PATH INTEGRALS AND SUPERCOHERENT STATES

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ABSTRACT

For the real supergroup $Osp(1|2 : \mathbb{R})$ with both its compact and non-compact versions, supercoherent states are introduced with a method close to the one by Perelomov for the even subgroups $SU(2)$ or $SU(1, 1)$. These states labelled by a complex c -number and Grassmann variable minimize the uncertainty of the quadratic Casimir operator of the group. A path integral formalism is developed for the transition amplitude between supercoherent states for a Hamiltonian linear in the generators of the superalgebra, which leads to a super-Riccati equation. Finally, in the classical limit the canonical equations of motion are derived which involve a generalized super-Poisson bracket.

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I. Introduction

Coherent states¹ are one of the most important elements in quantum physics since its early beginning². They have been used in almost every area of quantum mechanics and they have facilitated the understanding of the correspondence with classical mechanics^{3,4}. Their generalization to groups other than the Heisenberg-Weyl (H-W) group is well established^{5,6,7} and widely used^{8,9}. Of particular importance is the connection of coherent states to path integrals^{3,10} for the H-W group and the $SU(2)$ ¹¹ and $SU(1,1)$ groups¹². Recently coherent states for supergroups have been introduced¹³⁻¹⁶. They are states defined using Perelomov's definition of coherent states for arbitrary group to the case of supergroups and they are labelled by coordinates of a supercoset space. Also they have the property of being the minimum uncertainty states in the sense that they minimize the dispersion of the quadratic Casimir operator of the group.

In this work we first introduce coherent states for the supergroup $Osp(1|2; \mathbb{R})$ in its compact version, with $SU(2)$ being the even subgroup and noncompact form, with $SU(1,1)$ as the even subgroup. The method (Sect. II) we use to do this is close to the method by Perelomov which defines coherent states as the orbit of a "vacuum state" obtained by the action of a certain group element (displacement operator) on the "vacuum". In our derivation of the overlap of two coherent states and of the action of an arbitrary group element on the coherent state, we need only the fundamental 3×3 matrix representation of the group. Our coherent states coincide with those of Ref. 13 and upon restriction to the even subgroup reduce to the usual $SU(1,1)$ ⁵ and $SU(2)$ ^{8,9} coherent states. The property of any coherent state being the most classical one is here manifested for the supercoherent states of $Osp(1|2; \mathbb{R})$ as well. Specifically the quantum fluctuations of the expectation value between these states of the Casimir operator of the group takes its minimum and so $Osp(1|2; \mathbb{R})$ coherent states are the minimum-uncertainty states¹⁷ (Sect. III).

Second, the transition amplitude between two supercoherent states for a Hamiltonian which is a linear combination of the group generators is considered and using the resolution of unity of the coherent states we arrive at an expression which is usually interpreted as the path integral for the propagator. However, since the path integration for coherent states other than those of the H-W group is not a straightforward multiple Gaussian integration, we use an alternative method. The evolution of the system from its initial coherent states is considered, by taking the successive actions on it, for the $Osp(1|2; \mathbb{R})$ group elements for infinitesimal time evolutions. Using the group composition rule for these actions, one involves matrix multiplications of the 3×3 images of the group elements and thus we arrive at a recursion relation for the group parameters, which in the limit of infinite time resolution becomes a coupled super-Riccati equation. Such equations have been studied before¹⁸ and has it been shown to be possible to turn them to a system of equations which contains an ordinary Riccati equation decoupled from the rest of the equations, which are linear and easy to solve once we have a solution of the Riccati equation itself. Again reducing the whole structure to the even subgroups we obtain the results for $SU(1,1)$ ¹⁹ and $SU(2)$ coherent states path integrals.

Finally, the classical limit yields canonical equations of motion in a curved super-

phase space which is a Grassmannian extension of the sphere^{11,20} for the compact group and a Grassmannian extension of the hyperboloid^{12,20} for the case of the noncompact $Osp(1|2; \mathbb{R})$ group (Sect. IV).

II. Construction of coherent states

We shall first describe a general approach to the construction of coherent states for the case of a Lie group G . Our approach is close to the one described in Ref. 5. The extension of it to the case of supergroups is given in the next section.

Let $T(g), g \in G$ be a unitary irreducible representation of the group G in the Hilbert space $\mathcal{H}$:

$$T(g_1)T(g_2) = T(g_1g_2), \quad T(g^{-1}) = T^\dagger(g), \quad (2.1)$$

where $g, g_1 \in G$. Let $|x_0\rangle$ be a fixed vector (called ground state) in $\mathcal{H}$. We define the function $w(g)$ as

$$w(g) = \langle x_0 | T(g) | x_0 \rangle \quad (2.2)$$

and for $w(g) \neq 0$ we define

$$d(g) = \frac{\omega(g)}{|w(g)|}. \quad (2.3)$$

Let H be the stability group of the vector $|x_0\rangle$, i.e. H is the maximal subgroup with the property

$$T(h)|x_0\rangle = d(h)|x_0\rangle, \quad (2.4)$$

where $h \in H$. The function $d(h)$, $h \in H$, is a unitary character (phase factor) of the group H . The function $w(g)$ has the important properties

$$\omega(gh) = \omega(g)d(h),$$

for $g \in G$, $h \in H$ and

$$\Sigma a_i a_j \omega(g_i^{-1} g_j) = \|\Sigma a_i T(g_i)|x_0\rangle\|^2 \geq 0. \quad (2.5)$$

To any coset element $z \in G/H$ we assign a coherent state $|z\rangle \in \mathcal{H}$ as follows: we choose a representative $g_z \in G$ corresponding to z and we define,

$$T(g_z)|x_0\rangle = d(g_z)|z\rangle. \quad (2.6)$$

If we choose a different representative $g'_z = g_z h$, then the phase factor in eq.(2.6) is changed as

$$T(g'_z)|x_0\rangle = d(g_z h)|z\rangle = d(h)d(g_z)|z\rangle. \quad (2.7)$$

The operator $T(g)$ acts on the coherent state $|z\rangle$ as

$$T(g)|z\rangle = d^{-1}(g_z)d(g_z)|g \cdot z\rangle, \quad (2.8)$$

where $g \cdot z$ denotes the coset obtained by the action of g on z . The r.h.s. of eq.(2.8) does not depend on the choice of g_z . The proof of eq.(2.8) is simply as:

$$\begin{aligned} T(g)|z\rangle &= d^{-1}(g_z)T(g)T(g_z)|x_0\rangle \\ &= d^{-1}(g_z)T(gg_z)|x_0\rangle = d^{-1}(g_z)d(gg_z)|g \cdot z\rangle, \end{aligned}$$

where we have used the relation $gg_z x_0 = g z$ (here x_0 denotes a coset corresponding to the unity in eq.(2.6)).

The scalar product of two coherent states,

$$\begin{aligned} \langle x|y\rangle &= d(g_z)d^{-1}(g_y) \langle x_0|T^{-1}(g_z)T(g_y)|x_0\rangle \\ &= d(g_z)d^{-1}(g_y) \langle x_0|T(g_z^{-1}g_y)|x_0\rangle \\ &= d(g_z)d^{-1}(g_y)\omega(g_z^{-1}g_y), \end{aligned} \quad (2.9)$$

does not depend on the choice of representatives g_z, g_y of the cosets z and y . The same is true for the coherent state matrix element,

$$\langle x|T(g)|y\rangle = d(g_z)d^{-1}(g_y)\omega(g_z^{-1}gg_y). \quad (2.10)$$

As an example we study the case of the groups $SU(2)$ and $SU(1,1)$.

