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GAUGE DEPENDENCE IDENTITIES AND THEIR APPLICATION AT FINITE TEMPERATURE

R. KOBES

*Physics Department, University of Winnipeg,
515 Portage Ave., Winnipeg, Manitoba, Canada R3B 2E9*

G. KUNSTATTER*

*L.P.T.H.E., Université de Paris-Sud,
Bât. 211, F-91405 Orsay, France*

and

A. REBHAN

*Theory Division, CERN
CH-1211 Geneva 23, Switzerland*

ABSTRACT

We derive a general set of identities that determine the gauge dependence of the effective action and all associated n -point vertex functions in a wide class of gauge theories. These identities are then used to resolve the recent controversy concerning the dispersion relations in high-temperature QCD and quantum gravity. In particular, we show that the leading temperature contributions to all vertex functions in QCD, and the 1- and 2-point functions in gravity are gauge covariant and gauge fixing independent. Beyond leading order we establish gauge independence of the QCD plasma dispersion relations and show how gauge independent dispersion relations can in principle be extracted from the thermal curved-space graviton propagator.

*Permanent address: *Physics Department, University of Winnipeg, Manitoba,
Canada R3B 2E9*

1. Introduction

The question of how to extract physical quantities from the expectation values of gauge variant operators has been around for many years. One of the original contexts in which this problem arose was spontaneous symmetry breaking by radiative corrections. Shortly after the seminal paper of Coleman and Weinberg [1] showing that radiate corrections could indeed induce spontaneous symmetry breaking, it was shown by Jackiw [2] that the effective potential $V[\phi]$ used in such calculations was gauge fixing dependent, and that the physical quantities one calculated, such as masses, were also in principle gauge fixing dependent at the one-loop level. This triggered a long controversy whose resolution was found in the existence of a set of identities (sometimes called Nielsen identities) [3, 4] that determine the dependence of the effective potential on the gauge parameter, α :

$$\frac{\partial V[\phi]}{\partial \alpha} + \frac{d\phi}{d\alpha} \frac{\partial V[\phi]}{\partial \phi} = 0. \quad (1.1)$$

These identities state that the total derivative of the effective potential with respect to the gauge parameter α vanishes when the corresponding shift in the expectation value of the quantum field is taken into account. As shown by Nielsen [3], this in turn implies that spontaneous symmetry breaking through radiative corrections, if it exists, is a gauge fixing independent phenomenon, despite the gauge fixing dependence of the effective potential from which it is derived. These identities can also be used to show that the masses etc. are gauge fixing independent, if calculated accurately.

More recently, there has been a great deal of controversy about the correct definition of the plasma parameters in high-temperature QCD [5]. At temperatures above about 200 MeV, QCD is thought to undergo a deconfining phase transition to a plasma of asymptotically free quarks and gluons. For non-gauge theories, the plasma parameters would be simply extracted from the poles of the particle propagators using linear response theory [6], the plasmon mass being determined by the real part of the pole, and the damping constant coming from the imaginary part. In gauge theories, the linear response analysis appears to be problematic due to the gauge dependence of both the gluon and quark propagators. Nonetheless, the plasmon mass obtained from the gluon propagator at the one-loop level turns

out to be gauge fixing independent. However, the one-loop damping constant is gauge fixing dependent [7]. This fact has led to the introduction of new, improved definitions for the plasmon dispersion relations based on modified, manifestly gauge fixing independent propagators, which, however, again led to mutually disagreeing results [7].

The correct resolution of this problem was in fact suggested several years ago by several authors. Pisarski [8] in particular has stated that the one-loop result for the damping constant is not accurate: a resummation of an infinite number of loop diagrams is required to obtain the correct answer to leading order in the coupling constant, and that gauge independence should be restored by the higher-loop contributions. However, it was only recently that a definite proof was provided. First of all, Braaten and Pisarski [9, 10] did the resummation in the Coulomb gauge and gave arguments suggesting that in covariant gauges the gauge fixing parameter would cancel in a resummed series. Independently, the present authors [11] used a set of identities which in essence are generalizations of eq. (1.1) above, to prove that, modulo certain technical restrictions, the locations of the physical poles (to be precisely defined in the following) in the gluon propagator are gauge fixing independent if calculated using a self-consistent approximation scheme. Given this result, it follows that the gauge dependence of previous one-loop calculations indeed just signaled the breakdown of the ordinary loop expansion, and not the need for a modified, manifestly gauge independent linear response theory.

The purpose of the present paper is not only to provide the details of the proof outlined in Ref. 11, but also to extend the discussion to include e.g. background field gauges, fermions, and also gravity (where the literature is also in disagreement on the correct definition of the finite-temperature dispersion relations). By doing so we hope to emphasize the general utility of the gauge dependence identities with regard to questions concerning gauge dependence of physical quantities when obtained within an inherently gauge dependent framework.

This paper is organized as follows: In Sect. 2 a set of general gauge dependence identities is derived for a large class of gauge theories and gauge fixing conditions. The diagrammatic expansion of the quantities which occur in the identities is discussed, and the proof is extended to background field gauges. As a first application

at finite temperature, in Sect. 3 these identities are used to explain the gauge independence of the so-called hard thermal loops in both QCD and gravity. In Sect. 4, the gauge fixing identities are applied to the dispersion relations derived from gauge dependent propagators in finite temperature gauge theories. First, the resolution to the problem of gauge dependence of the one-loop QCD plasmon damping constant is presented. Then the analogous issue for the quark propagator is treated, and finally the question of the correct definition of dispersion relations in finite temperature quantum gravity is discussed. Sect. 5 contains the conclusions. Several technical issues have been relegated to Appendices: In Appendix A an alternative derivation of the gauge fixing identities using BRS techniques is presented. Appendix B extends the gauge fixing identities, which initially apply to infinitesimal variations of the gauge parameters, to finite variations, and gives some explicit solutions for the abelian case. Finally, in Appendix C, the gauge independence proof of the gluon plasma dispersion relations is extended to gauges which break the spatial symmetries of the plasma.

2. The Gauge Dependence of the Effective Action

2.1. THE GAUGE DEPENDENCE IDENTITIES

The following will be formulated entirely from the point of view of the effective action, which in perturbation theory is the generating functional of one-particle irreducible (1pi) vertex functions. The main advantage over using the generating functional for connected Green functions is that through the former it will be possible to also cover the case of background field gauges which require special care.

Also, in order to derive the basic relations in their most general form which is the same in all gauge theories (with an off-shell closed gauge algebra), we shall extensively use the compact notation of DeWitt [12], where the complete set of fields prior to gauge fixing is denoted by φ^i , and the index i comprises all discrete and continuous labels of the fields. The summation convention then also implies integration of the space-time variables with appropriate measure. Functional derivations with respect to φ^i are abbreviated by a comma and subsequent indices, where for simplicity we take all classical fields as commuting; the extension to Grassmann-valued fields is mostly straightforward (see Ref. 13).

Let the classical action of the gauge theory under consideration be $S[\varphi]$ with a complete and independent set of generators of the (local) gauge transformations, denoted by $D_\alpha^i[\varphi]$,

$$S[\varphi] = S[\varphi + \delta\varphi], \quad \delta\varphi^i = D_\alpha^i[\varphi]\delta\xi^\alpha. \quad (2.1)$$

(Greek indices in the compact notation label the gauge group and also include in general a space-time variable.) The only restriction we have to impose is that the gauge generators form an off-shell closed algebra, that is, the commutator of two gauge transformations is again a gauge transformation (in an off-shell open algebra this need be true only up to terms proportional to the field equations $S_{,i}[\varphi]$),

$$D_{\alpha,j}^i[\varphi]D_\beta^j[\varphi] - D_{\beta,j}^i[\varphi]D_\alpha^j[\varphi] = D_\gamma^i[\varphi]c_{\alpha\beta}^\gamma[\varphi], \quad (2.2)$$

where $c_{\alpha\beta}^\gamma = -c_{\beta\alpha}^\gamma$. Notice that we do not exclude the case of field-dependent structure functions $c_{\alpha\beta}^\gamma$. The following discussion will therefore be quite general, and only later on we shall specify to the important special cases of Yang-Mills theories and Einstein gravity.

Gauge invariance of the classical action is expressed by the identities

$$S_{,i}[\varphi]D_\alpha^i[\varphi] \equiv 0. \quad (2.3)$$

When expanded in powers of φ , this gives all the tree-level Ward identities among the classical vertices. In particular, eq. (2.3) implies that the kinetic kernel $S_{,ij}[\varphi]$ possesses on-shell zero modes: the gauge generators $-$,

$$S_{,ij}[\varphi]D_\alpha^i[\varphi] = 0, \quad \text{when } S_{,i}[\varphi] = 0, \quad (2.4)$$

hence $S_{,ij}[\varphi]$ is a singular operator. In order to be able to define its Green function, one needs to fix the gauge freedom, even at the present classical level. This can be done by adding a quadratic gauge breaking term

$$S[\varphi] \rightarrow S_{gf}[\varphi] = S[\varphi] + \frac{1}{2}\eta_{\alpha\beta}F^\alpha[\varphi]F^\beta[\varphi], \quad (2.5)$$

with a functional F^α such that the Faddeev-Popov operator $F^\alpha_i[\varphi]D^\dagger_\beta[\varphi]$ is invertible,* i.e. possesses a Green function $\mathcal{G}$,

$$(F^\alpha_i D^\dagger_\beta[\varphi])\mathcal{G}^\beta_\gamma[\varphi] = -\delta^\alpha_\gamma, \quad (2.6)$$

where δ is to be understood as the appropriate product of Kronecker and Dirac deltas. $\eta_{\alpha\beta}$ is an arbitrary invertible matrix operator independent of φ . An alternative way is to introduce a (Nakanishi-Lautrup) Lagrange multiplier field to implement a strict gauge condition $F^\alpha[\varphi] = 0$, but this is contained in (2.5) through $\eta_{\alpha\beta} \rightarrow \lambda\eta_{\alpha\beta}$ and taking the limit $\lambda \rightarrow \infty$.

