causes the machine to do the following operation: Spectrum on $\mathcal{J}_1$ is multiplied by $z_1$, spectrum on $\mathcal{J}_2$ by $z_2$; they are added and the result is placed on sector $\mathcal{J}_3$. In this operation the spectra on $\mathcal{J}_1$ and $\mathcal{J}_2$ are preserved.

This again can be done as often as desired.

Once the spectra are read in, all the operations are done without further order three times at once – carrying out the three versions given by the three sets of interaction volumes. Therefore one spectrum will not occupy one sector as assumed so far for simplicity, but three neighbouring sectors.
once and the sector number given is that of the first of the three sectors which belong together.

At the end of the tape the machine finds two sector numbers, namely the first and the last of those, which are considered as keeping the final results.

Reading these two numbers causes the machine to do, with all sectors between and including them, the following operations:

\( \alpha \) Normalizing. The sum over the spectrum is divided by the sum of all \( P_i \) and that yields according to (4) [P.10] the mean particle number of that kind.

\( \beta \) The mean kinetic energies are calculated and placed in the last register of the corresponding sector.

\( \gamma \) The mean particle numbers, normalized spectra and the mean kinetic energies are punched out three by three together for printed layout.

\( \delta \) The smoothing out by the above described least square Fourier series takes place and the results are punched again together with the interpolating smooth curve in a form which is directly accepted by an automatic graph plotter.

This whole calculation is a matter of less than one hour and most of the time is taken by reading and punching data.
ACKNOWLEDGEMENT

This work grew out of what was done earlier by Dr. F. Cerulus and the author in common, using a desk calculating machine.

The author is grateful to the members of the CERN computer group, in particular to Dr. Lipps, for help and discussions.
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