Ground state instability in a gauge theory model of a doped planar antiferromagnet

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Abstract

We study the spectrum of bound states in the $(2 + 1)$ -dimensional CP^1 model coupled to N flavours of Dirac fermions. The case $N = 2$ corresponds to the long-wavelength limit of a microscopic model of the dynamics of holes in a planar antiferromagnet. Fermions of opposite charge with respect to the CP^1 gauge field are found to be subject to attractive forces corresponding to a $1/R$ two-body potential in the large- N limit. For $N < N_c = 8/\pi$ the corresponding quantum mechanical model is closely related to the problem of an electron moving in a supercritical Coulomb field. At zero temperature the formation of fermion bound-states is energetically favoured and this is interpreted as leading to hole-pair condensation and a stable superconducting groundstate. The zero temperature mass-gap and coherence length are calculated in this approach. Finite temperature corrections to the two-body potential are derived and the existence of a critical temperature at which superconductivity is extinguished is demonstrated. The model is generalised to include additional four-fermion operators which arise naturally in the underlying statistical model. These are found to have a large effect on the coherence length, reducing it by a factor of about 50. The possible relation of this model to high- T_c superconductivity is discussed.

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The correct theoretical description of high temperature superconductivity is still unknown. In many of the high- T_c materials (such as YBaCuO and La(Sr)CuO) superconductivity is thought to arise primarily in layers of copper oxide and recent theoretical models have introduced concepts such as fractional spin and statistics which are special to two dimensions [1]. These anyonic models are formulated in terms of a statistical gauge field which provides attractive forces between the charge carriers and it has been suggested in Ref[2] that anyonic superconductivity can be understood as a dynamical mass generation phenomenon in three spacetime dimensions due to this interaction. This immediately suggests an analogy with the conventional BCS theory in which the quantum of the statistical gauge field plays the role of the phonon. In a subsequent paper [3] this idea was generalised to a new model in which fractional statistics do not occur. This was motivated both by general energetics arguments for parity conservation [2,4] and the fact that, so far, parity violation has not been observed experimentally. The purpose of this letter is to develop the analogy with the BCS theory further and illustrate the mechanism of hole-pair condensation in an effective quantum mechanical picture. In addition we examine the finite temperature behaviour of the theory and demonstrate the existence of a critical temperature. We report several new results for the model of Ref[3] and comment briefly on their possible relation to the high- T_c phenomenon. A more detailed account of this work is in preparation.

The model introduced in [3] is based on Shankar's model of dynamical holes in a planar quantum antiferromagnet [5]. The holes are described by Grassmann fields on a bipartite lattice which represents the Néel ordered state at half-filling. Shankar demonstrates that the transition of holes between the two sublattices is suppressed in the limit of large spin and this effectively defines two 'colours' of holes. The Grassmann fields are assembled as staggered fermions and the usual species doubling leads to two fermion 'flavours'. A naive continuum limit of the theory is derived in the large $-S$ and low doping approximation [6] and leads to an effective theory in which the holes are described by relativistic Dirac fermions coupled to a $U(1)$ statistical gauge field. The complete effective action for the system is then obtained by adding these terms to the CP^1 -model [6] that describes the spin degrees of freedom ;

$$S_{eff} = \int d^3x \left[\frac{1}{\gamma_0} |(\partial_\mu - ig a_\mu)z|^2 + \bar{\psi}_i (i\not{\partial} + S\not{\tau}_3) \psi_i \right] \quad (1)$$

where $\bar{z}z = 1$ and γ_0 is positive (at the microscopic level it is associated with the Heisenberg interaction, doping concentration etc [3,5,6]). The relationship $g = -S$ [3] and the absence of a Maxwell term for the gauge field means that both these couplings can be eliminated by rescaling a_μ and play no further role. The spinors in (1) are two-component in Dirac space, two-component in colour space, because of the antiferromagnetic sublattice structure mentioned above, and two-component in the flavour space ($i = 1, 2$) characterizing the staggered fermions. The τ_3 matrix is $Diag(1, -1)$ acting on colour indices.