Although the gauge dependence of $S_{g,f}$ is unmistakable, it can be expressed by the following identity: An arbitrary infinitesimal variation $F^\alpha[\varphi] \rightarrow F^\alpha[\varphi] + \delta F^\alpha[\varphi]$ and $\eta_{\alpha\beta} \rightarrow \eta_{\alpha\beta} + \delta\eta_{\alpha\beta}$ induces

$$\delta(S_{g,f}) = -(S_{g,f})_i D^\dagger_\alpha \mathcal{G}^\alpha_\beta \delta' F^\beta, \quad \delta' F^\alpha \equiv \delta F^\alpha + \frac{1}{2}\eta^{\alpha\beta}\delta\eta_{\beta\gamma}F^\gamma \quad (2.7)$$

which shows that gauge independence will be achieved again only "on-shell", defined by $(S_{g,f})_i = 0$. The proof of eq. (2.7) is trivial: Writing out $(S_{g,f})_i = S_{g,i} + \eta_{\gamma\delta}F^\gamma F^\delta_i$, the first term is annihilated because of (2.3), and with eq. (2.6) the r.h.s. of eq. (2.7) then equals $-\eta_{\gamma\delta}F^\gamma F^\delta_i D^\dagger_\alpha \mathcal{G}^\alpha_\beta \delta' F^\beta = \eta_{\gamma\delta}F^\gamma \delta' F^\beta$ as it should.

We shall now turn to quantum field theory and show that an identity analogous to eq. (2.7) can be found in terms of an effective action $\Gamma = S + \text{rad. corr.}$. This functional can be defined as the Legendre transformed path integral expression of the generating functional of connected Green's functions

$$\exp iW[J] = \int \mathcal{D}\varphi \mu[\varphi] \det(F^\alpha_i[\varphi]D^\dagger_\beta[\varphi]) \exp i\{S_{g,f}[\varphi] + J_i\varphi^i\}, \quad (2.8)$$

*We assume that any residual gauge invariance can be fixed by consistent boundary conditions. In covariant gauges at zero temperature, this would be the usual Feynman boundary conditions, whereas e.g. in axial gauges the so-called generalized Leibbrandt-Mandelstam prescription [14] would have to be used. We exclude, however, such procedures as for example superaxial gauges [15] which cannot be implemented into the present functional approach in a straightforward fashion. How to correctly treat the temporal axial gauge at finite temperature has recently been established in Ref. 16.

where J_i are external sources, through

$$W[J] = \Gamma[\varphi] + J_i\varphi^i, \quad \varphi^i \equiv \frac{\delta W[J]}{\delta J_i}. \quad (2.9)$$

$\mu[\varphi]$ is an appropriate path integration measure which can be determined by comparison with the canonical formalism. Eqs. (2.8, 2.9) can be summarized in one implicit functional integro-differential equation [17, 18] for $\Gamma[\varphi]$,

$$\exp i\Gamma[\varphi] = \int \mathcal{D}\varphi \mu[\varphi] \det(F^\alpha_i[\varphi]D^\dagger_\beta[\varphi]) \exp i\{S_{g,f}[\varphi] - \Gamma_i[\varphi](\varphi - \varphi^i)\}, \quad (2.10)$$

where there is no longer any explicit reference to external sources - these are now substituted by $\Gamma_i[\varphi] = -J_i$.

Considering again arbitrary variations δF^α and $\delta\eta_{\alpha\beta}$, the classical gauge dependence identity (2.7) suggests that such a change in eq. (2.10) can be compensated by performing the (non-local) gauge transformation

$$\delta\varphi^i = D^\dagger_\alpha[\varphi]\delta\Xi^\alpha[\varphi], \quad \delta\Xi^\alpha[\varphi] = -\mathcal{G}^\alpha_\beta[\varphi]\delta' F^\beta[\varphi], \quad (2.11)$$

on the path integration variable φ . The combined effects of this special gauge transformation and the explicit variations of the gauge fixing functionals leave $S_{g,f}$ invariant, and it remains to verify that the path integration measure together with the Faddeev-Popov determinant is also invariant.

This is indeed the case provided that the measure μ is proportional to $\det^{1/2}(\eta_{\alpha\beta})$ and is gauge invariant in the sense

$$\mu_i[\varphi]D^\dagger_\alpha[\varphi] + \mu[\varphi]D^\dagger_{\alpha,i}[\varphi] \equiv 0, \quad (2.12)$$

and provided that $c^\beta_{\beta\alpha} \equiv 0$. Then one has

$$\begin{aligned} & \delta(\mathcal{D}\varphi \mu[\varphi] \det(F^\alpha_i D^\dagger_\beta)) \\ &= \mathcal{D}\varphi \mu[\varphi] \det(F^\alpha_i D^\dagger_\beta) \left(\text{tr} \frac{\delta(\delta\varphi)^i}{\delta\varphi^j} + \delta \text{tr} \log(F^\alpha_i D^\dagger_\beta) + \delta \log \mu \right) \\ &= \mathcal{D}\varphi \mu[\varphi] \det(F^\alpha_i D^\dagger_\beta) \\ & \times \left(D^\dagger_i \mathcal{G}^\alpha_\gamma (F^\gamma_j D^\dagger^\dagger_i + F^\gamma_j D^\dagger^\dagger_i) \mathcal{G}^\delta_{\beta\delta'} F^{\beta\delta'} + D^\dagger_i \mathcal{G}^\alpha_\beta \delta' F^\beta \right. \\ & \quad \left. - \mathcal{G}^\beta_\alpha [\delta F^\alpha_i D^\dagger_\beta + F^\alpha_{ij} D^\dagger_j \mathcal{G}^\gamma_{\delta'} \delta' F^\delta D^\dagger_\beta + F^\alpha_{ij} (D^\dagger_i \mathcal{G}^\gamma_{\delta'} D^\dagger_j + \mathcal{G}^\delta_{\beta\gamma} D^\dagger_i) D^\dagger_j \mathcal{G}^\delta_{\delta'} F^{\delta'}] \right. \\ & \quad \left. + \frac{1}{2} \eta^{\alpha\beta} \delta\eta_{\alpha\beta} \right) = 0 \end{aligned} \quad (2.13)$$

by virtue of eqs. (2.2, 2.6).

The only term in (2.10) which is not invariant is the source term $\Gamma_i[\bar{\varphi}(\varphi - \bar{\varphi})^i]$, and therefore the quantum version of eq. (2.7) turns out to be

$$\delta\Gamma[\bar{\varphi}] = -\Gamma_i[\bar{\varphi}] \langle D_a^i[\varphi] G^\alpha_\beta[\varphi] \delta^\alpha F^\beta[\varphi] \rangle \equiv \Gamma_i[\bar{\varphi}] \delta X^i[\bar{\varphi}], \quad (2.14)$$

where we generally define

$$\langle \mathcal{O}[\varphi] \rangle = e^{-i\Gamma[\bar{\varphi}]} \int \mathcal{D}\varphi \mu[\varphi] \mathcal{O}[\varphi] \det(F^\alpha_\beta D^\beta_i) e^{i(S_g[\varphi] - \Gamma_i[\bar{\varphi}] \langle \varphi - \bar{\varphi} \rangle^i)}, \quad (2.15)$$

which naturally is a functional of $\bar{\varphi}$.

Eq. (2.14) is a generalization of the so-called Nielsen identities originally used to resolve the issue of gauge dependence of the minima of the effective potential of broken gauge theories [3]. In their general form they have been first obtained by Fukuda and Kugo [4]. The modern way of deriving them is to use (extended) BRS techniques without reference to the path integral [19]. However, there one usually only considers variations of constant parameters within a preassigned class of gauge choices, while the derivation using the functional manipulations due to DeWitt [12] we have followed here is the same for all gauges and arbitrary variations thereof. In Appendix A we show how the latter can be repeated using BRS invariance in the path integral.

Differentiating eq. (2.14) with respect to $\bar{\varphi}$ finally yields the gauge dependence identities for 1pi-vertex functions,

$$\delta\Gamma_{j_1 \dots j_n} = \sum_{k=0}^n \binom{n}{k} \Gamma_{i(j_1 \dots j_k} \delta X^i_{j_{k+1} \dots j_n)}, \quad (2.16)$$

where the bracket around the indices denotes symmetrization.

All of these identities can also be derived in terms of the generating functional $W[J]$. The basic identity (2.14) can be Legendre transformed immediately to read

$$\delta W[J] = J_i \langle D_a^i[\varphi] G^\alpha_\beta[\varphi] \delta^\alpha F^\beta[\varphi] \rangle [J], \quad (2.17)$$

however differentiations with respect to J do not quite give eq. (2.16), for there $\bar{\varphi}$ rather than J is the independent variable.

In the following we shall stick to the gauge dependence identities formulated in terms of the effective action $\Gamma[\bar{\varphi}]$ rather than $W[J]$. This will enable us to cover background field gauges as well. Also the diagrammatic structure of the functional δX and its derivatives turns out to be more transparent this way.

2.2. DIAGRAMMATIC EXPANSIONS

In order to investigate the consequences of the basic relation (2.14) within perturbation theory, in particular for perturbative thermal gauge theories to be considered in the following sections, it is useful to spell out its general diagrammatic content.