The group $G = SU(2)$ in its fundamental representation consists of complex matrices of the form

$$g = \begin{pmatrix} u & v \\ \mp v^* & u^* \end{pmatrix}, \quad |u|^2 \pm |v|^2 = 1. \quad (2.11)$$

Throughout the text double signs e.g. $\pm$ imply unless otherwise stated, that the upper sign corresponds to the compact group or algebra ($SU(2)$, $Osp(1|2; \mathbf{R})$) and the lower sign to its noncompact counterpart. These matrices act in the space of vectors $\psi = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbf{C}^2$ by the simple multiplication

$$\psi \mapsto g\psi, \quad \psi^* \mapsto \psi^* g^*, \quad (2.12)$$

where the star denotes the Hermitian conjugate. This action leaves invariant the sesquilinear form

$$|a|^2 \pm |b|^2,$$

i.e. the group G acts in $\mathbf{C}^2$ on orbits

$$|a|^2 \pm |b|^2 = \text{const.}$$

Let us choose the orbit $|a|^2 \pm |b|^2 = 1$. On this orbit we can introduce a homogeneous coordinate $z = a^{-1}b$ (we assume the chart $a \neq 0$). The group action,

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} u & v \\ \mp v^* & u^* \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ua + vb \\ \mp v^* a + u^* b \end{pmatrix},$$

induces a Möbius transformation for the homogenous coordinate of the projective space, $SU(2)/U(1) \approx S^2$, $SU(1,1)/U(1) \approx S^{1,1}$:

$$z \rightarrow z' = \frac{u^* z \mp v^*}{\mp z + u}. \quad (2.13)$$

The function $\omega(g)$ we can choose as

$$\omega(g) = (u^*)^{\pm 2\tau}, \quad (2.14)$$

where 2τ is an integer and τ is the Casimir (Bargmann) index. One can easily show that

$$\omega(gh) = \omega(g)d(h) = \omega(g)e^{2i\tau\varphi}, \quad (2.15)$$

for all matrices of the kind

$$h = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}. \quad (2.16)$$

To any complex z , we assign a group element

$$g_z = \exp(zK_+) \exp(sK_0) \exp(-z^*K_-), \quad (2.17)$$

where

$$K_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad K_- = \begin{pmatrix} 0 & 0 \\ \pm 1 & 0 \end{pmatrix}, \quad K_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.18)$$

are the generators of the $su(2)$ ($su(1,1)$) Lie algebra. We have,

$$g_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mp z^* & 1 \end{pmatrix} = \begin{pmatrix} e^{s/2} \mp z z^* e^{-s/2} & z e^{-s/2} \\ \mp z^* e^{-s/2} & e^{-s/2} \end{pmatrix}. \quad (2.19)$$

This is an element of the $SU(2)$ ($SU(1,1)$) group provided

$$z' = z, \quad e^s = 1 \pm |z|^2. \quad (2.20)$$

For g_z we obtain then a simple expression

$$g_z = (1 \pm |z|^2)^{-1/2} \begin{pmatrix} 1 & z \\ \mp z^* & 1 \end{pmatrix} \quad (2.21)$$

and we see that $d(g_z) = 1$.

To any z , is assigned a coherent state according to eq.(2.6),

$$|z\rangle = T(g_z)|x_0\rangle, \quad (2.22)$$

i.e. the ground state is assigned to $z = 0$.

Taking g in the forms (2.11) and using eq.(2.21) we obtain

$$gg_* = (1 \pm |z|^2)^{1/2} \begin{pmatrix} \mp vz^* + u & uz + v \\ \mp u^* z^* + v^* & \mp v^* z + u^* \end{pmatrix}. \quad (2.23)$$

The eq.(2.8) then gives

$$T(g)|z\rangle = \left(\frac{u^* \mp v^* z}{u \mp vz^*} \right)^{\pm r} \left| \frac{u^* z \mp v^*}{vz + u} \right\rangle. \quad (2.24)$$

The inverse matrix element g_*^{-1} is given as

$$g_*^{-1} = (1 \pm |z|^2)^{-1/2} \begin{pmatrix} 1 & -z \\ \pm z^* & 1 \end{pmatrix}. \quad (2.25)$$

The eqs. (2.10), (2.14), (2.23) and (2.25) then give

$$\langle z_1 | T(g) | z_2 \rangle = \omega(g^{-1} g g_*) = \left(\frac{u^* \pm vz_1^* \mp v^* z_2 \pm uz_1^* z_2}{\sqrt{1 \pm z_1^* z_1} \sqrt{1 \pm z_2^* z_2}} \right)^{\pm 2r}. \quad (2.26)$$

The overlap of two normalized coherent states is obtained by setting $u = u^* = 1$ and $v = v^* = 0$:

$$\langle z_1 | z_2 \rangle = \left(\frac{1 \pm z_1^* z_2}{\sqrt{1 \pm z_1^* z_1} \sqrt{1 \pm z_2^* z_2}} \right)^{\pm 2r}. \quad (2.27)$$

III. Coherent states for the supergroup $osp(1|2; \mathbf{R})$

The group $osp(1|2; \mathbf{R})$ ^{21,22,23} is formed in the fundamental representation by 3×3 matrices of the form

$$g = \begin{pmatrix} u & v & \mp \theta \\ \mp v^* & u^* & -\bar{\theta} \\ \chi & \mp \bar{\chi} & \lambda \end{pmatrix}, \quad (3.1)$$

where $u, v, u^*, v^*, \lambda = \lambda^*$ are even elements of some complex Grassmann algebra Λ , whereas $\theta, \bar{\theta}, \chi, \bar{\chi}$ are odd elements of Λ . The star $*$ is an involution in Λ with the property

$$(\alpha\beta)^* = \beta^* \alpha^* \quad (3.2)$$

and it means Hermitian conjugation.

The matrix (3.1) should obey the conditions

$$g \Sigma g^{ST} = \Sigma, \quad \text{where } \Sigma = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.3)$$

and

$$g^* \tilde{\Sigma} g = \tilde{\Sigma}, \quad \text{where } \tilde{\Sigma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.4)$$

The supertranspose and the conjugate of a graded matrix $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ are defined correspondingly as²⁴:

$$g^{ST} = \begin{pmatrix} A^T & C^T \\ -B^T & D^T \end{pmatrix} \quad (3.5)$$

and

$$g^* = \begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix}. \quad (3.6)$$

The condition (3.3) characterises the $osp(1|2; \mathbf{C})$ group and eq.(3.4) specifies its restriction to the real form $osp(1|2; \mathbf{R})$.

Imposing the condition (3.3) on the group element in eq.(3.1), we obtain the relations

$$uu^* \pm vv^* \mp \theta \bar{\theta} = 1, \quad (3.7a)$$

$$\mp u \bar{\chi} - v \chi \mp \lambda \theta = 0, \quad (3.7b)$$

$$-u^* \chi + v^* \bar{\chi} - \lambda \bar{\theta} = 0, \quad (3.7c)$$

$$\lambda^2 \mp 2\chi \bar{\chi} = 1. \quad (3.7d)$$

These relations allow to express $\chi, \bar{\chi}$ and $\lambda = \lambda^*$ in terms of u, v and θ as follows:

$$\chi = -u \bar{\theta} - v^* \theta, \quad (3.8a)$$

$$\bar{\chi} = -u^* \theta \pm u \bar{\theta}, \quad (3.8b)$$

$$\lambda = 1 \pm \chi \bar{\chi} = 1 \pm \bar{\theta} \theta. \quad (3.8c)$$

We see that the elements of the $osp(1|2; \mathbf{R})$ group are determined by the parameters u, v, θ subjected to the constraint (3.7a).

