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The topological phase of the Aharonov-Casher effect and
the anyon behaviour of charged particles in 2+1 dimensions

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ABSTRACT

We discuss the topological phase of the Aharonov-Casher effect and its implications for anyons in 2+1 dimensions. We clarify the conditions for developing the topological phase and discuss the interference of the Aharonov-Bohm and Aharonov-Casher effects. In 2+1 dimensions, we find that charged particles with non zero anomalous magnetic dipole moment behave like anyons.

In 1984 Aharonov and Casher (AC) [1] (see also, [2]) pointed out that in the non relativistic limit the wave function of a neutral particle with non zero magnetic dipole moment μ develops a topological phase when travelling in a closed path which encircles an infinitely long filament carrying a uniform charge density. This phenomenon is similar to the Aharonov-Bohm (AB) effect [3] for a charged spin half particle. The AC phase has been measured experimentally. [4] The similarities and the differences of these two phenomena [5] and possible classical interpretations of the AC effect [6,7] have been discussed by several authors. Recently in Ref [8] the equivalence of the AB and AC effects was studied in the relativistic case for spin half particles. It was shown that if one fixes the vector potential to be two dimensional, $A = (A_1(x, y), A_2(x, y), 0)$, in the AB case and the electric field also to be two dimensional, $E = (E_1(x, y), E_2(x, y), 0)$, in the AC case, the equations of motion for the particles take the same form. One can obtain the equation of motion for the AC effect from the AB effect by changing A_i to $se_i E_j$ and e to μ (where $\epsilon_{12} = -\epsilon_{21} = 1$, e is the electric charge of the particle in the AB case, and $s = \pm$ for spin up and down respectively).

This fact leads Ref[8] to the conclusion that the AB and AC effects are equivalent. This way of treating the AC effect does not address the underlying principle for the generation of the topological phase directly. Some further clarification of the physics necessary for the development of an AC topological phase is desirable. In the following we will discuss the conditions for developing the AC topological phase without imposing the non relativistic limit. We point out that when certain conditions are satisfied there is an additional gauge symmetry for the system and the electric field becomes a pure gauge vector potential of this symmetry. We then apply our formulation of the AC effect to two interesting cases. We discuss interference between the AB and AC effects. Further, we show that in 2+1 dimensions charged particles with non zero anomalous magnetic dipole moment behave like anyons.

Let us recall how the gauge invariance principle determines the topological phase of the AB effect for a charged spin half particle. In an external electromagnetic vector potential A_μ , the equation of motion is,

$$(\not{\partial} - ie \not{A} + m)\psi = 0, \quad (1)$$

where m is the mass of the particle. We will use the conventions of Ref [9] for γ -matrices. In a static external magnetic field, one can choose a gauge with $A_4 = 0$. The equation of motion becomes,

$$(\not{\partial} - ie\gamma \cdot A + m)\psi = 0, \quad (2)$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ and A is the spatial component of the vector potential. Under a

gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu f$, the gauge invariance of the theory requires the wave function ψ to transform as $\psi \rightarrow e^{ief}\psi$. Consider now the situation where $B = 0$ but $A \neq 0$ in some regions of space. Since $B = \nabla \times A = 0$, A can be written as a gradient of a scalar function f , that is, $A = \nabla f$. In this case A is a pure gauge vector potential and the equation of motion can be written as,

$$(\not{\partial} + m)\psi' = 0, \quad (3)$$

where $\psi' = e^{-ie \int A \cdot dx} \psi$. Eq.(3) is the equation of motion for a force free particle.