In what follows we will study a more general continuum model in which the flavour index $i = 1, 2$ is extended to $i = 1, 2 \dots N$ and consider results for the physical case $N = 2$

at the end of the calculation. Other values of N may be realised in different physical models [3]. In addition we will also consider the effect of four-fermion interactions of the form $(\bar{\psi}_i \tau_3 \psi_i)^2$. Such terms have a natural origin in the underlying statistical model [3] and are known to be relevant operators in (2+1)-dimensions [12], at least in the context of the large- N expansion. Our starting point then is the field theory described by the Lagrangian;

$$\mathcal{L} = \frac{1}{\gamma_0} |(\partial_\mu - i \frac{1}{\sqrt{N}} a_\mu) z|^2 + \bar{\psi}_i (i \not{\partial} - \frac{1}{\sqrt{N}} \not{a} \tau_3) \psi_i - \frac{G_0}{2N} (\bar{\psi}_i \tau_3 \psi_i)^2 \quad (2)$$

where the gauge field has been rescaled by a factor of $\sqrt{N}$. Initially we will consider only the case $G_0 = 0$, the effect of non-zero G_0 on our results will be described subsequently. In Ref. [3] we studied the occurrence of superconductivity in this model. In particular, using a large- N approximation to the Schwinger-Dyson equations, we demonstrated the existence of a zero-temperature mass gap for $N < N_c^{SD} = 32/\pi^2$ which leads to superfluidity and hence superconductivity when the system is coupled to electromagnetic fields. This was interpreted in terms of the transition of the system to a phase in which the global $U_V(1)$ symmetry of the action (1) under ordinary phase rotations, $\psi \rightarrow \exp(i\theta)\psi$, is implemented in the Kosterlitz-Thouless (KT) mode, according to the general analysis of (2+1)-dimensional gauge symmetries given in [7]. This symmetry is gauged when the system is coupled to electromagnetism and its spontaneous breaking leads directly to the Meissner effect. Below T_c , due to the mixed Chern-Simons term between statistical and electromagnetic gauge fields [3], any configuration for the electromagnetic potential with non-zero flux is wiped out as a result of the non-fixed boundary condition requirement for the monopoles [8,2]. This implies that electromagnetic vortices (which are monopoles in (2+1)-dimensions) are confined in the superconducting phase in agreement with the KT nature of the phase transition [2]. This is in contrast to the situation in conventional superconductors where finite penetration depth for external magnetic fields prevents vortex confinement [9]. Details will be given in a future publication. Hence the planar superconductors considered here evade the Mermin-Wagner theorem [10] and do not require an interplanar coupling.

In order to examine the mechanism which gives rise to the superconducting ground state we will again employ a large- N approximation and hope that the results are at least qualitatively reliable in the case $N = 2$. To leading order in $1/N$, the term in the effective action quadratic in the gauge field a_μ is given by the 1PI two-point vacuum polarisation diagram. Evaluating this using a gauge-invariant regulator gives,

$$S_{eff}^a = \int \frac{d^3 p}{(2\pi)^3} \frac{\Pi_a(p)}{2} a(p)^\mu (p^2 \delta_{\mu\nu} - p_\mu p_\nu) a(-p)^\nu \quad (3)$$

where $\Pi_a(p) = N/8p$. The fermions describing the propagation of holes on the two sublattices (the two ‘colours’) have opposite charges with respect to the gauge field and are subject to attractive forces. At leading order in $1/N$ the effective static potential between

two fermions of opposite charge is given by

$$V(R) = - \int_{-\infty}^{\infty} dt \Delta_{00}(R, t) = - \frac{4}{N\pi R} \quad (4)$$

where R is the spatial separation between the holes and $\Delta_{\mu\nu}(R, t)$ is the position-space propagator for the a_μ field derived from the effective action (3). Gauge fields in (2+1)-dimensions usually give rise to logarithmic potentials which confine at large distances, the $1/R$ behaviour seen here is due to the non-trivial momentum dependance of the vacuum polarisation [13] and the absence of a bare Maxwell term. A novel feature of the model is that in this approximation the potential is independent of coupling constants and the role of the dimensionless coupling which occurs in four dimensions is played by the flavour number N . This already suggests the occurrence of strong pairing forces between fermions in the physical case $N = 2$. In a more complete treatment holes carry not only statistical charge but also a real (three-dimensional) electric charge e and an ordinary Coulomb repulsion term must be added to $V(R)$. Assuming these effects do not interfere significantly, then the $1/R$ behaviour allows a direct comparison of these terms and given the small numerical value of the electromagnetic fine structure constant in the units in which we are working ($\hbar = v_F = 1$) it is easy to see that, for $N = 2$, the statistical attraction is the stronger effect by many orders of magnitude! A more complete analysis of electromagnetic effects in this model is in progress. In what follows we shall show that, for $N < N_c = 8/\pi$, it is energetically preferable for holes of opposite statistical charge to form s -wave bound-states. These composite bosonic states have electric charge $2e$ and are the analogues of Cooper pairs in our model: they are the charge carriers of the superconducting groundstate. The s -wave pairing has to be contrasted with the p -wave pairing occurring in anyon models with P-violation [14]. There, due to the Chern-Simons term, there is a ‘magnetic’ attraction which binds the pairs in p -waves [1,14].