Now we turn to the algebra $osp(1|2; \mathbf{R})$ the generators of which in their fundamental representations are given by the matrices

$$K_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_- = \begin{pmatrix} 0 & 0 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$F_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \mp 1 \\ 0 & 0 & 0 \\ 0 & \mp 1 & 0 \end{pmatrix}, \quad F_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (3.9)$$

The $K_{\pm 0}$ are the even generators in $osp(1|2; \mathbf{R})$ and $F_{\pm}$ are the odd ones. They satisfy the commutation/anticommutation relations:

$$[K_0, K_+] = K_+, \quad [K_0, K_-] = -K_-, \quad [K_+, K_-] = \pm 2K_0, \quad (3.10a)$$

$$[K_0, F_+] = \frac{1}{2}F_+, \quad [K_0, F_-] = -\frac{1}{2}F_-, \quad [K_+, F_+] = 0, \quad [K_-, F_-] = 0, \quad (3.10b)$$

$$[K_-, F_+] = F_-, \quad [K_+ F_-] = \pm F_+ \quad (3.10c)$$

and

$$\{F_+, F_+\} = K_+, \quad \{F_-, F_-\} = \mp K_-, \quad \{F_+, F_-\} = \mp K_0. \quad (3.10d)$$

Denoting the Casimir indices by τ for the $osp(1|2; \mathbf{R})$ Casimir operator and by k for the $SU(2)$, $SU(1, 1)$ Casimir operator, we have two irreducible representations labelled by $k = \tau$ and $k = \tau \pm \frac{1}{2}$ for any given integer or halfinteger value of τ . If we consider for the $SU(1, 1)$ the discrete series of representations only, then the states are labelled by τ, k and by the eigenvalue of the K_0 generator m . Then we obtain the following matrix realizations for the Casimir operators:

$$C_2(osp(1|2; \mathbf{R})) = \tau \left(\tau \pm \frac{1}{2} \right) 1, \quad (3.11a)$$

$$C_2(SU(2); SU(1, 1)) = k(k \pm 1)1. \quad (3.11b)$$

Similarly for the generators of the compact algebra:

$$K_0 = \sum_{m=-k}^k m |\tau; k, m\rangle \langle \tau; k, m|, \quad (3.12a)$$

$$K_{\pm} = \sum_{m=-k}^k \sqrt{(k \pm m + 1)(k \mp m)} |\tau; k, m \pm 1\rangle \langle \tau; k, m|, \quad (3.12b)$$

$$F_{\pm} = \mp \frac{1}{\sqrt{2}} \sum_{m=-k}^k \sqrt{\tau \mp m} |\tau; k = \tau - \frac{1}{2}, m \pm \frac{1}{2}\rangle \langle \tau; k = \tau, m|, \quad (3.12c)$$

or

$$F_{\pm} = -\frac{1}{\sqrt{2}} \sum_{m=-k}^k \sqrt{\tau \pm m + \frac{1}{2}} |\tau; k = \tau, m \pm \frac{1}{2}\rangle \langle \tau; k = \tau - \frac{1}{2}, m|, \quad (3.12d)$$

while for the noncompact case we have:

$$K_0 = \sum_{m=k}^{\infty} m |\tau; k, m\rangle \langle \tau; k, m|, \quad (3.13a)$$

$$K_{\pm} = \sum_{m=k}^{\infty} \sqrt{(m \pm k)(m \mp k \pm 1)} |\tau; k, m \pm 1\rangle \langle \tau; k, m|, \quad (3.13b)$$

$$F_{\pm} = \frac{1}{\sqrt{2}} \sum_{m=k}^{\infty} \sqrt{m \mp \tau \pm \frac{1}{2}} |\tau; k = \tau, m \pm \frac{1}{2}\rangle \langle \tau; k = \tau + \frac{1}{2}, m|, \quad (3.13c)$$

or

$$F_{\pm} = \frac{1}{\sqrt{2}} \sum_{m=k}^{\infty} \sqrt{m \pm \tau} |\tau; k = \tau + \frac{1}{2}, m \pm \frac{1}{2}\rangle \langle \tau; k = \tau, m|. \quad (3.13d)$$

The elements of $osp(1|2; \mathbf{R})$ are linear combinations of the matrices of eq.(3.9) with coefficients from Λ of the type

$$H = \alpha_+ K_+ + \alpha_- K_- + \alpha_0 K_0 + \theta_+ F_+ + \theta_- F_-, \quad (3.11)$$

where $\alpha_{\pm}$, α_0 are even elements in Λ , while $\theta_{\pm}$ are odd elements in Λ . We shall assume that all generators commute with the coefficients, i.e. $osp(1|2; \mathbf{R})$ is a Grassmann envelope of the first kind²⁴.

Any element of the group $osp(1|2; \mathbf{R})$ is the exponent of some element (3.11):

$$\exp H = 1 + \frac{1}{1!} H + \frac{1}{2!} H^2 + \dots \in osp(1|2; \mathbf{R}). \quad (3.12)$$

We will frequently need the formulae

$$\exp(sK_0) = \begin{pmatrix} e^{s/2} & 0 & 0 \\ 0 & e^{-s/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.13)$$

$$\exp(-z''K_- - \bar{\varphi}'F_-) = \begin{pmatrix} 1 & 0 & 0 \\ \mp z'' & 1 & \frac{1}{\sqrt{2}}\bar{\varphi}' \\ -\frac{1}{\sqrt{2}}\bar{\varphi}' & 0 & 1 \end{pmatrix}, \quad (3.14)$$

$$\exp(zK_+ + \varphi F_+) = \begin{pmatrix} 1 & z & \mp \frac{1}{\sqrt{2}}\varphi \\ 0 & 1 & 0 \\ 0 & \mp \frac{1}{\sqrt{2}}\varphi & 1 \end{pmatrix}. \quad (3.15)$$

Simple multiplication then gives:

$$\begin{aligned} g_{z\varphi} &= \exp(zK_+ + \varphi F_+) \exp(sK_0) \exp(-z''K_- - \bar{\varphi}'F_-) \\ &= \begin{pmatrix} e^{s/2} \mp e^{-s/2} z z'' \pm \frac{1}{\sqrt{2}} \varphi \bar{\varphi}' & \frac{1}{\sqrt{2}} z e^{-s/2} \bar{\varphi}' \mp \frac{1}{\sqrt{2}} \varphi \\ \mp e^{-s/2} z'' & e^{-s/2} & \frac{1}{\sqrt{2}} e^{-s/2} \bar{\varphi}' \\ \frac{1}{\sqrt{2}} e^{-s/2} \varphi - \frac{1}{\sqrt{2}} \bar{\varphi}' & \mp \frac{1}{\sqrt{2}} e^{-s/2} \varphi & 1 \mp \frac{1}{2} e^{-s/2} \varphi \bar{\varphi}' \end{pmatrix}. \end{aligned} \quad (3.16)$$

This expression has the form of eq.(3.1) if

$$z'' = z^*, \quad (3.17)$$

$$\bar{\varphi}' = (\mp \bar{\varphi} + z^* \varphi) e^{-s/2} \quad (3.18)$$

and

$$e^s = 1 \pm z^* z - \frac{1}{2} \bar{\varphi} \varphi. \quad (3.19)$$

Using eqs.(3.17)-(3.19) we can simplify eq.(3.16) as

$$g_{z\varphi} = (1 + z^* z - \frac{1}{2} \bar{\varphi} \varphi)^{-1/2} \begin{pmatrix} 1 & z & \mp \frac{1}{\sqrt{2}}(z \bar{\varphi} + \varphi) \\ \mp z^* & 1 & \frac{1}{\sqrt{2}}(1 \pm z^* z)(z^* \varphi \mp \bar{\varphi}) \\ \pm \frac{1}{\sqrt{2}} \bar{\varphi} \pm \frac{1}{\sqrt{2}} \varphi & 1 \pm z^* z - \frac{1}{2} \bar{\varphi} \varphi & 1 \end{pmatrix}. \quad (3.20)$$

The parametrization of a group element as in eq.(3.20) has four independent coordinates, z, φ , which constitute the group manifold of the super-group; the fifth coordinate has been set equal to zero by assuming that the s variable is a real number.