The implicit equation for the effective action (2.10) can be solved by iteration so that $\Gamma[\bar{\varphi}]$ is given by a sum of 1pi primary (or vacuum) diagrams, which are displayed in Fig. 1. Full lines stand for the propagator of φ -fields,

$$G^{ij}[\bar{\varphi}] = -[(S_{g,f.})_{,ij}[\bar{\varphi}]]^{-1}, \quad (2.18)$$

and dashed lines correspond to the Faddeev-Popov ghost propagator introduced in eq. (2.6). Since the latter is not symmetric, it carries an orientation marked by small arrows. The propagators are connected by vertices representing $(S_{g,f.})_{,ijk}[\bar{\varphi}]$ for φ -lines, and $(F^\alpha_\beta D^\beta_i)_{,i}[\bar{\varphi}]$ for a ghost line with inserted φ -lines. If both, the gauge generators and the gauge conditions are at most linear in φ , which often is the case, only one φ -line can couple to a ghost line at one time through the vertex $F^\alpha_\beta D^\beta_i$, and the last diagram shown in Fig. 1 does not exist.

The only exception to these rules are the one-loop contributions, where the single closed loop without vertices represents the trace of the logarithm of the respective propagator, rather than only the trace as the diagram itself would suggest.

All the quantities involved are to be considered as functionals in $\bar{\varphi}$, and the 1pi vertex diagrams are obtained by functional differentiation. Because of

$$(G^{ij})_{,k} = G^{im}(S_{g,f.})_{,mnk} G^{nj} \quad (2.19)$$

and

$$(G^\alpha_\beta)_{,k} = G^\alpha_\gamma (F^\gamma_\beta D^\beta_i)_{,k} G^\delta_\beta, \quad (2.20)$$

differentiations with respect to $\bar{\varphi}$ either increase the number of prongs of an existing vertex or insert new vertices into an existing line. Since

$$(\text{tr log } G[\bar{\varphi}])_{,k} = -G^{ij}(S_{g,f.})_{,jik}[\bar{\varphi}] \quad (2.21)$$

and analogously for the ghost loop, this holds true also for the one-loop objects.

A similar expansion into primary diagrams can also be obtained for the functional $\delta X^i[\bar{\varphi}]$ in eq. (2.14), for which one has to introduce two additional types of

vertices, one for $D_\alpha^i[\bar{\varphi}]$ and one for $\delta F^\alpha[\bar{\varphi}]$ and their derivatives, as we have done in Fig. 2. Such an expansion is given up to one-loop order in Fig. 3. It consists of primary diagrams which are 1pi with respect to all lines apart from the basic ghost line attached to δF^α . This peculiar rule arises because the operator insertion $D_\alpha^i G^\alpha \delta F^\beta$ into the effective action, which yields δX^i , has to be expanded in a power series in $(\varphi - \bar{\varphi})$, whose coefficients play the role of additional vertices inserted in all possible ways into the primary diagrams of the effective action. Thus these additional vertices are to be treated as irreducible entities, although from the point of view of the ordinary Feynman rules they are 1p-reducible, for they contain propagators, to wit, the basic ghost line ending at δF^α .

In the case of linear gauge conditions

$$F^\alpha = F_1^\alpha \varphi^i \quad (2.22)$$

(which still allows for background field gauges (or more precisely: mean-field dependent gauges) [12, 20], see below) an important simplification is achieved by subtracting the tree-level gauge breaking term from the effective action.* Defining

$$\bar{\Gamma}[\bar{\varphi}] = \Gamma[\bar{\varphi}] - \frac{1}{2} \eta_{\alpha\beta} F_1^\alpha F_1^\beta \bar{\varphi}^i \bar{\varphi}^j, \quad (2.23)$$

the 'source term' in eq. (2.10) becomes

$$\Gamma_{,i}[\bar{\varphi}](\varphi - \bar{\varphi})^i = \bar{\Gamma}_{,i}[\bar{\varphi}](\varphi - \bar{\varphi})^i - \eta_{\alpha\beta} F_1^\alpha F_1^\beta \bar{\varphi}^j (\varphi - \bar{\varphi})^i, \quad (2.24)$$

so that $\bar{\Gamma}$ satisfies formally the same equation as Γ , eq. (2.10), if one makes the following substitution in the gauge fixing functional:

$$F_1^\alpha \varphi^i \rightarrow F_1^\alpha (\varphi - \bar{\varphi})^i. \quad (2.25)$$

As a consequence $F^\alpha[\varphi]|_{\varphi=\bar{\varphi}} = 0$, and one can drop all diagrams from δX^i which contain an undifferentiated vertex F^α . Assuming furthermore that the gauge generators are at most linear in the fields, $D_{\alpha,j}^i \equiv 0$, the remaining primary diagrams of δX^i up to 2-loop order are those given in Fig. 4.

*This will greatly simplify the discussion in Chapt. 3, but will not be imperative elsewhere.

Since the gauge dependence at the tree level is subtracted out, the diagrammatic expansion of δX^i now starts only at 1-loop order. Thus the gauge dependence of an ℓ -loop n -point 1pi vertex function $\delta\Gamma_{i_1 \dots i_n}$ is determined through eq. (2.16) in terms of products of a vertex function with $\leq (n+1)$ prongs and loop order $< \ell$, and a function obtained by up to n vertex insertions into the primary diagrams of δX^i .

As an example, the gauge dependence of a 1-loop 2-point vertex function $\Gamma_{ij}[0]$ is given in Fig. 5, where for simplicity we have assumed $S_i[0] = 0$.

All that is left to do for a concrete field theory is to specify the various objects like gauge generators and vertices. In QED, for instance, the gauge generators $D_\alpha^i[\varphi]$ of the compact notation have to be expanded as

$$D_\alpha^i \rightarrow \frac{\partial}{\partial x_\mu^{(i)}} \delta(x^{(i)} - x^{(\alpha)}) \quad \text{when } i \leftrightarrow A_\mu \quad (2.26a)$$

$$\rightarrow -ie\psi(x^{(i)})\delta(x^{(i)} - x^{(\alpha)}) \quad \text{when } i \leftrightarrow \psi \quad (2.26b)$$

$$\rightarrow +ie\bar{\psi}(x^{(i)})\delta(x^{(i)} - x^{(\alpha)}) \quad \text{when } i \leftrightarrow \bar{\psi}. \quad (2.26c)$$

One immediate result for the case of QED with linear gauge conditions is that vertex functions with only external photon lines are gauge independent to all loop orders, for $D_{\alpha,j}^i$ is non-zero only for fermionic fields and apart from this no ghost vertices exist, hence one cannot build diagrams for $\delta X_{j\ell}^i$ which are 1pi (with exception of the basic ghost line) as required.

Notice that this does not hold as soon as external fermion lines are involved due to the field dependence of (2.26b, c). Indeed, the fermion self-energy admits a gauge dependence as exhibited in Fig. 6.

Nonetheless, the above result for photon vertex functions already reveals the power (and simplicity) of diagrammatic reasonings based on the gauge dependence identities (2.14, 2.16). We are not aware of any other, similarly simple and general proof of the all-order gauge independence of purely photonic vertex functions.

2.3. BACKGROUND FIELD GAUGES

The above diagrammatic expansions remain the same when the gauge condition is allowed to depend parametrically on another external field $\tilde{\varphi}$, $F_i^\alpha = F_i^\alpha[\tilde{\varphi}]$ and possibly $\eta_{\alpha\beta} = \eta_{\alpha\beta}[\tilde{\varphi}]$. This holds true also if one identifies $\tilde{\varphi} = \varphi$ after having solved the implicit equation (2.10) for $\Gamma[\tilde{\varphi}]$, or that for $\delta X^i[\tilde{\varphi}]$. However, in order to obtain vertex functions, the primary diagrams have to be functionally differentiated, and the external vertices inserted by this will differ from purely internal ones. Frequently, the former are dubbed background-quantum vertices as opposed to ordinary (quantum) vertices.

Such a procedure is in fact followed when one employs background covariant gauges (or more precisely, mean-field gauges) [12, 20]. There one uses gauge conditions of the form (2.25) which are covariant with respect to gauge transformations of $\tilde{\varphi}$, that is

$$F_{i,j}^\alpha[\tilde{\varphi}] D_\beta^j[\tilde{\varphi}] = c_{\beta\gamma}^\alpha F_i^\gamma[\tilde{\varphi}] - D_{\beta,i}^\gamma F_j^\alpha[\tilde{\varphi}], \quad (2.27a)$$

$$\eta_{\alpha\beta,i}[\tilde{\varphi}] D_i^\beta[\tilde{\varphi}] = -c_{\gamma\alpha}^\delta \eta_{\delta\beta}[\tilde{\varphi}] - c_{\gamma\beta}^\delta \eta_{\alpha\delta}[\tilde{\varphi}]. \quad (2.27b)$$

Performing a simultaneous gauge transformation of the external fields $\tilde{\varphi}$ and $\tilde{\varphi}$ as well as of the path integration variable φ , one can then show that the effective action $\Gamma[\tilde{\varphi}; \tilde{\varphi}]$, which through the gauge conditions now depends parametrically on $\tilde{\varphi}$, satisfies

$$\Gamma_{,i}[\tilde{\varphi}; \tilde{\varphi}] D_\alpha^i[\tilde{\varphi}] + \frac{\delta \Gamma[\tilde{\varphi}; \tilde{\varphi}]}{\delta \tilde{\varphi}^i} D_\alpha^i[\tilde{\varphi}] \equiv 0, \quad (2.28)$$

provided the gauge generators are at most linear in φ . Identifying finally $\tilde{\varphi}$ with $\tilde{\varphi}$ then results into a gauge invariant effective action,

$$\Gamma'[\tilde{\varphi}] D_\alpha^i[\tilde{\varphi}] \equiv 0, \quad \Gamma'[\tilde{\varphi}] \equiv \Gamma[\tilde{\varphi}; \tilde{\varphi}]. \quad (2.29)$$

It should be noted, however, that this gauge invariance does not yet guarantee gauge fixing independence, $\delta \Gamma / \delta F = 0$, which indeed does not follow from the former.