For a physically non trivial example, the region in which $B = 0$ but $A \neq 0$ must have a hole in it, such that $B \neq 0$ inside the hole. If one now transports the particle around a closed path in the $B = 0$ region which encloses regions with $B \neq 0$, the wave function develops a non trivial topological phase,

$$\phi_{AB} = -e \oint A \cdot dr = -e \int_S (\nabla \times A) \cdot dS = -e\Phi, \quad (4)$$

where Φ is the magnetic flux passing through the area enclosed by the path. It is important to notice that in order that a topological phase develop in the AB effect there must be a topologically non trivial region in which the vector potential is pure gauge, that is, $\nabla \times A = 0$. This is the necessary condition that the effect of the vector potential can be transformed into the phase of the wave function. The phrase "topologically non trivial" means that one can traverse a closed path in the $\nabla \times A = 0$ region which is such that any surface whose boundary is the closed path intersects a $\nabla \times A \neq 0$ region.

Let us come back to the AC effect. The equation of motion for a neutral spin half particle with a non-zero magnetic dipole moment moving in a static electric field E is

$$(\not{\partial} + \frac{1}{2}\mu\sigma_{\mu\nu}F^{\mu\nu} + m)\psi = 0, \quad (5)$$

which can be written as

$$(\not{\partial} + \mu\gamma \cdot E\gamma_4 + m)\psi = 0, \quad (6)$$

where $F^{\mu\nu}$ is the field strength of the electric field. Eqs.(2) and (6) look similar. However, due to the appearance of γ_4 in the second term of Eq.(6) and the different nature of the curls of E and A , these two equations are in fact quite different. In the AC effect, what makes the space topologically non trivial is the fact that there are

regions where $\nabla \cdot \mathbf{E} = 0$ and some other regions in which $\nabla \cdot \mathbf{E} \neq 0$, in contrast to the AB case where the topology of the space is determined by $\nabla \times \mathbf{A}$. This gives us a hint of how to find the possible topological phase of the AC effect. We now discuss the AC effect following the analysis in Ref[1] but without taking the non relativistic limit. We will then give the conditions for developing an AC topological phase in a more general form.

In Ref[1], the topological phase of the AC effect was derived by considering a neutral particle with a non zero magnetic dipole moment moving in a electric field produced by an infinitely long filament carrying a uniform charge density. In this case the electric field is two dimensional in the x-y plane and there is no z dependence, $\mathbf{E} = (E_1(x, y), E_2(x, y), 0)$, and $\partial_z \mathbf{E} = 0$. Writing the equation of motion in the two component form, one obtains

$$\frac{1}{2m} \sigma \cdot (\mathbf{P} - i\mu \mathbf{E}) \sigma \cdot (\mathbf{P} + i\mu \mathbf{E}) \psi_A = \frac{1}{2m} (\epsilon^2 - m^2) \psi_A, \quad (7)$$

where $\psi_A = (u_1, u_2)^T$ represents the upper two components of $\psi = (u_1, u_2, u_3, u_4)^T$, and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. Changing μ to $-\mu$, one obtains the equation of motion for the lower two components. The above equation can be written as,

$$\frac{1}{2m} \{ (\mathbf{P} - \mu \mathbf{E} \times \sigma)^2 + \mu (\nabla \cdot \mathbf{E}) - \mu^2 \mathbf{E}^2 \} \psi_A = \frac{1}{2m} (\epsilon^2 - m^2) \psi_A. \quad (8)$$

In a charge free region, $\nabla \cdot \mathbf{E} = 0$. The canonical momentum is $\mathbf{P} - \mu \mathbf{E} \times \sigma$. Comparing this equation with the equivalent equation describing the AB effect, one might conclude that the topological phase in the AC effect is given by

$$\phi_{AC} = \oint \hat{\mu} \times \mathbf{E} \cdot d\mathbf{r} \quad (9)$$

where $\hat{\mu} = \mu \sigma$. This is not correct in the relativistic case because $\nabla \times (\hat{\mu} \times \mathbf{E}) = \hat{\mu} (\nabla \cdot \mathbf{E}) - (\hat{\mu} \cdot \nabla) \mathbf{E} \neq 0$, even in a charge free region, that is, $\hat{\mu} \times \mathbf{E}$ can not be identified with the pure gauge vector potential $\mathbf{A}$ in the AB effect and therefore this term cannot be absorbed into the phase of the wave function in a charge free region. Care must be taken and the correct conditions for the development of a topological phase must be specified. In the particular case discussed above it is possible to arrange that the particle to move in the x-y plane, that is, $P_3 = 0$. In this case, the equation of motion

becomes

$$\frac{1}{2m} ((P_1 - \mu E_2 \sigma_3)^2 + (P_2 + \mu E_1 \sigma_3)^2 + \mu (\nabla \cdot \mathbf{E})) \psi_A = \frac{1}{2m} (\epsilon^2 - m^2) \psi_A. \quad (10)$$