Next we utilize this group parametrization to construct the supercoherent states as follows. Let $\psi = \begin{pmatrix} a \\ b \\ \gamma \end{pmatrix}$ be an element of the flat superspace $C^{2,1}$ (element of $C^{2,1}$ are smooth functions of even variables a, b and of odd one γ). The $Osp(1|2; \mathbb{R})$ acts in $C^{2,1}$ by the rule

$$\psi \mapsto g\psi, \quad \psi^* \mapsto \psi^* g^*. \quad (3.23)$$

According to eq.(3.4), the form

$$\psi^* \bar{\Sigma} \psi = a^* a \pm b^* b + \bar{\gamma} \gamma \quad (3.24)$$

is invariant, i.e. the group acts in $C^{2,1}$ on orbits:

$$\psi^* \bar{\Sigma} \psi = \text{const.} \quad (3.25)$$

The orbit $\psi^* \bar{\Sigma} \psi = 1$ can be identified with the projective space $CP^{1|2}$ of complex lines

$$[\psi] = \{\lambda \psi; \psi = \begin{pmatrix} a \\ b \\ \gamma \end{pmatrix}, a \neq 0, \lambda \in \mathbb{C}\}. \quad (3.26)$$

In the complex projective superspace $CP^{1|2}$ we define now the homogenous coordinates,

$$z = a^{-1} b, \quad \varphi = \sqrt{2} a^{-1} \gamma. \quad (3.27)$$

The group action in eq.(3.23),

$$[\psi] = \begin{pmatrix} a \\ b \\ \gamma \end{pmatrix} \rightarrow [g\psi] = \begin{pmatrix} ua + vb \mp \theta \gamma \\ \mp v^* a + u^* b - \bar{\theta} \gamma \\ \chi a \mp \bar{\chi} b + \lambda \gamma \end{pmatrix}, \quad (3.28)$$

induces an action in $CP^{1|2}$. In terms of the homogenous coordinates, this takes the form of a super-Möbius transformation:

$$z \rightarrow z' = \left(vz + u \mp \frac{1}{\sqrt{2}} \theta \varphi \right)^{-1} \left(u^* \mp v^* - \frac{1}{\sqrt{2}} \bar{\theta} \varphi \right), \quad (3.29a)$$

$$\varphi \rightarrow \varphi' = \left(vz + u \mp \frac{1}{\sqrt{2}} \theta \varphi \right)^{-1} \left(\mp \bar{\chi} z + \frac{1}{\sqrt{2}} \lambda \varphi + \chi \right). \quad (3.29b)$$

Let us take the element $g \in Osp(1|2; \mathbb{R})$ in the form (3.1). The function $\omega(g)$ we define as

$$\omega(g) = (u^*)^{\pm 2r}. \quad (3.30)$$

One can easily shows that

$$\omega(gh) = \omega(g)d(h) = \omega(g)e^{2r \cdot \ell}, \quad (3.31)$$

for any matrix of the kind

$$h = \begin{pmatrix} e^{i\ell} & & \\ & e^{-i\ell} & \\ & & 1 \end{pmatrix}. \quad (3.32)$$

These matrices leave the line $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ invariant. Any complex line in $\mathbb{C}P^{1|2}$ can be

obtained by acting with $g_{i\varphi}$ on the line $X_{00} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. According to eqs.(3.20), (3.28)

we obtain a line,

$$X_{i\varphi} = \begin{pmatrix} 1 \\ \mp z^* \\ \pm \frac{1}{\sqrt{2}} \bar{\varphi} \end{pmatrix}, \quad (3.33)$$

so the elements of $\mathbb{C}P^{1|2}$ are uniquely characterized by z, φ . To any (z, φ) we assign (according to eq.(2.6)) a coherent state

$$|z\varphi\rangle = T(g_{z,\varphi})|00\rangle, \quad (3.34)$$

where we have used the fact that, $d(g_{z,\varphi}) = 1$.

The action of a general group element $T(g)$ on $|z\varphi\rangle$ is then given by (see eq.(2.8)):

$$T(g)|z\varphi\rangle = \left[\left(u \mp v z^* - \frac{1}{\sqrt{2}} \theta \bar{\varphi} \right)^{-1} \left(u^* \mp v^* z - \frac{1}{\sqrt{2}} \bar{\theta} \varphi \right) \right]^{\pm r} \times \left[\left(v z + u \mp \frac{1}{\sqrt{2}} \theta \varphi \right)^{-1} \left(u^* z \mp v^* - \frac{1}{\sqrt{2}} \bar{\theta} \varphi \right), \left(v z \mp \frac{1}{\sqrt{2}} \theta \varphi + u \right)^{-1} \left(\mp \bar{\lambda} z + \frac{1}{\sqrt{2}} \lambda \varphi + \chi \right) \right]. \quad (3.35)$$

Moreover the matrix elements of any group element in the supercoherent states basis are:

$$\begin{aligned} \langle z_1 \varphi_1 | T(g) | z_2 \varphi_2 \rangle &= \omega(g_{z_1 \varphi_1}^{-1} g_{z_2 \varphi_2}) \\ &= \left(u^* \pm v z_1^* \mp v^* z_2 \pm u z_1^* z_2 \pm \frac{1}{\sqrt{2}} \bar{\lambda} \bar{\varphi}_1 \mp \frac{1}{\sqrt{2}} \lambda \bar{\varphi}_2 \pm \frac{1}{\sqrt{2}} \varphi_2 \bar{\theta} \pm \frac{1}{\sqrt{2}} \varphi_1 \theta z_1^* - \frac{1}{2} \lambda \bar{\varphi}_1 \varphi_2 \right)^{\pm 2r} \\ &\quad \times \left(1 \pm z_1^* z_2 - \frac{1}{2} \bar{\varphi}_2 \varphi_2 \right)^{\mp r} \left(1 \pm z_1^* z_1 - \frac{1}{2} \bar{\varphi}_1 \varphi_1 \right)^{\mp r}. \end{aligned} \quad (3.36)$$

The overlap of coherent states, we obtain from eq.(3.40) by putting $u = \lambda = 1$, $v = \theta = \chi = 0$:

$$\langle z_1 \varphi_1 | z_2 \varphi_2 \rangle = \left(1 \pm z_1^* z_2 - \frac{1}{2} \bar{\varphi}_2 \varphi_2 \right)^{\pm 2r} \left(1 \pm z_1^* z_1 - \frac{1}{2} \bar{\varphi}_1 \varphi_1 \right)^{\mp r} \left(1 \pm z_1^* z_2 - \frac{1}{2} \bar{\varphi}_2 \varphi_2 \right)^{\mp r} \left(1 \pm z_1^* z_1 - \frac{1}{2} \bar{\varphi}_1 \varphi_1 \right)^{\mp r}. \quad (3.37)$$

We see that the coherent states are normalized to unity. Also from the above form of the overlap we can readily deduce the following:

$$\langle z_1 \varphi_1 | z_2 \varphi_2 \rangle = \langle z_1 | z_2 \rangle > \left[1 \pm \frac{\tau}{2(1 \pm |z_1|^2)} \bar{\varphi}_1 \varphi_1 \pm \frac{\tau}{2(1 \pm |z_2|^2)} \bar{\varphi}_2 \varphi_2 \mp \frac{\tau}{1 \pm z_1^* z_2} \bar{\varphi}_1 \varphi_2 + \frac{\tau^2}{4(1 \pm |z_1|^2)(1 \pm |z_2|^2)} \bar{\varphi}_1 \varphi_1 \bar{\varphi}_2 \varphi_2 \right], \quad (3.38)$$

which factorizes the overlap of the coherent states of the supergroup $osp(1|2; \mathbb{R})$ as the overlap of the coherent states of its even subgroup $SU(2)(SU(1,1))$ times a $N =$

4 Grassmann element, formed by the Grassmannian variables which parametrise the supercoherent states.

To relate the coherent states described here with those of Ref. 13 for the noncompact group, we must identify the ground state $|00\rangle$, with the lowest weight of the irreducible representation $k = \tau$, which is $|\tau; k = \tau; m = \mp \tau\rangle$. Then the action in eq.(3.16) gives the expansion of the coherent state $|z\varphi\rangle = T(g_{z,\varphi})|\tau; k = \tau; m = \mp \tau\rangle$ into the $osp(1|2; \mathbb{R})$ basis states.