The gauge dependence identities (2.14) also pertain to the case of background covariant gauges, the only difference being that everything depends parametrically on $\tilde{\varphi}$,

$$\delta \Gamma[\tilde{\varphi}; \tilde{\varphi}] = \Gamma_{,i}[\tilde{\varphi}; \tilde{\varphi}] \delta X^i[\tilde{\varphi}; \tilde{\varphi}]. \quad (2.30)$$

But after making the identification $\tilde{\varphi} = \varphi$ this analogy seems to be lost because $\Gamma_{,i}[\tilde{\varphi}; \tilde{\varphi}]|_{\tilde{\varphi}=\varphi}$ will be different from $\Gamma'_{,i}[\varphi]$.

However, since the dependence of $\Gamma[\tilde{\varphi}; \tilde{\varphi}]$ on $\tilde{\varphi}$ enters only through the gauge conditions, it is determined by an identity which is completely analogous to eq. (2.14):

$$\begin{aligned} \frac{\delta \Gamma[\tilde{\varphi}; \tilde{\varphi}]}{\delta \tilde{\varphi}^i} &= -\Gamma_{,j}[\tilde{\varphi}; \tilde{\varphi}] \langle D_\alpha^j[\tilde{\varphi}] \mathcal{G}_\beta^\alpha[\varphi; \tilde{\varphi}] \rangle \\ &\times \left(\frac{\delta F_\beta^\alpha[\varphi; \tilde{\varphi}]}{\delta \tilde{\varphi}^i} + \frac{1}{2} \eta_{\beta\gamma}^\alpha[\varphi] \frac{\delta \eta_{\gamma\delta}[\tilde{\varphi}]}{\delta \tilde{\varphi}^i} F^\delta[\varphi; \tilde{\varphi}] \right). \end{aligned} \quad (2.31)$$

Therefore

$$\begin{aligned} \Gamma'_{,i}[\varphi] &= \Gamma_{,i}[\tilde{\varphi}; \tilde{\varphi}]|_{\tilde{\varphi}=\varphi} + \frac{\Gamma[\tilde{\varphi}; \tilde{\varphi}]}{\delta \tilde{\varphi}^i}|_{\tilde{\varphi}=\varphi} \\ &= \Gamma_{,j}[\tilde{\varphi}; \tilde{\varphi}]|_{\tilde{\varphi}=\varphi} C^j_{,i}[\varphi], \end{aligned} \quad (2.32)$$

where

$$C^j_{,i}[\varphi] = \delta^j_i - \langle D_\alpha^j \mathcal{G}_\beta^\alpha \left(\frac{\delta F_\beta^\alpha}{\delta \tilde{\varphi}^i} + \frac{1}{2} \eta_{\beta\gamma}^\alpha \frac{\delta \eta_{\gamma\delta}}{\delta \tilde{\varphi}^i} F^\delta \right) \rangle|_{\tilde{\varphi}=\varphi}. \quad (2.33)$$

The second term on the r.h.s. of eq. (2.33) has a perturbative expansion which is similar to δX^i and which also starts only at 1-loop order, hence $C^j_{,i}[\varphi]$ can be regarded as a perturbatively invertible operator, and therefore the gauge invariant effective action $\Gamma'[\varphi]$ also satisfies a gauge dependence identity of the form (2.14):

$$\delta \Gamma'[\varphi] = \Gamma'_{,i}[\varphi] \delta X^i[\varphi], \quad \delta X'^i[\varphi] = (C^{-1})^i_j[\varphi] \delta X^j[\varphi; \varphi]|_{\tilde{\varphi}=\varphi}. \quad (2.34)$$

3. Gauge Invariance and Gauge Independence of Hard Thermal Loops

As a first application of the gauge dependence identities derived in the previous chapter to perturbative thermal gauge theories, we shall consider the leading temperature corrections to 1-loop vertex functions in a high-temperature expansion, for which we shall restrict ourselves to the imaginary-time formalism.

It has been observed most recently by Braaten and Pisarski [9] and Frenkel and Taylor [21] that in high-temperature QCD these so-called "hard thermal loops" [9], obey tree-level-type ghost-free Ward identities, and moreover appear to be gauge fixing independent. The latter (which does not already follow from the former!) has been checked for general covariant gauges, the Coulomb gauge and axial gauges. However, this gauge independence did not arise in an obvious way and had to be verified gauge by gauge so far.

As will be shown, the gauge dependence identities derived in the previous sections can be used to verify the gauge independence (and as a corollary also the gauge invariance) of these hard thermal loops in a very simple manner, if we restrict ourselves to linear and background field gauges, and it is also possible to verify the analogous statement in quantum gravity at finite temperature, where the leading terms in the thermal graviton self-energy have been obtained recently by one of the present authors [22].

The only further ingredients needed are the temperature-power-counting rules as given in Ref. 9, which, roughly, read as follows:* in a Feynman diagram, loop momenta in the numerator give a factor T , each propagator $1/\square$ gives T^{-1} , and the integration over the loop momentum (including summation over the Matsubara frequencies) contributes T^3 unless there are two or more propagators and all of them of the same statistics, in which case one only receives T^2 .

*These rules pertain to the imaginary time formalism only. The real time formalism can give rise to higher powers in T , which, however, cancel if appropriate combinations of the 2-field vertex functions are taken. [23]

3.1. HIGH-TEMPERATURE QCD

Using the above temperature-power-counting rules one can show [9] that the leading temperature contribution to a 1-loop vertex function in QCD (or QED) goes like T^2 . Vertex functions with more than two external fermion lines do not contribute to this order.

Restricting ourselves to vertex functions with only external gluons, all of the potential gauge dependence of hard thermal loops is now contained in the 1-loop contributions to δX_{ij}^i , whose diagrammatic structure is shown in Fig. 7.

Starting from Feynman gauge and considering variations of the gauge condition which are at most linear in derivatives (hence do not bring in inverse mass parameters which would complicate the power-counting), but arbitrary otherwise, the gauge independence of the hard thermal loops is seen immediately, without the need to investigate the tensorial structure of the loop integrals in great detail.

The diagrams of Fig. 7 with m gluon vertex insertions contain $(m+2)$ propagators, m vertices which are at most linear in the loop momentum and one F_{ij}^a insertion which by hypothesis adds at most one further loop momentum. For $m > 0$, this adds up to $\leq 2 - (m+2) + m + 1 = 1$ in powers of T . Without any gluon vertex inserted, $3 - 2 + 1 = 2$ powers of T appear to be possible, but such a diagram vanishes by virtue of the unbroken global color symmetry. Thus gauge dependence can occur only in the subleading order $\propto T$.

This argument is not changed when background gauges are considered, the only difference being that gluon vertices can now also be inserted into F_{ij}^a itself. Gauge invariance (i.e. tree-level-type Ward identities) is thus a corollary of gauge independence, as the former can always be achieved through background field gauges.

Although these considerations do not yield additional physical insight into why the off-shell hard thermal loops have to be gauge independent, they at least simplify the derivation of this result.*

*Unfortunately the simplicity of our argument does not persist for vertex functions with external fermion lines. In this case one still has to analyse the internal structure of the δX_{ij}^i ...

3.2. PERTURBATIVE QUANTUM GRAVITY AT FINITE TEMPERATURE

In finite temperature quantum gravity, the actual high-temperature limit cannot be studied by perturbation theory, one instead has to require that $T \ll G^{-1/2} \sim m_{\text{Planck}}$, in order that higher loop orders fall off by powers of $GT^2 \ll 1$. However, since collective phenomena generally occur for momenta small compared to the temperature, the corresponding calculations in quantum gravity turn out to be technically a high-temperature expansion, as worked out in [22] for the thermal 1- and 2-point graviton vertex functions. Although the presence of a plasma necessarily induces curvature, a flat-space perturbation theory is again applicable in order to determine the leading temperature contributions for temperatures below the Planck temperature (which is $\sim 10^{32} K$). Curvature corrections appear at the expense of powers of T , and because of $R^{\mu\nu} \sim GT^4 \ll T^2$, they are subleading.

Now the graviton loop constitutes a potential source of gauge dependence, and to check gauge independence by explicit calculations would be quite tedious. However, the analogous reasoning as above turns out to work also in this case, at least as far as the 1- and 2-point functions are concerned, which determine the spectrum of collective modes of a gravitational plasma.

Starting from Feynman-de Donder gauge (background covariant or not), the main difference from the case of QCD is that vertices among gravitons always contribute up to 2 loop momenta and so do the graviton-ghost vertices. This entails that at 1-loop order the leading temperature behaviour according to the superficial power-counting laws of Ref. 9 is

$$\Gamma_{(1)} \sim T^4, \quad \Gamma_{(2)} \sim T^4, \quad \Gamma_{(n)} \sim T^{4+(n-2)}, \quad n \geq 3, \quad (3.1)$$

where the index in brackets stands for the number of external graviton lines. However, in the case $n \geq 3$, the estimate (3.1) turns out to be too crude. An explicit calculation [24] reveals that also $\Gamma_{(3)} \sim T^4$, because the superficially leading term cancels by virtue of total symmetry in the Lorentz indices.

Fortunately, for the 1- and 2-point vertex functions superficial power-counting suffices, and the argument for gauge independence of their leading temperature contributions is almost as simple as before. For the gauge variation of the 1- and 2-point 1-loop graviton vertex functions, one has to consider again the diagrams of Fig. 7 with $m = 0$ or 1 vertex insertion, respectively, replacing gluons by gravitons. The

difference from QCD is just that a vertex insertion adds twice as many loop momenta, and now also the structure functions in $D_{\alpha,j}^i$ contribute one loop momentum. This gives $(2 + \delta_{0m}) - (m + 2) + 2m + 1 + 1 = 3$ powers of T for the potential gauge dependences, which are thus down by one power of T . In contrast to QCD, the case $m = 0$ fits into this power counting argument, and indeed the 1-point vertex function, which gives the energy-momentum tensor of the plasma, does not vanish in finite temperature quantum gravity.