The derivation of the effective action (3) and potential (4) may be easily generalised to non-zero temperature by repeating the calculation of the one-loop vacuum polarisation graph using the standard finite- T formalism (see for example [15]). In what follows we will only need the leading short distance/low temperature correction to $V(R)$. Here we merely state the result of a somewhat lengthy calculation,

$$V(R, \beta) = V(R) + \frac{A}{\beta} + O\left(\frac{R}{\beta^2}\right) \quad (5)$$

where $\beta = (k_B T)^{-1}$ and the coefficient A may be expressed as

$$A = \frac{2}{\pi} \int_0^\infty dy \frac{y I_1(y) + 2 I_2(y) - y K}{y I_1(y) + 2 I_2(y)} \quad (6)$$

with

$$I_1(y) = \int_0^1 dx \sqrt{x(1-x)} \tanh\left(\frac{y}{2} \sqrt{x(1-x)}\right) \quad I_2(y) = \int_0^1 dx \log(2 \cosh(\frac{y}{2} \sqrt{x(1-x)}))$$

and $K = \pi/4$. A numerical evaluation of the integrals yields the result $A \simeq 0.6$. The occurrence of an effective two-body potential which is strongly dependent on temperature is a distinctive feature of three space-time dimensions and may have important physical consequences for planar superconductivity.

Now that we have described the interactions of the holes in terms of an effective potential we can formulate a simple quantum mechanical model of hole pairing. The (accidental) ‘relativistic’ form of the continuum action leads us to consider the effective problem of a relativistic fermion interacting with a point-like Coulomb center in the plane. The wavefunctions $\psi(R, \theta)$ obey the two dimensional Dirac equation:

$$(\sigma \cdot \partial + W(R))\psi(R, \theta) = E\psi(R, \theta) \quad (7)$$

where $W(R) = V(R, \beta)$ for $R > a$ where a is the intersublattice spacing. The physical condition that holes can not be brought closer together than neighbouring lattice sites is imposed by setting $V(R) = \infty$ for $R < a$.

The problem of a Dirac fermion moving in a strong Coulombic potential is not a new one: the corresponding equation in three spatial dimensions arises in the case of an electron moving in the field of superheavy nucleus, with $Z > 137$, and was first studied by Popov [16]. More recently, interest in this phenomena has been renewed by the work of Miransky et.al. [17] due to its relation to chiral symmetry breaking in strongly coupled QED. In that case the phenomenon of ‘short-distance collapse’ occurs which can only be interpreted by introducing an ultra-violet cut-off and performing a rather mysterious renormalization of the coupling. Here the lattice spacing acts as the natural short distance cut-off for the problem and leads to physical results. After separating the equation in polar coordinates and identifying the appropriate eigenstate of orbital angular momentum zero, the radial component of the wavefunction, $\psi(R)$, is found to obey a differential equation of Whittaker form [18];

$$\frac{d^2\psi}{d\rho^2} + \left[-\frac{1}{4} + \frac{\kappa}{\rho} + \frac{(\frac{1}{4} - \mu^2)}{\rho^2} \right] \psi = 0 \quad (8)$$

where $\rho = 2iER$, $\kappa = -4i/N\pi$ and $\mu = \sqrt{(N_c^2/N^2 - 1)}/2$ with $N_c = 8/\pi$. The corresponding solution with the correct asymptotic behaviour for large R is proportional to the Whittaker function $W_{\kappa, \mu}(2iER)$. As usual, the boundary condition for the infinite potential barrier at $R = a$ is $\psi(a) = 0$. Using the expansion of the Whittaker function for small arguments this condition gives the relation,

$$\frac{\Gamma(-2i\mu)}{\Gamma(\frac{1}{2} - \kappa - i\mu)}(2iEa)^{\frac{1}{2}+i\mu} + \frac{\Gamma(2i\mu)}{\Gamma(\frac{1}{2} - \kappa + i\mu)}(2iEa)^{\frac{1}{2}-i\mu} = 0 \quad (9)$$

For $N < N_c$, the corresponding energy eigenvalues are complex and the energy levels are quasi-stationary [16]. In the case $N = 2$, the ground state energy may be calculated from (9) in terms of the the lattice spacing and is given by

$$E(0) \simeq \frac{0.49}{a}(\cos(\gamma) + i\sin(\gamma)) \exp\left(-\frac{\pi}{\mu}\right) \quad (10)$$

where $\gamma = (1.05)\pi$. Evaluating this and putting in the units gives $\text{Re}E(0) \simeq -0.16 \times 10^{-3}(\hbar v_F/a)$.