First, for the compact case we obtain the expansion:

$$\begin{aligned} |z\varphi\rangle &= \frac{1}{(1+|z|^2)^{\frac{\tau}{2}}} \sum_{n=-\tau}^{\tau} \left(\frac{2\tau}{n+\tau} \right)^{\frac{1}{2}} z^{n+\tau} |\tau; k = \tau; m = n\rangle \\ &\quad - \frac{1}{\sqrt{2}} \frac{\varphi}{(1+|z|^2)^{\frac{\tau}{2}}} \sum_{n=-\tau}^{\tau} \left(\frac{2\tau}{n+\tau} \right)^{\frac{1}{2}} \sqrt{\tau-n} z^{n+\tau} |\tau; k = \tau - \frac{1}{2}; m = n + \frac{1}{2}\rangle, \end{aligned} \quad (3.39a)$$

while the corresponding expression for the noncompact group is¹³:

$$\begin{aligned} |z\varphi\rangle &= \sum_{n=0}^{\infty} \left[\frac{\Gamma(2\tau+n)}{\Gamma(n+2)\Gamma(2\tau)} \right]^{\frac{1}{2}} z^n |\tau; k = \tau; m = \tau+n\rangle \\ &\quad + \frac{1}{\sqrt{2}} \varphi \sum_{n=0}^{\infty} \left[\frac{\Gamma(2\tau+n)}{(2\tau+1)\Gamma(n)} \right]^{\frac{1}{2}} \sqrt{2\tau+n} z^n |\tau; k = \tau + \frac{1}{2}; m = \tau + \frac{1}{2} + n\rangle. \end{aligned} \quad (3.39b)$$

The states $|z\varphi\rangle$ can be normalized and they become

$$|z\varphi\rangle = \frac{1}{\sqrt{\langle z\varphi | z\varphi \rangle}} |z\varphi\rangle, \quad (3.40)$$

where $\langle z\varphi | z\varphi \rangle = [\text{sdet} M_{\pm}(z^*, z; \bar{\varphi}, \varphi)]^{\mp 2r}$ stands for the normalization factor and the matrix $M_{\pm}$ reads:

$$M_{\pm}(z_i^*, z_j; \bar{\varphi}, \varphi_j) = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \varphi_j \\ \frac{1}{\sqrt{2}} \bar{\varphi}_i & 1 \pm z_i^* z_j \end{pmatrix} \quad (3.41)$$

and the superdeterminant (or Berezinian) is defined²⁴ for a graded matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as:

$$\text{sdet} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C) / \det D. \quad (3.42)$$

Next we can evaluate the matrix elements of the generators of the $osp(1|2; \mathbb{R})$ algebra and of products of them in the supercoherent states. The results are simple if we consider the unnormalized states. Indeed making multiple differentiation of $\langle z_1 \varphi_1 | z_2 \varphi_2 \rangle = [\text{sdet} M_{\pm}(z_1^*, z_2; \bar{\varphi}_1, \varphi_2)]^{\mp 2r} = w_0 \mp \tau w_1 \bar{\varphi}_1 \varphi_2$ (with $w_i \equiv (1 \pm z_1^* z_2)^{\pm 2r+i}$), we obtain

$$\langle z_1 \varphi_1 | K_+ | z_2 \varphi_2 \rangle = 2\tau z_1^* w_1, \quad (3.43a)$$

$$\langle z_1 \varphi_1 | K_- | z_2 \varphi_2 \rangle = 2\tau z_2 w_1, \quad (3.43b)$$

tensor and X_i stands for the group generators. As such, they minimize the dispersion of the quadratic Casimir operator^{16,17},

$$\Delta C_2 = g^{ij} < z\varphi | X_i X_j | z\varphi > - g^{ij} < z\varphi | X_i | z\varphi > X_j. \quad (3.44b)$$

The supercoherent states form a overcomplete set and we can write the following resolution of unity¹³:

$$\begin{aligned} \int d\mu_{\pm}(z^*, z; \bar{\varphi}, \varphi) |z\varphi\rangle \langle z\varphi| = \sum_{\substack{m=k=\tau, \tau+1, \tau+2, \dots \\ m=\tau, \tau+1, \tau+2, \dots}} |\tau, k = \tau, m > \tau, k = \tau, m| \\ + \sum_{\substack{m'=\tau, \tau+1, \tau+2, \dots \\ m'=\tau, \tau+1, \tau+2, \dots}} |\tau, \tau \mp \frac{1}{2}, m > \tau, \tau \mp \frac{1}{2}, m| = 1, \end{aligned} \quad (3.45)$$

where the measure is given by

$$d\mu_{\pm}(z^*, z; \bar{\varphi}, \varphi) = \frac{4}{\pi} d\bar{\varphi} d\varphi d^2 z \text{ sdet} M_{\pm}(z^*, z; \bar{\varphi}, \varphi). \quad (3.46)$$

IV. Path integral representation for the transition amplitude

The time evolution of the quantum mechanical system is determined by the operator

$$\begin{aligned} U(t'', t') &= T \exp \left[\int_{t'}^{t''} -\frac{i}{\hbar} H(t) dt \right] \\ &= \lim_{N \rightarrow \infty} \exp \left[-\frac{i}{\hbar} H(t_N) \epsilon \right] \cdots \exp \left[-\frac{i}{\hbar} H(t_1) \epsilon \right] \\ &= \lim_{\substack{N \rightarrow \infty \\ t \rightarrow 0}} \prod_{j=1}^N U_{\epsilon}(t_j) \end{aligned} \quad (4.1)$$

with

$$U_{\epsilon}(t_j) = \exp \left[-\frac{i}{\hbar} \epsilon H(t_j) \right], \quad \epsilon = (t'' - t')/N \quad (4.2)$$

and $t_j = (j - \frac{1}{2})\epsilon + t'$ are the times at the midpoint of the intervals. The propagator for the coherent states of the supergroup $Osp(1|2; \mathbf{R})$ is then

$$\begin{aligned} \mathbf{K} &= \mathbf{K}(z'', \varphi'', t''; z', \varphi', t') = < z'', \varphi'' | U(t'', t') | z', \varphi' >, \\ &\text{with the defining property} \end{aligned} \quad (4.3)$$

$$\bar{\Psi}(z'', \varphi'') = \int d\mu_{\pm}(z'', \varphi'') \mathbf{K}(z'', \varphi'', t''; z', \varphi', t') \bar{\Psi}(z', \varphi'). \quad (4.4)$$

Using the resolution of unity of the supercoherent states N times, the propagator is written as

$$\begin{aligned} \mathbf{K}(z'', \varphi'', t''; z', \varphi', t') \\ = \lim_{N \rightarrow \infty} \int \cdots \int \prod_{j=1}^{N-1} d\mu_{\pm}(z_j, \varphi_j) \prod_{j=1}^N < z_j \varphi_j | 1 - \frac{i}{\hbar} \epsilon H(t_j) | z_{j-1} \varphi_{j-1} >, \end{aligned} \quad (4.5)$$

$$(z_1 \varphi_1 | K_0 | z_2 \varphi_2) = \mp \tau w_0 + \tau^2 \bar{\varphi}_1 \varphi_2 w_1, \quad (3.43c)$$

$$(z_1 \varphi_1 | F_{\pm} | z_2 \varphi_2) = \pm \tau \bar{\varphi}_1 w_1, \quad (3.43d)$$

$$(z_1 \varphi_1 | F_{\pm} | z_2 \varphi_2) = \mp \tau \varphi_1 w_1, \quad (3.43e)$$

$$(z_1 \varphi_1 | K_{\pm}^2 | z_2 \varphi_2) = 2\tau(2\tau \mp 1)z_1^2 w_2 \mp \tau(2\tau - 1)(2\tau \mp 2)z_1^2 \bar{\varphi}_1 \varphi_2 w_3, \quad (3.43f)$$

$$(z_1 \varphi_1 | K_{\pm}^2 | z_2 \varphi_2) = 2\tau(2\tau \mp 1)z_2^2 w_2 \mp \tau(2\tau - 1)(2\tau \mp 2)z_2^2 \bar{\varphi}_1 \varphi_2 w_3, \quad (3.43g)$$

$$\begin{aligned} (z_1 \varphi_1 | K_0^2 | z_2 \varphi_2) &= \tau^2 w_0 \mp \tau^3 \bar{\varphi}_1 \varphi_2 w_1 + 2\tau(1 \mp 2\tau)z_1^2 z_2 w_1 \mp \frac{1}{4}\tau(1 \mp 4\tau)\bar{\varphi}_1 \varphi_2 w_1 \\ &\mp \tau(2\tau - 1)z_1^2 z_2 \bar{\varphi}_1 \varphi_2 w_2 + 2\tau(2\tau \mp 1)z_1^2 z_2^2 w_2 \mp \tau(2\tau - 1)(2\tau \mp 2)z_1^2 z_2 \bar{\varphi}_1 \varphi_2 w_3, \end{aligned} \quad (3.43h)$$