Again, since tree-level-type Ward identities hold in background covariant gauges, they generally hold for the gauge independent leading temperature parts of the 1- and 2-point vertex functions. These are thus related by

$$\Pi_{\alpha\beta\mu\nu}(Q)(\eta^{\mu\epsilon}Q^\nu + \eta^{\nu\epsilon}Q^\mu) = -\frac{1}{2}T_{\mu\nu}(\delta_{\alpha\beta}^{\nu\epsilon}Q^\mu + \delta_{\alpha\beta}^{\mu\epsilon}Q^\nu - \delta_{\alpha\beta}^{\mu\nu}Q^\epsilon), \quad (3.2)$$

as has been found by explicit calculation in Feynman-de Donder gauge in Ref. 22.

In QCD the Ward identities entail that up to an overall factor gluons and quarks give rise to the same polarization tensor (and in fact also to like n -point gluon vertex functions) [9]

$$\begin{aligned} \Pi^{\mu\nu}(Q) &\propto I^{\mu\nu} - \frac{1}{2}\delta^{\mu\nu}\delta_{\alpha\beta}I^{\alpha\beta}, \\ I^{\mu\nu} &= \int \frac{d^4K}{(2\pi)^4} \frac{K^\mu K^\nu}{K^2(K-Q)^2}. \end{aligned} \quad (3.3)$$

A similar thing happens in gravity: all the different (massless) fields contribute the same gravitational polarization tensor, the differing overall factors corresponding to the different energy densities associated with these fields. This universal result can be given in terms of the integral

$$I^{\mu\nu\rho\sigma}(q_0, \vec{q}) = \int \frac{d^4K}{(2\pi)^4} \frac{K^\mu K^\nu K^\rho K^\sigma}{K^2(K-Q)^2} \quad (3.4)$$

and reads [22]

$$\begin{aligned} \Pi^{\alpha\beta\mu\nu} &= \frac{1}{2}I^{\alpha\beta\mu\nu} - \frac{1}{4}(\delta_{\sigma\rho}^{\alpha\mu}\delta_{\rho}^{\nu\beta} + \delta_{\sigma\rho}^{\alpha\nu}\delta_{\rho}^{\mu\beta} + \\ &\quad \delta_{\sigma\rho}^{\beta\mu}\delta_{\rho}^{\nu\alpha} + \delta_{\sigma\rho}^{\beta\nu}\delta_{\rho}^{\mu\alpha})\eta_{\gamma\delta}I^{\gamma\delta\sigma\rho}, \end{aligned} \quad (3.5)$$

omitting contractions of I which give zero. It can in fact be shown that the combination (3.5) is the only one satisfying the identity (3.2).

4. Gauge Independence of Plasma Dispersion Relations

In linear response theory [6], the spectrum of collective phenomena of a finite temperature field theory is usually read from the poles of the propagators. Since in gauge theories the propagators are inherently gauge dependent objects, the gauge independence of this procedure has to be verified, and this will be the aim of this chapter. In particular we shall investigate the potential gauge dependences of the poles of the gluon and quark propagators in QCD.

In the simpler case of QED we have already seen that gauge independence holds to all orders for the photon propagator within linear gauges, but already the electron propagator contains non-trivial gauge fixing dependences.

At the end of this chapter, we briefly consider the case of quantum gravity at finite temperature, where the very presence of a plasma requires inclusion of a curved background so that a flat-space graviton propagator does not make sense.

4.1. GAUGE DEPENDENCE IDENTITIES FOR THE GLUON PROPAGATOR

In a quark-gluon plasma, which is expected to exist at sufficiently high temperatures, the dispersion relations of colour oscillations are contained in the finite temperature gluon propagator

$$\begin{aligned} (\text{Tr } A_\mu^\alpha(x) A_\nu^\beta(y))_T &= \text{Tr} \left[e^{-H/T} \text{Tr } A_\mu^\alpha(x) A_\nu^\beta(y) \right] / \text{Tr} \left[e^{-H/T} \right] \\ &= \delta^{\alpha\beta} \Delta_{\mu\nu}(x-y) \end{aligned} \quad (4.1)$$

with $T \gg \Lambda_{\text{QCD}}$.

Since Lorentz symmetry is broken by the presence of the plasma with rest-frame velocity n_μ , there are now four structure functions in the full gluon propagator, unless a non-covariant choice of gauge further reduces the 3-dimensional symmetry (see Appendix C for this case):

$$\Delta_{\mu\nu}(Q) = \frac{1}{a(Q)} A_{\mu\nu} + \frac{1}{b(Q)} B_{\mu\nu} + \frac{1}{c(Q)} C_{\mu\nu} + \frac{1}{d(Q)} D_{\mu\nu} \quad (4.2)$$

in momentum space ($Q_\mu = (q_0, \mathbf{q}_i)$), where we employ the following tensorial basis,

$$A_{\mu\nu} \equiv P_{\mu\nu} - \frac{\tilde{n}_\mu \tilde{n}_\nu}{\tilde{n}^2}, \quad (4.3a)$$

$$B_{\mu\nu} \equiv \frac{\tilde{n}_\mu \tilde{n}_\nu}{\tilde{n}^2}, \quad (4.3b)$$

$$C_{\mu\nu} \equiv \tilde{n}_\mu Q_\nu + Q_\mu \tilde{n}_\nu, \quad (4.3c)$$

$$D_{\mu\nu} \equiv \frac{Q_\mu Q_\nu}{Q^2}, \quad (4.3d)$$

with

$$\tilde{n}_\mu \equiv P_{\mu\nu} n^\nu, \quad P_{\mu\nu} \equiv g_{\mu\nu} - \frac{Q_\mu Q_\nu}{Q^2}. \quad (4.4)$$

In (4.2) only $A_{\mu\nu}$ and $B_{\mu\nu}$ are transverse with respect to the 4-vector Q_μ , and they correspond to spatially transverse and longitudinal modes, respectively.

Decomposing the bare inverse propagator according to

$$(\Delta_{(0)}^{-1})_{\mu\nu} = Q^2 P_{\mu\nu} + f_b B_{\mu\nu} + f_c C_{\mu\nu} + f_d D_{\mu\nu}, \quad (4.5)$$

where the gauge fixing term is parametrized by the f 's, which are constrained by $f_b f_d = \tilde{n}^2 Q^2 f_c^2$, and writing the self-energy corrections as

$$\Pi_{\mu\nu}(Q) = \Pi_a(Q) A_{\mu\nu} + \Pi_b(Q) B_{\mu\nu} + \Pi_c(Q) C_{\mu\nu} + \Pi_d(Q) D_{\mu\nu}, \quad (4.6)$$

the structure functions of the full propagator are given by [25]

$$a(Q) = Q^2 + \Pi_a, \quad (4.7a)$$

$$b(Q) = Q^2 + \Pi_b + \tilde{n}^2 Q^2 \left[\frac{f_c^2}{f_d} - \frac{(f_c + \Pi_c)^2}{f_d + \Pi_d} \right], \quad (4.7b)$$

$$c(Q) = -\frac{f_d + \Pi_d}{f_c + \Pi_c} b(Q), \quad (4.7c)$$

$$d(Q) = \frac{f_d + \Pi_d}{Q^2 + \Pi_b + f_b} b(Q). \quad (4.7d)$$

At the scale $Q \sim gT$, the dispersion relations are completely determined by the leading temperature contributions to the 1-loop self-energy, the "hard thermal loops", which according to the discussion of the previous section are gauge covariant and gauge fixing independent. Gauge covariance of the self-energy implies $\Pi_c = \Pi_d = 0$, and gauge independence of the transverse parts Π_a and Π_b is thus sufficient for gauge independence of the leading order dispersion relations $a(Q) = 0$ and $b(Q) = 0$ which have been studied extensively in the literature [5, 26, 27]. In fact, at leading order (and only there) the functions $a(Q)$ and $b(Q)$ are gauge independent as a whole, including the generally gauge dependent residues of the dispersion relations [28].

Beyond leading order, however, one cannot expect gauge covariance and gauge independence of $\Pi_{\mu\nu}$, and explicit calculations exhibit that gauge dependence starts at order T , which persists at one-loop order even at the location of the plasma dispersion relations and which has given rise to the puzzle of a seemingly gauge dependent plasmon damping constant [7]. Much of the ensuing work has concentrated on the quest for a manifestly gauge independent substitute for the inherently gauge dependent gluon propagator, with the result that even different gauge independent schemes have yielded different answers. However, as Pisarski and others [9, 10, 29] have recently demonstrated, the real problem is that the ordinary loop expansion is not complete and needs resummation of the leading temperature corrections.

This conclusion can also be drawn by applying the gauge dependence identity, eq. (2.14), to the issue of potential gauge dependences of the plasma dispersion relations [11].

Returning for the moment to the condensed notation of the previous sections, the gauge dependence of the propagator $\Delta^{ij}, \Delta^{ij}\Gamma_{jk} = -\delta^i_k$, is related to the gauge dependence of the self-energy through

$$\delta\Delta^{ij}|_{J=0} = \Delta^{im}\delta\Gamma_{mn}\Delta^{nj}|_{J=0}. \quad (4.8)$$

Differentiating the basic relation (2.14) twice at fixed $J_i = 0$,

$$\begin{aligned} \delta\Gamma_{mn}|_{J=0} &= (\Gamma + \delta\Gamma)_{mn}|_{(\Gamma+\delta\Gamma)_{j=0}} - \Gamma_{mn}|_{\Gamma_j=0} \\ &= (\Gamma_{in}\delta X^i_{,m} + \Gamma_{im}\delta X^i_{,n})|_{\Gamma_j=0} \end{aligned} \quad (4.9)$$

gives

$$\delta\Delta^{ij}|_{J=0} = -(\Delta^{im}\delta X^j_{,m} + \delta X^i_{,m}\Delta^{mj})|_{J=0}. \quad (4.10)$$

Here one has to work with the effective action without subtraction of the tree-level gauge fixing term - otherwise Γ_{ij} would not be perturbatively invertible. Consequently in what follows δX^i also contains primary diagrams with vertices $\delta F^\alpha[\vec{p}]$, as in Fig. 3.