The occurrence of a ground state energy level with a negative real part and a small phase signals the formation of a quasi-stationary s -wave boundstate of two holes. It can easily be checked that introducing a mass into the Dirac equation reduces the phase and consequently stabilises the bound-state. In field-theoretic terms this corresponds to the transition of the system to a new vacuum state in which a mass for the fermions is dynamically generated; the superconducting ground state. This is analogous to the Cooper instability [19] in a degenerate fermi gas. As in the case of QED₄ [17], there is a direct relationship between the quantum mechanical problem and the corresponding Schwinger-Dyson equation analysis in the field theory, which in this case is closely related to QED₃ as considered in [20]. The absolute value of the real part of the energy is identified with the superconducting mass-gap $2\Delta(0)$ and using typical values of the lattice spacing and Fermi velocity gives values of the same order of magnitude as the BCS theory. Specifically for $N = 2$, with $a = 10^{-9}\text{m}$ and $E_F \simeq \hbar v_F/a \simeq 10\text{eV}$, we have, $2\Delta(0) \simeq 1.6\text{meV}$. Identifying the coherence length, λ , of the superconductor with the radius of the bound state gives $\lambda \simeq 6000a \simeq 6 \times 10^{-6}\text{m}$, again comparable with the BCS theory. While the occurrence of such a large natural length scale for the model justifies our use of an effective continuum description, it is disappointing from a phenomenological point of view considering that experimentally observed coherence lengths in high- T_c superconductors are typically less than 10 lattice spacings [21]. Consequently, it is of interest to identify an effect that drastically reduces the coherence length in our model. As mentioned above we consider the effect of additional four-fermion operators, in particular the case $G_0 > 0$ in (2). This interaction provides an additional attraction between fermions of opposite statistical charge which has the desired effect of reducing the coherence length.

Following the standard procedure, the four-fermion interaction may be linearized by introducing an auxiliary scalar field ϕ ,

$$\mathcal{L} = \mathcal{L}_{gauge} + \bar{\psi}_i i \not{\partial} \psi_i + \phi \bar{\psi}_i \tau_3 \psi_i + \frac{N}{2G_0} \phi^2. \quad (11)$$

The propagator, $D(p^2)$, for the scalar field may be calculated to leading order in $1/N$ by summing fermion bubble graphs,

$$D^{-1}(p^2) = \frac{N}{G_0} + N \int_{k^2 < \Lambda^2} \frac{d^3 k}{(2\pi)^3} \text{Tr} \left(\frac{1}{\not{k} + M} \cdot \frac{1}{\not{p} - \not{k} + M} \right) \quad (12)$$

where M is a dynamically generated fermion mass which is non-zero at leading order in $1/N$ for $G_0 \Lambda > \pi^2/2$. Renormalizing the coupling to remove the linear divergence in (12) gives a scalar propagator [12];

$$D(p^2) = \frac{2\pi}{N} \frac{p}{(p^2 + 4M^2) \tan^{-1}(p/2M)} \quad (13)$$

The effective static potential now receives an additional contribution at leading order in $1/N$ from scalar exchange. For $M \ll \Lambda$, the new potential $\bar{V}(R)$ is given by $(1.5)V(R)$ (this remains true for $T > 0$). The effect of this is simply to change the values of the parameters of the Whittaker function. In particular $\bar{\kappa} = (1.5)\kappa$ and $\bar{N}_c = (1.5)N_c$. The value of the groundstate energy is exponentially sensitive to a change in the critical value of N and consequently, in the case $N = 2$, the coherence length is reduced by a factor of about 50.

Including the lowest order term in the low-temperature/short-distance expansion of the potential $V(R, \beta)$ (eqn(5)) shifts the energy levels by a constant. Thus, finite temperature corrections to the mass-gap are given by the simple relation $2\Delta(\beta) = 2\Delta(0) - A/\beta$. This indicates the existence of a critical temperature above which the stability of the normal ground state is restored and superconductivity is extinguished. This effect is entirely due to the temperature dependence of screening effects in (2+1)-dimensions and has no analog in ordinary BCS theory and it is tempting to speculate that it leads to an increase in the mass-gap to critical temperature ratio beyond that predicted by mean-field theory in accord with the experimental situation. The true value of this ratio can only be found when the effects of entropy are fully included in a many-body calculation, this is in progress.

In conclusion, we have presented a field theoretic model of a relativistic superconductor in three spacetime dimensions and examined the formation of charged fermion bound states in an approximate quantum mechanical approach. In the case $N = 2$ the model is the long-wavelength limit of a theory of dynamical holes in a quantum antiferromagnet and our results indicate that a superconducting groundstate arises as a result of the statistical gauge attraction between holes belonging to the two sublattices defined by the antiferromagnetic order at low temperature. Within our approximate approach we have calculated the coherence length, mass gap and finite temperature properties of the model. It is important to check these results beyond the $1/N$ expansion and in particular they should be tested by lattice simulations.

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