$$(z_1 \varphi_1 | K_{\pm} F_{\pm} | z_2 \varphi_2) = \pm \tau(2\tau - 1)z_1^2 \bar{\varphi}_1 w_2, \quad (3.43i)$$

$$(z_1 \varphi_1 | K_{-} K_{+} | z_2 \varphi_2) = 2\tau w_1 + 2\tau(2\tau \mp 1)z_1^2 w_2 \mp \tau(2\tau - 1)(2\tau \mp 1)z_2 \bar{\varphi}_1 \varphi_2 w_3, \quad (3.43j)$$

$$(z_1 \varphi_1 | F_{-} F_{+} | z_1 \varphi_2) = \pm \tau w_1. \quad (3.43k)$$

In the derivation of the matrix elements of K_0 and K_0^2 we have used its commutation relations with $K_{\pm}$ and $F_{\pm}$. By multiplying the above formula with the normalizations of the $|z_1, z_2 \varphi_1, z_2 \varphi_2\rangle$ states, we obtain the matrix elements between normalized coherent states.

It is finally easy to see that these supercoherent states are the minimum uncertainty states, which means that they minimize the dispersion of the Casimir operator¹³:

$$\Delta C_2 = < K_0^2 > \pm \frac{1}{2}(< K_{+} K_{-} > + < K_{-} K_{+} >) + < F_{+} F_{-} > - < F_{-} F_{+} > \quad (3.44a)$$

$$- < K_0 >^2 \mp \frac{1}{2}(< K_{+} > < K_{-} > + < K_{-} > < K_{+} >) - < F_{+} > < F_{-} > + < F_{-} > < F_{+} >.$$

This is due to the fact that the supercoherent states are eigenstates of the linearized Casimir operator of the group, $g^{ij} < X_i > X_j$, where g^{ij} is the Cartan-Killing metric

with $z_0 \equiv z'$, $\varphi_0 \equiv \varphi'$ and $z_N \equiv z''$, $\varphi_N \equiv \varphi''$.

In the usual path integral formalism the next step is to exponentiate eq.(4.5) and to use multiple time integration of the overlap of two coherent states. These integrations are straightforward in the case of Gaussian integrals. Such is the case when we deal with coherent states of the Heisenberg-Weyl group. However for generalized coherent states of the $SU(2)$, $SU(1,1)$ groups we have overlaps that are not simple exponential functions (see eq.(2.27)) and the multiple integrations in the path integral are not possible. The propagator of the $Osp(1|2; \mathbf{R})$ supergroup with $SU(2)$ or $SU(1,1)$ as even subgroups faces of course the same difficulty and in fact is even worse due to the additional odd part (see eq. (3.41-3.46)). Due to this fact, we turn to another method of calculating the propagator which in a strict sense deviates from the usual path integration. However it is efficient for the generalized coherent states.

We consider again eq.(4.5) for the most general Hamiltonian which belongs to the $osp(1|2; \mathbf{R})$ superalgebra with time dependent coefficients:

$$H(t_j) = A(t_j)K_0 + f(t_j)K_+ + f(t_j)^*K_- + \theta(t_j)F_+ - \bar{\theta}(t_j)F_- \quad (4.6)$$

We can now rewrite the propagator in eq.(4.5) as

$$\mathbf{K}(z'', \varphi'', t''; z', \varphi', t') = \lim_{N \rightarrow \infty} \int \prod_{j=1}^{N-1} d\mu_{\pm}(z_j, \varphi_j) \prod_{j=1}^N < z_j \varphi_j | T(g_j) | z_{j-1} \varphi_{j-1} >, \quad (4.7)$$

where $T(g_j) = 1 - \frac{i}{\hbar} \epsilon H(t_j)$ is the $Osp(1|2; \mathbf{R})$ group element which corresponds to the infinitesimal time evolution of the system and is parametrized by the matrix elements of the group element g_j in the 3×3 fundamental representation of the group and by using eqs.(4.6) and (3.9) reads:

$$g_j = g_j(\epsilon) = \begin{pmatrix} u_j' & v_j' & \bar{\theta}_j' \\ \mp v_j'^* & u_j'^* & -\bar{\theta}_j'^* \\ \chi_j' & \mp \bar{\chi}_j' & \lambda_j' \end{pmatrix}, \quad (4.8)$$

where, $u_j' = 1 - \frac{i}{\hbar} \frac{\epsilon}{2} A(t_j)$, $v_j' = -\frac{i}{\hbar} \epsilon f(t_j)$, $\theta_j' = -\frac{i}{\hbar} \frac{\epsilon}{2} \theta(t_j)$, $\lambda_j' = 1$ and $\chi_j' = \bar{\theta}_j'$.

It is now straightforward to express the matrix elements occurring in the superpropagator of eq.(4.7) using eq.(3.40). Indeed by substitution, we obtain

$$\begin{aligned} < z_j \varphi_j | T(g_j) | z_{j-1} \varphi_{j-1} > \\ = & \left(u_j'^* \pm v_j'^* z_j' \mp v_j'^* z_{j-1}' \pm u_j'^* z_j' \mp \frac{1}{\sqrt{2}} \bar{\chi}_j' \varphi_j \mp \frac{1}{\sqrt{2}} \bar{\chi}_j' \varphi_{j-1} \right) \\ & - \frac{1}{\sqrt{2}} \varphi_{j-1} \bar{\theta}_j' \pm \frac{1}{\sqrt{2}} \varphi_{j-1} \bar{\theta}_j' z_j'^* - \frac{1}{2} \lambda_j' \varphi_j \varphi_{j-1} \Big)^{\pm 2r} \\ & \times \left(1 \pm z_{j-1}' z_{j-1} - \frac{1}{2} \bar{\varphi}_{j-1} \varphi_{j-1} \right)^{\mp r} \left(1 \pm z_j' z_j - \frac{1}{2} \bar{\varphi}_j \varphi_j \right)^{\mp r}. \end{aligned} \quad (4.9)$$

Further, the products of matrix elements in eq.(4.7) can be handled using the group composition rule of eq.(2.1) which together with the resolution of unity provides the important formula

$$< z' \varphi' | T(g_3) | z \varphi > = \int d\mu_{\pm}(z'', \varphi'') < z' \varphi' | T(g_2) | z'' \varphi'' > < z'' \varphi'' | T(g_1) | z \varphi >, \quad (4.10)$$

for any g_1 , g_2 , g_3 successive group element where $g_3 = g_2 g_1$ and $T(g_3) = T(g_2) T(g_1)$. Again eq.(3.40) is used to give for the above matrix element the expression

$$\begin{aligned} < z' \varphi' | T(g_3) | z \varphi > \\ = & \left(u_3'^* \pm v_3'^* z' \pm u_3'^* z'' \mp \frac{1}{\sqrt{2}} \bar{\chi}_3' \varphi' \mp \frac{1}{\sqrt{2}} \bar{\chi}_3' \varphi'' \mp \frac{1}{\sqrt{2}} \varphi \theta_3' \pm \frac{1}{\sqrt{2}} \varphi \theta_3'' \pm \frac{1}{2} \lambda_3' \varphi' \varphi'' \right)^{\pm 2r} \\ & \times \left(1 \pm z'' z' - \frac{1}{2} \bar{\varphi} \varphi \right)^{\mp r} \left(1 \pm z'' z' - \frac{1}{2} \bar{\varphi}' \varphi' \right)^{\mp r}, \end{aligned} \quad (4.11)$$

where

$$\begin{pmatrix} u_3' & v_3' & \bar{\theta}_3' \\ \mp v_3'^* & u_3'^* & -\bar{\theta}_3'^* \\ \chi_3' & \mp \bar{\chi}_3' & \lambda_3' \end{pmatrix} = \begin{pmatrix} u_2' & v_2' & \bar{\theta}_2' \\ \mp v_2'^* & u_2'^* & -\bar{\theta}_2'^* \\ \chi_2' & \mp \bar{\chi}_2' & \lambda_2' \end{pmatrix} \begin{pmatrix} u_1' & v_1' & \bar{\theta}_1' \\ \mp v_1'^* & u_1'^* & -\bar{\theta}_1'^* \\ \chi_1' & \mp \bar{\chi}_1' & \lambda_1' \end{pmatrix}. \quad (4.12)$$