Notice that the $\mathbf{E}^2$ term has disappeared, without introducing any approximations. Let us define the effective vector potential $\mathbf{A}' = \mu \sigma_3 (E_2, -E_1, 0)$. We find

$$\nabla \times \mathbf{A}' = -\mu \sigma_3 \nabla \cdot \mathbf{E} \quad (11)$$

This quantity is the quantity to be identified with the vector potential in the AB effect. In a charge free region because $\nabla \times \mathbf{A}' = 0$, $\mathbf{A}'$ is equivalent to some effective pure gauge vector potential. The equation of motion can be rewritten as

$$\frac{1}{2m} (P_1^2 + P_2^2) \psi_A' = \frac{1}{2m} (\epsilon^2 - m^2) \psi_A', \quad (12)$$

where $\psi_A' = e^{-i\mu \sigma_3 \int \mathbf{E} \cdot \hat{k} \cdot d\mathbf{r}} \psi_A$ with $\hat{k}$ being the unit vector in the z direction. Here we have used the fact that $\mathbf{A}' = \mu \sigma_3 \mathbf{E} \times \hat{k}$. Eq.(12) is an equation of motion for a force free particle. It is now evident that if the particle travels a closed path which includes the filament, a topological phase will result. The phase is given by

$$\phi_{AC} = -\mu \sigma_3 \oint (\mathbf{E} \times \hat{k}) \cdot d\mathbf{r} = \mu \sigma_3 \int_S \nabla \cdot \mathbf{E} dS = \mu \sigma_3 \lambda, \quad (13)$$

where λ is the linear charge density of the filament. This phase is purely quantum mechanical and has no classical interpretation. The appearance of σ_3 in the phase represents the spin degrees of freedom. We see that different components acquire phases with different signs. This is also one of the points that distinguishes the AC effect from the AB effect. What has been demonstrated here is that when certain conditions are satisfied the system of a neutral particle with a non zero magnetic dipole moment moving in a electric field configuration in the above case has an additional gauge symmetry and the electric field in the charge free region is a pure gauge vector potential of that symmetry.

Let us now study a more general case without assuming the electric field to be two dimensional and without the assumption of z independence, to see what conditions must be satisfied in order to obtain a nontrivial topological phase. We will formulate the problem using the four component form of the Dirac equation. We should look for

solutions of Eq.(6) which can be written, in a charge free region, as solutions of a force free equation of motion

$$(\partial + m)\psi' = 0, \quad (14)$$

where $\psi' = e^{i\mathbf{A} \cdot \mathbf{r}} \psi$ with ψ a solution of Eq.(6). Here $\mathbf{A}'$ is a linear function of $\mathbf{E}$ and $\mathbf{a}$ is a matrix which will be determined in the following. Of course, in order that the phase be topologically non trivial, there must be region in which $\nabla \times \mathbf{A}' \neq 0$ and in the charge free region it is zero. In the particular case discussed above, $\psi' = e^{-i\mu\sigma_{12}\gamma_4} \int \mathbf{E} \cdot \mathbf{k} d\mathbf{r}$ with $\mathbf{E}$ being two dimensional in the $x-y$ plane. Writing Eq.(14) in terms of ψ in the charge free region, we obtain

$$(e^{-i\mathbf{A} \cdot \mathbf{r}} \int \mathbf{A}' \cdot d\mathbf{r} \gamma_\mu e^{i\mathbf{A} \cdot \mathbf{r}} \int \mathbf{A}' \cdot d\mathbf{r} \partial_\mu + \gamma \cdot \mathbf{A}' i\mathbf{a} \gamma_4 + m)\psi = (\varepsilon^2 - m^2)\psi \quad (15)$$