Restricting to QCD again and extracting the transverse structure functions from (4.10), one obtains

$$\delta a(Q) = -a(Q)A_\mu^\mu(Q)\delta X_\mu^\nu(Q) \equiv a(Q)\delta Y(Q) \quad (4.11)$$

for the gauge dependence of the spatially transverse structure function, and

$$\delta b(Q) = -2b(Q) \left[\frac{\tilde{n}^\mu}{\tilde{n}^2} - \frac{f_c + \Pi_c}{f_d + \Pi_d} Q^\mu \right] \tilde{n}_\nu \delta X_\mu^\nu \equiv b(Q)\delta Z(Q) \quad (4.12)$$

for that of the spatially longitudinal one.

Now if $\delta Y(Q)$ and $\delta Z(Q)$ are well-behaved at the location of the plasma dispersion relations, these particular gauge dependence identities imply that the solutions to $a(Q) = 0$ and $b(Q) = 0$, which determine the dispersion relations, are also solutions of $(a + \delta a)(Q) = 0$ and $(b + \delta b)(Q) = 0$, hence gauge independent.

No such relations hold for the nontransverse structure functions c and d . Indeed, the factor $(f_d + \Pi_d)$ contained in them is obviously gauge dependent already at tree-level, and in general according to

$$\delta[f_d + \Pi_d] = 2 \left[(f_c + \Pi_c)\tilde{n}_\mu + (f_d + \Pi_d)\frac{Q_\mu}{Q^2} \right] Q^\nu \delta X_\nu^\mu. \quad (4.13)$$

What decides the gauge (in)dependence of the dispersion relations obtained from $a(Q)$ and $b(Q)$ is the pole structure of δY and δZ . This is determined by that of δX_μ^ν , and in the case of δZ in addition by a simple pole $1/Q^2$ contained in the $\tilde{n}$'s as well as by possible poles of $1/(f_d + \Pi_d)$.^{*} Thus one has to exclude these two kinematical poles from the gauge independence proof for the spatially longitudinal mode b . Indeed, $b(Q)$ turns out to be proportional to Q^2 , and this factor cancels in the Coulomb gauge propagator, but not in the covariant one, while the potential pole $1/(f_d + \Pi_d)$ is evidently a gauge artefact, as exhibited by eq. (4.13).

The more important question is whether the potential pole structure of $\delta X_\mu^\nu(Q)$ could possibly coincide with the expectantly physical dispersion relations given by $a(Q) = 0$ and $b(Q)/Q^2 = 0$. The diagrammatic representation of δX_μ^ν is obtained by inserting one vertex in all possible ways into the primary diagrams of Fig. 4, where all resulting tadpole-like diagrams vanish due to unbroken global colour symmetry. Since the primary diagrams were 1-particle reducible only with respect to the basic ghost line attached to δF_i^α , the diagrams for δX_μ^ν are built from ghost propagators and their radiative corrections carrying the external momentum Q , and in addition

^{*}As for $\delta a(Q^2)$, notice that the matrix $A_{\mu\nu}$ (defined in (4.3)) which is contained in δY does not have a factor $1/Q^2$.

from 1pi subdiagrams whose internal propagators are off the (plasmon) mass shell even when the external momentum is on it.

Hence the only possible gauge dependences in the solutions of $a(Q) = 0$ and $b(Q)/Q^2 = 0$ are those given by the poles of the (full) ghost propagator, which is clearly a gauge dependent object – its pole structure is gauge dependent already at the tree-level. A coincidence of the ghost poles with the physical dispersion relations from $a(Q)$ and $b(Q)$ will thus occur only exceptionally, and gauge independence of the zeros of a and b is ascertained by Eqs. (4.11, 4.12) if one finds at least one gauge where such a coincidence does not happen.

According to the discussion of the previous sections, at finite temperature the ghost propagator as well as the 1pi-subdiagrams of $\delta X''_\mu$ receive no leading temperature corrections $\sim g^2 T^2$, so their “dispersion relations” will be independent of the physical ones given by transverse structure functions of the gluon propagator, whose leading order is $Q^2 \sim g^2 T^2$. This renders plausible the assumption that δY and δZ are well-behaved at the location of the physical plasma dispersion relations.

The identities (4.11) and (4.12) also reveal that the gauge dependence found in the one-loop calculations at sub-leading order is in fact to be expected. Truncating at one-loop order, $\delta a(Q)$ for example is determined by

$$\delta a^{(1)}(Q) = a^{(0)}(Q) \delta Y^{(1)}(Q) \quad (4.14)$$

($\delta Y^{(0)}$ vanishes identically). However, the leading order of the plasma dispersion relations is given by $a^{(0)} + a^{(1)} = 0$, and the temperature correction $a^{(1)} \sim g^2 T^2$ cannot be considered small compared to $a^{(0)} = Q^2$, but rather set the scale for the dispersion relations through $Q^2 \sim g^2 T^2$. Since $\delta Y \sim g^2 T$, this implies that the r.h.s. of eq. (4.14) does not vanish at the order $g^2 T$, where one is looking for plasmon damping.

As Pisarski [8, 9] has pointed out, the conventional loop expansion has to be substituted by an expansion which resums all the hard thermal loops $\sim g^3 T^2$, so that it becomes an expansion not in g alone, but in the excess of powers of g over powers of T 's. Such a resummation of perturbation theory is technically achieved by writing

$$S \rightarrow S + (1 - \ell) \Gamma_{\text{leading}}, \quad (4.15)$$

where ℓ is identified with a loop counting parameter (i.e. whose powers correspond to the loop order) and which is set to one only after having performed a loop expansion to a given loop order. The new lowest-order Feynman rules then contain the leading effects contained in Γ_{leading} as they are defined at $\ell = 0$, but nonperturbatively nothing has been changed [29].

This program has been successfully completed in Ref. 10 within Coulomb gauge to the first subleading order, and the above argument shows that this result can be regarded as a final, gauge independent answer, and indeed gauge independence has been checked explicitly in a certain class of gauges in Ref. 9. Here we have proven that gauge independence of the resummed loop expansion is a natural consequence of the fact that there corrections to $a_{\text{resum}}^{(0)}$ and $b_{\text{resum}}^{(0)}$ are indeed mere corrections, making the r.h.s. of (4.14) vanish at the resummed 1-loop order.

4.2. BACKGROUND FIELD GAUGES

A slight technical complication arises in the case of background field gauges, which has to do with the fact that there the (gauge invariant but still gauge dependent) effective action is no longer the Legendre transform of a conventional generating function for connected Green functions, and thus the connection with ordinary linear response theory appears to be lost.

However, this complication is introduced through a modification of the gauge fixing term and so should not be able to affect physical quantities. A corresponding statement has been proven for the S -matrix in the background field method in Ref. 30. In the following we shall show that gauge independent dispersion relations can also be obtained from the gluon propagator within the background field method, and that they in fact coincide with the ordinary ones when calculated self-consistently.

As shown in Sect. 2.3, the gauge dependence identities for background field gauges have a similar structure to the ordinary ones, the essential modification being that the functional $\delta X''[\vec{\varphi}]$ is replaced by

$$\delta X''[\vec{\varphi}] = (C^{-1})^i_j[\vec{\varphi}] \delta X^j[\vec{\varphi}; \vec{\varphi}]_{\vec{p}=\vec{\varphi}}.$$

In order that a propagator can be defined as the inverse of the 2-point vertex function, the (background covariant) gauge condition has to be introduced in the

form (2.22) rather than (2.25), which only means that the tree-level gauge breaking term is not removed from the effective action. In this case C^i remains a perturbatively invertible operator, and eq. (2.33) which determines its diagrammatic expansion remains valid. The latter now just includes vertices coming from undifferentiated gauge fixing functionals $F[\varphi]$, in analogy to the diagrammatic expansion of the ordinary δX .

As seen from the explicit expression (2.33), the diagrammatic expansion of the operator C is identical in structure to the one of δX – it also contains only ghost propagators as possible 1-particle reducible subdiagrams. Consequently the gauge independence proof for the dispersion relations based on the gluon propagator carries over immediately. This gauge independence furthermore implies that the dispersion relations obtained in the background field method are equivalent to those from the conventional framework, for when $F_i^\alpha[\varphi]$ is a background covariant gauge condition, one can use $F_i^\alpha[\lambda\varphi]$ with $\lambda \in [0, 1]$ for interpolation and λ is again just a gauge parameter. Note that the argument leading to gauge independence did not require background gauge invariance, hence an interpolation through background dependent but background non-covariant gauges is allowed.

This equivalence of background field gauges furthermore implies, at least in the present case of Yang-Mills theories, that the manifestly gauge independent Vilkovisky-DeWitt effective action [18, 17] will give the same result as the conventional framework if the dispersion relations are calculated within a self-consistent expansion scheme, for it can be shown that in Yang-Mills theories the Vilkovisky-DeWitt effective action is equivalent to a certain background field gauge invariant effective action [18].

4.3. GAUGE DEPENDENCE IDENTITIES FOR THE QUARK PROPAGATOR

Collective excitations also exist in the fermionic sector, sometimes called “plasmino” [28]. The poles in the quark propagator can be conveniently summarized by the equation

$$\det(\Gamma_{\alpha\bar{\beta}}(Q)) = 0, \quad (4.16)$$

where $\Gamma_{\alpha\bar{\beta}}$ stands for the fermionic 2-point-function and α and $\bar{\beta}$ are spinor indices. The determinant is only with respect to the spinor indices.