We now use a multiple of times the composition rule constituted from eqs.(4.10)-(4.12) to integrate the propagator in eq.(4.7) which is written as

$$\mathbf{K}(z'', \varphi'', t''; z', \varphi', t') = \lim_{N \rightarrow \infty} < z'' \varphi'' | T(g_N) | z' \varphi' >. \quad (4.13)$$

The group parameters g_N are given according to eq.(4.12) by recursion relations which provide with expression for the g_j parameters when the g_{j-1} are known, in the following way:

$$\begin{pmatrix} u_j' & v_j' & \bar{\theta}_j' \\ \mp v_j'^* & u_j'^* & -\bar{\theta}_j'^* \\ \chi_j' & \mp \bar{\chi}_j' & \lambda_j' \end{pmatrix} = \begin{pmatrix} 1 - \frac{i}{\hbar} \frac{\epsilon}{2} A(t_j) & -\frac{i}{\hbar} \epsilon f(t_j) & \mp(-\frac{i}{\hbar} \frac{\epsilon}{2} \theta(t_j)) \\ \mp(\frac{i}{\hbar} \epsilon f^*(t_j)) & 1 + \frac{i}{\hbar} \frac{\epsilon}{2} A(t_j) & -\frac{i}{\hbar} \frac{\epsilon}{2} \theta(t_j) \\ \frac{i}{\hbar} \frac{\epsilon}{2} \bar{\theta}(t_j) & \mp(-\frac{i}{\hbar} \frac{\epsilon}{2} \theta(t_j)) & 1 \end{pmatrix} \\ \times \begin{pmatrix} u_{j-1}' & v_{j-1}' & \bar{\theta}_{j-1}' \\ \mp v_{j-1}'^* & u_{j-1}'^* & -\bar{\theta}_{j-1}'^* \\ \chi_{j-1}' & \mp \bar{\chi}_{j-1}' & \lambda_{j-1}' \end{pmatrix}. \quad (4.14)$$

The seed of this iteration is the unit matrix which implies the choices, $u_0 = \lambda_0 = 1$ and $\chi_0 = \theta_0 = 0$. Taking the limits $\epsilon \rightarrow 0$ and $N \rightarrow \infty$ of an infinite number of refined insertions of the resolution of unity, turns the above recursion relations to

$$\dot{u}(t) = -\frac{i}{\hbar} A(t) u(t) \pm \frac{i}{\hbar} f(t) v(t)^* \pm \frac{i}{\hbar} \frac{\theta(t)}{\sqrt{2}} \chi(t), \quad (4.15a)$$

$$\dot{v}^*(t) = \frac{i}{\hbar} f(t)^* u(t) + \frac{i}{\hbar} \frac{A(t)}{2} v(t)^* \pm \frac{i}{\hbar} \frac{\bar{\theta}(t)}{\sqrt{2}} \chi(t), \quad (4.15b)$$

$$\dot{\chi}(t) = \frac{i}{\hbar} \frac{\bar{\theta}(t)}{\sqrt{2}} u(t) - \frac{i}{\hbar} \frac{\theta(t)}{\sqrt{2}} v(t)^*. \quad (4.15c)$$

These first order super-equations contain Grassmann variables as dependent variables, with the initial conditions $u(t') = 1$, $u(t'') = \chi(t'') = 0$. The propagator in eq. (4.13) is seen now to be the matrix element between the initial and the final super-coherent states of the $osp(1|2; \mathbb{R})$ group element parametrized by the variables which are obtained as solutions of eq. (4.15) and related with the remaining group parameters with eqs. (3.7), (3.8). If we denote explicitly the initial and final time dependence in the group parameters as $u(t'', t')$, $v(t'', t')$ etc. the final form of the propagator reads:

$$\begin{aligned} K(z'', \varphi'', t''; z', \varphi', t') &= \langle z'' \varphi'' | T(g(t'', t')) | z' \varphi' \rangle \\ &= \left(u^*(t'', t') \pm v(t'', t') z'' \pm u^*(t'', t') z' \pm u(t'', t') z'' z' \pm \frac{1}{\sqrt{2}} \bar{\chi}(t'', t') \bar{\varphi}' \pm \frac{1}{\sqrt{2}} \chi(t'', t') \varphi' z'' \right. \\ &\quad \left. - \frac{1}{\sqrt{2}} \varphi'' \theta(t'', t') \pm \frac{1}{\sqrt{2}} \varphi'' \bar{\theta}(t'', t') z'' - \frac{1}{2} \chi(t'', t') \bar{\varphi}' \varphi'' \right)^{\pm 2\tau} \\ &\quad \times \left(1 \pm z'' z' - \frac{1}{2} \bar{\varphi}'' \varphi'' \right)^{\mp \tau} \left(1 \pm z'' z' - \frac{1}{2} \bar{\varphi}' \varphi' \right)^{\pm \tau}. \end{aligned} \quad (4.16)$$

The difficulty with the evaluation of the propagator is therefore concentrated in solving eq. (4.15). Obviously they describe the time evolution of the components $(u(t), v^*(t), \chi(t))$ of a supervector in the space $\mathbb{C}^{1,2}$ and they satisfy eqs. (3.7), (3.8). If in this space we define homogeneous coordinates as

$$\alpha = u^{-1} v^*, \quad \Xi = u^{-1} \chi, \quad (4.17)$$

where α is a complex number and Ξ is a complex Grassmannian variable, then in these coordinates eq. (4.15) reads:

$$\dot{\alpha} = \frac{i}{\hbar} f + \frac{i}{\hbar} \alpha + \frac{i}{\hbar} f \alpha^2 \pm \frac{i}{\hbar} \frac{\bar{\theta}}{\sqrt{2}} \Xi \mp \frac{i}{\hbar} \frac{\theta}{\sqrt{2}} \alpha \Xi, \quad (4.18a)$$

$$\dot{\Xi} = \frac{i}{\hbar} \frac{\bar{\theta}}{\sqrt{2}} - \frac{i}{\hbar} \frac{\theta}{2} \alpha + \frac{i}{\hbar} \frac{A}{2} \Xi \mp \frac{i}{\hbar} f \alpha \Xi. \quad (4.18b)$$

We notice now that the even part of eq. (4.18a) is a Riccati equation. So by reduction to the even subgroup $SU(2)$ or $SU(1, 1)$ the evaluation of propagator amounts to the solution of that Riccati equation for the $SU(2)/U(1) \approx S^1$ or $SU(1, 1)/U(1) \approx S^{1,1}$ coset coordinate α . Indeed, this is what one encounters in path-integral¹⁹ or other treatments²⁵ of the $SU(2)$ or $SU(1, 1)$ dynamics. In the present case of the dynamics of the supergroup $osp(1|2; \mathbb{R})$, we have the super-coset space $\mathbb{C}^{1,2}$ with two coordinates (α, Ξ) and accordingly a super extension of the Riccati equation. Such equations have been studied before, independently of super path-integrals¹⁸ and also they appear in BCH theorems for the $osp(1|2; \mathbb{R})$ group²⁶.