We find $\mathbf{a}$ and $\mathbf{A}'$ must satisfy

$$\begin{aligned} \mu \gamma \cdot \mathbf{E} &= i\mathbf{a} \gamma \cdot \mathbf{A}' \\ \mathbf{a} \gamma_4 \gamma_\mu &= \gamma_\mu \mathbf{a} \gamma_4 \end{aligned} \quad (16)$$

$\mathbf{A}'$ can not simply be proportional to $\mathbf{E}$ because $\nabla \times \mathbf{E}$ is always zero in the static situation we are considering. The matrix $\mathbf{a}$ can be expressed by some linear combination of the complete set of 4×4 matrices $1, \gamma_5, \gamma_\mu, \gamma_\mu \gamma_5$, and $\sigma_{\mu\nu}$. The second equation of Eq.(16) can not be satisfied if all of γ_1, γ_2 and γ_3 are present in Eq.(15), but two of them are possible. This indicates that the AC topological phase can only arise when the particle is moving in two spatial dimensions. If one chooses $\mathbf{a}$ to be one of the matrices σ_{12}, σ_{23} , or σ_{13} , one can get two dimensional solutions. For other choices of $\mathbf{a}$ there is no desired solutions. To be specific, let us choose $\mathbf{a} = -\sigma_{12}$. In this case, γ_3 cannot appear in the equation which implies P_3 (i.e. $\partial_3 \psi$) and E_3 are zero. From the first of Eq.(16), we then have

$$\mu(\gamma_1 E_1 + \gamma_2 E_2) = -i\sigma_{12}(\gamma_1 A'_1 + \gamma_2 A'_2) \quad (17)$$

and therefore $A'_1 = \mu E_2, A'_2 = -\mu E_1$. At this stage one must check if in charge free region $\nabla \times \mathbf{A}' = 0$. From the above, we have $\nabla \times \mathbf{A}' = -\mu(\partial_1 E_1 + \partial_2 E_2)$. We conclude that $\partial_1 E_1$ must also be zero in order the phase developed to be purely topological.

From the above discussion we see that in order to develop a AC topological phase several conditions must be satisfied. We summarize the conditions in the following:

- i) a neutral spin half particle with a non zero magnetic dipole moment is moving in a plane, say the $x-y$ plane, in a external electric field $\mathbf{E}$;

- ii) On this plane, $E_3 = 0, \partial_3 E_3 = 0$; and $\partial_3 \psi = 0$, and

- iii) It is possible for this particle to travel a closed path in the $\nabla \cdot \mathbf{E} = 0$ region which is such that a surface whose boundary is the path intersects regions where $\nabla \cdot \mathbf{E} \neq 0$.

When conditions i) and ii) are satisfied, there is an additional gauge symmetry for the system. The electric field is a pure gauge vector potential of this symmetry. Condition iii) is the condition for the space to be topologically non trivial.

Having understood how the topological AC and AB phases are developed, we now consider the possibility of the interference of these two phases. This can happen if we let a charged particle with a non zero anomalous magnetic dipole moment μ_k , for example a proton, to move in the $x-y$ plane in the vector potential $\mathbf{A}$ produced by a infinitely long solenoid along the z axis and the electric field $\mathbf{E} = -\nabla V$, where V is the electric potential produced by a infinitely long filament carrying a uniform charge density parallel to the solenoid (not necessarily located at the same place as the solenoid). The equation of motion is

$$(\partial - ie \mathbf{A} + \frac{1}{2} \mu_k \sigma_{\mu\nu} F^{\mu\nu} + m)\psi = 0. \quad (18)$$