In contrast to the gluon propagator there is no need to separate nontransverse pieces, and the gauge dependence identity can be immediately derived from eq. (4.16):

$$\begin{aligned} \delta(\det(\Gamma_{\alpha\bar{\beta}}(Q))) &= \det(\Gamma_{\alpha\bar{\beta}}(Q)) \Gamma^{-176}(Q) \delta\Gamma_{\bar{\beta}}(Q) \\ &= -\det(\Gamma_{\alpha\bar{\beta}}(Q)) (\delta X_{\bar{\beta}}^\gamma(Q) + \delta X_{\bar{\beta}}^{\bar{\gamma}}(Q)). \end{aligned} \quad (4.17)$$

The functions $\delta X_{\bar{\beta}}^\gamma(Q)$ and $\delta X_{\bar{\beta}}^{\bar{\gamma}}(Q)$ are obtained by taking a Dirac trace on the corresponding diagrams shown in Fig. 6. Gauge independence of the solutions of eq. (4.16) follows in analogy to the discussion of the previous sections, with the simplification that now we do not need to exclude kinematical singularities brought about by projection operators as in (4.11) and (4.12). Only potential poles of $\delta X_{\bar{\beta}}^\gamma(Q)$ and $\delta X_{\bar{\beta}}^{\bar{\gamma}}(Q)$ could lead to gauge dependent solutions of (4.16), but the former are again determined by the poles of the ghost propagators, which, incidentally, now first appear as 1p-reducible parts of δX only at the 2-loop level.

In Ref. 15 it has been argued that in the case of superaxial gauges the spectrum of the fermion propagator is afflicted by hidden spurious sources which are associated with the gauge fixing procedure. According to the above argument, these will be reflected by singularities of δX . However, as remarked in Sect. 2.1, such gauges cannot be treated straightforwardly within the path integral framework and should therefore not be included in our discussion. In fact, the fermion propagator in a superaxial gauge should rather be viewed as the gauge covariant but path dependent object

$$(\psi(x) \text{P exp} \left[i \int_{C(x,y)} A_\mu(z) dz^\mu \right] \bar{\psi}(y)),$$

where $C(x,y)$ is a path connecting x and y , which thus is altogether different from the ordinary definition of the fermion propagator used here.

On the other hand, with the conventional definition of the fermion propagator it suffices to find one gauge in which the δX is regular for poles given by (4.16) in order to guarantee their gauge independence and to identify them as truly physical.

At $T = 0$, assuming a consistent regularization scheme, eq. (4.17) (or its generalization to arbitrary matter fields) implies the gauge independence of the on-shell mass counter-term.

contrast to the perturbative graviton propagator. These are given by

$$\begin{aligned} & \left\{ \frac{1}{2} (g^{\alpha\beta} g^{\mu\nu} - g^{\alpha(\mu} g^{\nu)\beta}) \nabla^2 - \frac{1}{2} (g^{\alpha\beta} \nabla^{\mu} \nabla^{\nu}) + g^{\mu\nu} \nabla^{\alpha} \nabla^{\beta} \right. \\ & \left. + g^{\alpha(\mu} \nabla^{\nu)} \nabla^{\beta} - G^{\alpha(\mu\nu)\beta} + \frac{1}{12} g^{\alpha\beta} g^{\mu\nu} R - \frac{1}{3} g^{\alpha(\mu} g^{\nu)\beta} R \right. \\ & \left. - 16\pi G \Pi^{\alpha\beta\mu\nu} (\nabla^0, \vec{\nabla}) \right\} \delta g_{\alpha\beta} = 8\pi G \delta T^{\mu\nu}, \end{aligned} \quad (4.18)$$

where $G^{\alpha\mu\nu\beta}$ is the conformal curvature (or Weyl) tensor.

In order to make these equations invertible, one needs to fix the diffeomorphism invariance, which because of the Ward identity (3.2) is not broken by the leading temperature contributions to the thermal graviton self-energy $\Pi^{\alpha\beta\mu\nu}$. If furthermore only massless matter (or massive matter with masses much smaller than the temperature) contributes to the plasma (in addition to the gravitons themselves), one can, as in the classical theory, require that the perturbation $\delta g_{\alpha\beta}$ be transverse and traceless, [22]

$$\nabla^{\alpha} \delta g_{\alpha\beta} = 0, \quad g^{\alpha\beta} \delta g_{\alpha\beta} = 0. \quad (4.19)$$

Using (4.19) and the Einstein equations, and assuming a vanishing conformal curvature tensor, which is undetermined by the Einstein equations, eq. (4.18) reduces to

$$\left\{ \frac{1}{2} g^{\alpha(\mu} g^{\nu)\beta} \nabla^2 + 16\pi G \Pi^{\alpha\beta\mu\nu} \right\} \delta g_{\alpha\beta} = -8\pi G \delta T^{\mu\nu}. \quad (4.20)$$

The thermal graviton self-energy thus indeed acts as a (non-constant) mass matrix for small perturbations $\delta g_{\alpha\beta}$, but because of the constraints (4.19), only the transverse-traceless parts contribute to eq. (4.20).

This fact has not been taken into account at all in Ref. 34, whereas in Ref. 35, because of their restriction to a zero momentum limit, the first constraint in (4.19) could not be covered.

At finite temperature there are now three transverse-traceless modes (instead of the two transverse ones in the QCD case), and their dispersion relations at leading order have been obtained in Ref. 22.

Gauge independence of the leading order results is covered by the considerations of Sect. 3.2. However, beyond leading order, the line of reasoning employed in the case of QCD does not apply in gravity at finite temperature. Technically this is

In a high-temperature expansion the leading order of the fermionic dispersion relations (4.16) is again $Q^2 \sim g^2 T^2$, and one can verify by explicit calculation that the functions δX in (4.17) are only of subleading order, making them still less likely to interfere with (4.16).

Exactly as in the case of the gluonic dispersion relations, mere one-loop calculations of corrections to the fermionic dispersion relations are expected to be gauge dependent, as indeed found in Ref. 31 and a consistent next-to-leading order evaluation again requires resummation of leading contributions.

4.4. DERIVATION OF DISPERSION RELATIONS IN QUANTUM GRAVITY AT FINITE TEMPERATURE

In the case of gravity at finite temperature there has been some controversy about the correct definition of dispersion relations even at the level of the leading temperature corrections [32, 33, 34, 35], in particular about the issue of the correct definition of the Jeans mass within thermal perturbative quantum gravity. The tachyonic Jeans mass is expected in place of the electric Debye mass of the QCD plasma, giving rise to instabilities of spatial inhomogeneities instead of spatial screening.

To this end only the static limit of the thermal graviton self-energy has been considered until recently. In Ref. 32 the static limit of Π_{0000} was interpreted as giving the Jeans mass, but there it was assumed that Π was transverse so that the only nonvanishing static component of Π would have to be just this 0000-component. However, this is not the case as has been shown above, eq. (3.2), because the 1-point function, the energy-momentum tensor associated with the plasma, does not vanish. Correspondingly, it was later shown by the authors of Ref. 33, that Π_{0000} by itself is dependent on the parametrization of the gravitational field (although its leading temperature contributions are gauge independent, as has been verified above). They instead proposed to focus on the 0000-component of the graviton propagator, but again difficulties appeared due to the nonvanishing 1-point function. Their approach was to simply neglect the singular contributions from the 1-point function to the propagator, but this certainly cannot be correct.

In refs. [34, 35] it was later pointed out that the equations for small disturbances obtained from the Einstein equations are well defined also at finite temperature, in

due to the non-vanishing one-point vertex function, which requires that one has to analyse the effective field equations rather than a propagator. At least in the type of analysis described above it is essential that the thermal graviton self-energy was gauge covariant, which it is at leading order. This property will generally be lost beyond that order as the simpler example of hot QCD shows, hence one will be forced to employ background covariant gauges.

The same requirement arises when one tries to analyse the dispersion relations of QCD along the lines followed here in the case of gravity. Instead of eq. (4.19) one would require transversality of δA_μ to project out the two (4-d) transversal structure functions a and b . This can be done at leading order in any case, and beyond leading order in background covariant gauges which guarantee that $\Pi_c = \Pi_d = 0$. Gauge independence of a and b can then be established directly from eq. (4.9) by projection onto the tensors $A_{\mu\nu}$ and $B_{\mu\nu}$ which are orthogonal to each other.

In the case of gravity the presence of a non-trivial one-point function severely complicates this type of analysis. The one-point function also carries potential gauge dependences beyond leading order and its contribution to the effective Einstein equations has to be taken into account in eq. (4.18). A gauge independence proof of the dispersion relations of the graviton plasma beyond leading order is thus still missing and is under investigation.

5. Conclusion

Summarizing, we have shown the existence of very general identities that govern the gauge dependence of the effective action and n -point functions in gauge theories. These identities apply to zero as well as finite temperature cases. We have mainly considered the latter, where the discussion is simplified by the fact that the potential renormalization of these identities need not be taken into account. From these gauge fixing identities we have proven the gauge independence and gauge covariance of the leading order contributions to vertex functions of high-temperature Yang-Mills theories, and in the case of Einstein gravity we have been able to establish the same result for the 1- and 2-point graviton functions. This implies gauge independence of the leading high-temperature dispersion relations as well as their residues.

Beyond leading order, gauge independence of the vertex functions no longer

holds in general, but the gauge dependence identities nevertheless allow one to show the gauge independence of the dispersion relations of the collective modes in finite temperature Yang-Mills theories, thus settling the controversy concerning the correct definition of these dispersion relations evoked by the apparent gauge dependence of subleading 1-loop results. As we have shown, gauge independent results will be obtained naturally from the gauge dependent gluon propagator if a consistent (resummed) perturbative scheme is employed, so that gauge dependence of the one-loop result does not provide a compelling reason to consider manifestly gauge independent alternatives. Such a resummed perturbation theory for high-temperature QCD which is able to determine the subleading contributions $\sim gT^2$ has been developed by Braaten and Pisarski [9, 10].