According to the general theory¹⁸, it is always possible to turn the super-Riccati equation obtained above to ordinary nonlinear equation by first specifying the number L of odd generators in the Grassmann algebra, where the Grassmannian elements of

the equations belong and then rewriting the equations in terms of the components. In general a Grassmann algebra²⁷ has a basis of elements $\{1, \Lambda_1, \dots, \Lambda_L\}$, $L \geq 1$ with anticommutation relations, $\Lambda_i \Lambda_j + \Lambda_j \Lambda_i = 0$, and we assume that it takes values over the field of complex numbers. Then its even and odd elements are expanded respectively as

$$B = b_0 1 + \sum_{i < j} b_{ij} \Lambda_i \Lambda_j + \dots, \quad (4.19a)$$

$$A = \sum_{i=1}^L a_i \Lambda_i + \sum_{i < j < k} A_{ijk} \Lambda_i \Lambda_j \Lambda_k + \dots \quad (4.19b)$$

For the case of eq. (4.18) we assume $L = 1$. So we use the basis $\{1, \Lambda \equiv \Lambda_1\}$ and thus all the odd variables are proportional to Λ and for the projective coordinates we have the expansions $\alpha = \alpha_{(0)} 1$ and $\Xi = \xi_{(1)} \Lambda$. Then eqs. (4.18) are written in component form as

$$\dot{\alpha} = \frac{i}{\hbar} f_{(0)} + \frac{i}{\hbar} \frac{A_{(0)}}{2} \alpha \mp \frac{i}{\hbar} f_{(0)} \alpha^2, \quad (4.20a)$$

$$\dot{\xi}_{(1)} = \frac{i}{\hbar} \frac{\theta_{(1)}}{\sqrt{2}} - \frac{i}{\hbar} \frac{\theta_{(1)}}{2} \alpha_{(0)} + \frac{i}{\hbar} \frac{A_{(0)}}{2} \xi_{(1)} \mp \frac{i}{\hbar} f_{(0)} \alpha_{(0)} \xi_{(1)}. \quad (4.20b)$$

Clearly $\alpha_{(0)}$ obeys a usual Riccati equation which has known solutions for a class of time dependent functions for $A_{(0)}(t)$ and $f_{(0)}(t)$; once $\alpha_{(0)}$ is known, eq. (4.20b) is a linear equation for $\xi_{(1)}$. This remarkable decomposition of even and odd variables holds true for $L > 1$ as well¹⁸. Elsewhere, specific solutions of the super-Riccati equation will be given for problems utilizing multiboson realizations of the $osp(1|2; \mathbb{R})$ generators with $L > 1$.

To close, let us derive the classical limit for the propagator and discuss the canonical structure of the problem. To this end, we return to eq. (4.5) which is now written as

$$\begin{aligned} K(z', \varphi', t'; z'', \varphi'', t'') &= \lim_{\hbar \rightarrow 0} \int \prod_{j=1}^{N-1} d\mu_{\pm}(z_j, \varphi_j) \exp \left[\sum_{k=1}^N (\ell_k < z_k \varphi_k | z_{k-1} \varphi_{k-1} > - \frac{i}{\hbar} \mathcal{H}) \right], \quad (4.21) \end{aligned}$$

with

$$\mathcal{H} = \mathcal{H}(z_k, \varphi_{k-1}, z_k^*, \varphi_{k-1}^*; \varphi_k, \varphi_{k-1}, \bar{\varphi}_k, \bar{\varphi}_{k-1}) = \frac{< z_k, \varphi_k | H | z_{k-1}, \varphi_{k-1} >}{< z_k, \varphi_k | z_{k-1}, \varphi_{k-1} >}. \quad (4.22)$$

For $z_{k-1} = z_k - \Delta z_k$ and $\varphi_{k-1} = \varphi_k - \Delta \varphi_k$ we calculate the logarithm of the overlap of eq. (3.41) and write the exponential integrand in eq. (4.21) up to second order for $\Delta z_k, \Delta \varphi_k$, as

$$\begin{aligned} \exp \left(\sum_{k=1}^N \epsilon \left\{ \frac{\pm \tau}{1 + |z_k|^2} \left[\left(\frac{\Delta z_k^*}{z_k} - \frac{\Delta z_k}{z_k^*} \right) \left(1 - \frac{1}{2} \frac{\bar{\varphi}_k \varphi_k}{1 \pm |z_k|^2} \right) + \frac{i}{2} \left(\frac{\Delta \bar{\varphi}_k \varphi_k - \bar{\varphi}_k \Delta \varphi_k}{\epsilon} \right) \right] \right. \right. \\ \left. \left. + \mathcal{O}(\Delta^2) \right\} \right). \end{aligned} \quad (4.23)$$

In the continuous limit we may write a formal functional integral for the propagator from eqs. (4.21) and (4.23),

$$\mathbf{K} = \int \mathcal{D}\mu_{\pm}[z(t), \varphi(t)] \exp \left[\frac{i}{\hbar} \int_0^{t''-t'} dt \mathcal{L}(z, \dot{z}, z^*, \dot{z}^*; \varphi, \dot{\varphi}, \bar{\varphi}, \dot{\bar{\varphi}}) \right], \quad (4.24)$$

where we have introduced the classical Lagrangian,

$$\mathcal{L} = \frac{\mp i \tau \hbar}{(1 \pm |z|^2)} \left[(z \dot{z}^* - z^* \dot{z}) \left(1 - \frac{1}{2} \frac{\bar{\varphi} \varphi}{1 \pm |z|^2} \right) + \frac{1}{2} (\bar{\varphi} \dot{\varphi} - \varphi \dot{\bar{\varphi}}) \right] - \mathcal{H}. \quad (4.25)$$

The classical limit is now understood as the case where the Planck constant is much less than the action integral, $\hbar \ll S \equiv \int_0^{t''-t'} dt \mathcal{L}$ and in such a case only the classical path in the supercoset space $(z(t), \varphi(t))$ which makes the action stationary, $\delta S = 0$, has the dominant contribution to the transition amplitude. By setting $\delta S = 0$, we obtain the following canonical equations of motion:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{z}} - \frac{\partial \mathcal{L}}{\partial z} = 0, \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{z}^*} - \frac{\partial \mathcal{L}}{\partial z^*} = 0 \quad (4.26a)$$

and

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0, \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\bar{\varphi}}} - \frac{\partial \mathcal{L}}{\partial \bar{\varphi}} = 0. \quad (4.26b)$$

The general Lagrangian in eq. (4.25) yields the equations²⁸

$$\dot{z} = \{z, \mathcal{H}\}_B - \frac{1}{2} z \bar{\varphi} \{\varphi, \mathcal{H}\}_F, \quad (4.27a)$$

$$\dot{z}^* = \{z^*, \mathcal{H}\}_B + \frac{1}{2} z^* \varphi \{\bar{\varphi}, \mathcal{H}\}_F, \quad (4.27b)$$

$$\dot{\varphi} = -\{\varphi, \mathcal{H}\}_F + \frac{z^* \varphi}{1 \pm |z|^2} \{z^*, \mathcal{H}\}_B, \quad (4.27c)$$

$$\dot{\bar{\varphi}} = -\{\bar{\varphi}, \mathcal{H}\}_F + \frac{z \bar{\varphi}}{1 \pm |z|^2} \{z, \mathcal{H}\}_B, \quad (4.27d)$$

where the bosonic (B) and fermionic (F) Poisson brackets are defined by

$$\{A, B\}_B = \pm \frac{i}{2\tau\hbar} (1 \pm z^* z + \frac{1}{4} \bar{\varphi} \varphi)^2 \left(\frac{\partial A}{\partial z^*} \frac{\partial B}{\partial z} - A \frac{\partial}{\partial z} \frac{\partial B}{\partial z^*} \right), \quad (4.28a)$$

$$\{A, B\}_F = \pm \frac{i}{\tau\hbar} (1 \pm z^* z) \left(\frac{\bar{\partial}}{\partial \bar{\varphi}} A \frac{\bar{\partial}}{\partial \varphi} B - A \frac{\bar{\partial}}{\partial \varphi} B \frac{\bar{\partial}}{\partial \bar{\varphi}} \right). \quad (4.28b)$$

We observe from the structure of the Poisson brackets that the phase space is generalized in a double sense of first, being curved^{11,12} (prefactors in eqs. (4.28)) and second being with Grassmannian extension. This case extends in fact, the considerations of Berezin²⁹ for the generalized phase space and physically may describe the motion of a superparticle.

Finally we remark that we can derive the coupled super-Riccati eqs. (4.18), using the classical Lagrangian (4.25) with the Hamiltonian (4.6) and eq. (4.22) as the equations of motion obtained from eq. (4.27).

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