In our case it becomes

$$(\partial' + m - ie\gamma_4 V - ie(\gamma_1 A_1 + \gamma_2 A_2) + \mu_k(\gamma_1 E_1 + \gamma_2 E_2)\gamma_4)\psi = 0, \quad (19)$$

where $\partial' = \gamma_4 \partial_4 + \gamma_1 \partial_1 + \gamma_2 \partial_2$. The above equation can be rewritten in the following form.

$$(\partial' + m - ie\gamma_4 V)\psi' = 0, \quad (20)$$

with $\psi' = e^{-i \int (\mathbf{eA} + \mu_k \sigma_{12} \gamma_4 (\mathbf{E} \times \mathbf{k})) \cdot d\mathbf{r}} \psi$. When the particle moves in the $x-y$ plane in a closed path, different topological phases will be developed depending on how the particle circles around the solenoid and the filament. If the path encloses both the solenoid and the filament, the topological phase will be $\phi = -e\oint + \mu_k \sigma_{12} \gamma_4 \lambda$ and if the path only encloses either the solenoid or the filament alone, the topological phase developed will be $\phi_B = -e\oint$ or $\phi_C = \mu_k \sigma_{12} \gamma_4 \lambda$, respectively. Even though the particle experiences a classical force when moves out side of the solenoid and in charge free region, the phases developed are still topological in nature. Interference of the AB and AC phases may be observed, at least in principle.

Let us now discuss the AC effect for a charged particle with a non zero anomalous magnetic dipole moment in 2+1 dimensions. In an external electric field $\mathbf{E} = (E_1, E_2)$, the equation of motion is

$$[\partial_0 - ieV(\mathbf{r}) + m\sigma_3 + \frac{1}{i}\sigma'_1 \cdot \nabla - s\mu_4 \sigma'_1 \cdot \mathbf{A}']\psi(s) = 0, \quad (21)$$

where $s = \pm 1$ are for spin up and down respectively, $\sigma' = (\sigma'_1, s\sigma'_2)$, $\mathbf{A}' = (E_2, -E_1)$ and $\mathbf{E} = -\nabla V$. In the charge free region, the above equation can be written as

$$[\partial_0 - ieV + m\sigma_3 + \frac{1}{i}\sigma'_1 \cdot \nabla]\psi(s) = 0, \quad (22)$$

with $\psi' = e^{-is\mu_4 \int \mathbf{A}' \cdot d\mathbf{r}} \psi$. When the particle circles the charge with a closed path in the charge free region, a topological phase $\phi = s\mu Q$ (where Q is the charge enclosed by the path) will result.

For a system which contains two identical charged Dirac- Fermi particles, the motion of one particle (1) around the other (2) is described by Eq.(22) with $\mathbf{E} = e\mathbf{r}/(2\pi r^2)$, the electric field produced by particle (2). This system is very similar to the one considered in [10] where the system contains two tightly bound fluxtube-charged- particle composites. If one replaces e by Φ and $s\mu_4$ by e in Eq.(22), one obtains the equation of motion for the system in Ref[10]. All the arguments in Ref[10] for the anyon behavior of the particles can be applied to our system. We express the two-body wave function Ψ in terms of the center-of-mass coordinates R, θ and the relative coordinates r, φ . When interchange these two particles, we have

$$\Psi(R, \theta; r, \varphi + \pi) = -e^{is\mu_4 e} \Psi(R, \theta; r, \varphi). \quad (23)$$

The minus sign is due to the fact that ψ is a fermion. Eq.(23) means that the particle has a fractional statistic phase, $\phi_s = \pi + s\mu_4 e$. We therefore have shown that charged particles with non zero anomalous magnetic dipole moment behave like anyons in 2+1 dimensions. This effect may have significant implications in high T_C superconductivity [11].

To summarise we have clarified the conditions for developing a topological AC phase, and have pointed out the possibility of observing the interference between the AB and AC effects. We have also shown that in 2+1 dimensions, charged particles with non zero anomalous magnetic dipole moment behave like anyons.

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