In Ref. 29 it has been demonstrated by explicit calculations that even a partial resummation (where higher loop orders do not necessarily fall off with powers of g) leads to gauge independent results at each loop order provided that the lowest order mass-shell is shifted to the location of the dispersion relation where one is evaluating corrections thereof. This finding is consistent with our general identities, eqs. (4.11, 4.12), and in fact constitutes a non-trivial check of them: Having resummed the leading order contributions (at the location of the dispersion relation under consideration but only there), one enforces that corrections to $a^{(0)}$ and $b^{(0)}$ are down by a factor of g , thereby guaranteeing gauge independence of the redefined perturbation series. However, this does not prevent a mixing of different loop orders starting at subleading orders, so the partially resummed results are not yet complete. This example highlights a point which sometimes appears to have been overlooked in the controversy around the ‘correct’ definition of the plasmon damping constant, namely that gauge fixing independence does not yet imply physicality. There have been already numerous attempts to set up a manifestly gauge independent perturbation theory in order to get rid of the gauge dependence of the 1-loop plasmon decay constant [7], and the discrepancy between various gauge independent results has only led to further confusion. It should now be clear, however, that by using a manifestly gauge independent framework in this case, one is just giving away a useful check on the completeness of the calculation, namely the cancellation of the gauge parameter in the final result. This in turn hides the fact that higher loop orders must also be included in the gauge independent approaches. In any case, the gauge

independence proof for the QCD plasma dispersion relations presented here and the explicit calculations of refs. [9, 10, 29] should settle this particular controversy.

Finally we would like to stress that the issue of potential gauge dependences in finite-temperature dispersion relations is just one application of the gauge dependence identities derived and analysed in the present paper. These identities will in general be useful whenever physical quantities are derived within an inherently gauge dependent framework so that gauge independence of the final result must be explicitly verified.

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Appendix A. Derivation of the Gauge Dependence Identities through BRS

In this Appendix we derive using BRS [19] invariance of the classical Lagrangian the identity of eq. (2.14) expressing how the generating functional changes with a change in gauge condition. We restrict ourselves here to the case of Yang-Mills fields. Starting with the path integral representation for $W(J)$ of eq. (2.8), we write the Faddeev-Popov determinant as an integral over Grassmann variables to find, suppressing the integration over space-time,

$$\exp[iW(J, F)] = \int \mathcal{D}A_\mu \mathcal{D}\eta \mathcal{D}\bar{\eta} \exp \{-[I(F) + J_a \cdot A^a]\}, \quad (A.1)$$

where

$$I(F) = S + \frac{\lambda}{2} [F^a(A)]^2 + \bar{\eta}^a \frac{\delta F^a}{\delta A_\mu^b} (D_\mu \eta)_b \quad (A.2)$$

and S is the classical action.

We now examine the effects of an infinitesimal change in gauge condition $F \rightarrow F + \delta F$, for which we have

$$I(F + \delta F) = I(F) + \lambda F^a \delta F^a + \bar{\eta}^a \frac{\delta(\delta F^a)}{\delta A_\mu^b} (D_\mu \eta)_b. \quad (A.3)$$

If we now define $\delta W \equiv W(J, F + \delta F) - W(J, F)$, then to first order in δF we have

$$\delta W = \left\langle \lambda F^a \delta F^a + \bar{\eta}^a \frac{\delta(\delta F^a)}{\delta A_\mu^b} (D_\mu \eta)_b \right\rangle. \quad (A.4)$$

We now exploit the fact that the integral over a Grassmann variable vanishes due to its odd character:

$$\langle \bar{\eta}^a \delta F_a \rangle = 0. \quad (A.5)$$

To this equation we perform the BRS transformation $A_\mu \rightarrow A_\mu + \delta_{\text{BRS}} A_\mu$, $\eta \rightarrow \eta + \delta_{\text{BRS}} \eta$, and $\bar{\eta} \rightarrow \bar{\eta} + \delta_{\text{BRS}} \bar{\eta}$, where

$$\begin{aligned} \delta_{\text{BRS}} A_\mu^a &= (D_\mu \eta)_a \delta \xi, \\ \delta_{\text{BRS}} \bar{\eta}_a &= \lambda F_a \delta \xi, \\ \delta_{\text{BRS}} \eta_a &= -\frac{1}{2} g f_{abc} \eta_b \eta_c \delta \xi, \end{aligned} \quad (A.6)$$

and $\delta \xi$ is a Grassmann variable. One can check that this transformation has unit Jacobian and that in the usual manner the action $I(F)$ is invariant. The effects of this transformation on eq. (A.5) are thus on the integrand $\bar{\eta}^a \delta F_a$ and on the source term $J^a \cdot A_a$, and we find to first order in δF

$$\langle \delta_{\text{BRS}}(\bar{\eta}^a \delta F_a) + \bar{\eta}^a \delta F_a (J_b^\mu \delta_{\text{BRS}} A_\mu^b) \rangle = 0. \quad (A.7)$$

One can now show that

$$\delta_{\text{BRS}}(\bar{\eta}^a \delta F_a) = \lambda F^a \delta F^a \delta \xi + \bar{\eta}^a \frac{\delta(\delta F^a)}{\delta A_\mu^b} (D_\mu \eta)_b \delta \xi, \quad (A.8)$$

and so comparing this with eq. (A.4) results in

$$\delta W = J_a^\mu \langle (D_\mu \eta)_a \bar{\eta}^b \delta F_b \rangle, \quad (A.9)$$

where we have used the anticommuting nature of η and $\bar{\eta}$ and the arbitrariness of the Grassmann parameter $\delta \xi$.

Use of the relationship

$$\int \mathcal{D}\eta \mathcal{D}\bar{\eta} \eta_a \bar{\eta}_b \exp[-\bar{\eta}_c M^{cd} \eta_d] = [M^{-1}]_{ab} \det M \quad (A.10)$$

to integrate out the Grassmann fields leads to

$$\delta W = J_a^\mu \langle (D_\mu)_a^b \mathcal{G}_{bc} \delta F_c \rangle, \quad (A.11)$$

where $\mathcal{G}$ is the Faddeev-Popov propagator. Up to the Legendre transformation $W[J] \rightarrow \Gamma[\bar{\varphi}]$, this is indeed identity (2.14).

Appendix B. Finite Gauge Variations

The identities we have derived for infinitesimal gauge variations can in principle be integrated to identities describing finite variations of the gauge conditions, since they were derived for arbitrary gauges. However, eq. (2.14) and its descendants are highly nonlinear, and they could at best be "bilinearized" through the introduction of additional sources which couple to all the composite operators appearing therein.

Formally linear relations can however be written down in terms of

$$Z[J] \equiv e^{iW[J]}.$$

Eq. (2.14) translates into

$$\delta Z[J] = iJ_i D_\alpha^i \left[\frac{\delta}{i\delta J} \right] G_\alpha^\beta \left[\frac{\delta}{i\delta J} \right] \delta' F^\beta \left[\frac{\delta}{i\delta J} \right] \cdot Z[J], \quad (B.1)$$

which formally integrates to

$$Z[J, F + \Delta F, \eta + \Delta\eta] = \exp \left\{ iJ_i D_\alpha^i \left[\frac{\delta}{i\delta J} \right] G_\alpha^\beta \left[\frac{\delta}{i\delta J} \right] \Delta' F^\beta \left[\frac{\delta}{i\delta J} \right] \right\} Z[J, F, \eta]. \quad (B.2)$$

In the case of abelian gauge theories eq. (B.2) can in fact be solved. To do this we first separate gauge and matter fields which we collectively denoted φ^i so far by reserving indices from the beginning of the alphabet, $a, b, \dots$ for matter fields, and those from the middle of the alphabet, $m, n, \dots$ for gauge fields. The gauge generators D_α^m corresponding to the abelian gauge fields are field independent, and so is the Faddeev-Popov propagator G_α^β , if the gauge condition F^α is linear in the gauge fields and independent of the matter fields. The group indices $\alpha, \beta, \dots$ now of course stand for space-time variables only. The gauge generators for the matter fields, on the other hand, read $D_\alpha^a[\varphi] \equiv D_{\alpha, \delta}^a \varphi^\delta$.

Therefore, if the sources of the matter fields are switched off, eq. (B.1) retains only one derivative with respect to J_m coming from $\delta' F[\frac{\delta}{i\delta J}]$. Considering first only a variation of F_m^α in the limit of a homogeneous gauge, $\eta^{\alpha\beta} \rightarrow 0$, one can furthermore exploit the Ward identity

$$\langle F^\alpha[\varphi] \rangle \equiv \eta^{\alpha\beta} J_i (D_i^\beta G^\alpha), \quad (B.3)$$

in

$$\begin{aligned} \delta Z[J_m, J_a = 0; F, \eta = 0] &= J_m D_\alpha^m G_\alpha^\beta \delta F^\beta \frac{\delta}{i\delta J_n} Z \\ &= J_m D_\alpha^m \delta \left(G_\alpha^\beta F_n^\beta \right) \frac{\delta}{i\delta J_n} Z. \end{aligned} \quad (B.4)$$

Because

$$\left[J_m D_\alpha^m \delta \left(G_\alpha^\beta F_n^\beta \right), \frac{\delta}{i\delta J_n} \right] = 0, \quad (B.5)$$

eq. (B.4) can be integrated so that a finite gauge variation corresponds to a pure substitution of J_m :

$$Z[J_m, J_a = 0; F + \Delta F, \eta = 0] = Z[J_m + J_n D_\alpha^n \Delta(F_m^\beta G_\beta^\alpha), J_a = 0; F, \eta = 0]. \quad (B.6)$$

A variation of $\eta^{\alpha\beta}$, on the other hand, gives

$$\begin{aligned} \delta Z[J_m, J_a = 0; F, \eta] &= \frac{i}{2} J_m D_\alpha^m G_\alpha^\beta \eta^\gamma \delta_\gamma F_n^\beta \frac{\delta}{i\delta J_n} Z \\ &= -\frac{i}{2} J_m D_\alpha^m G_\alpha^\beta \delta \eta^\gamma G_\gamma^\delta J_n Z \end{aligned} \quad (B.7)$$

in virtue of (B.3). This equation is integrated with equal ease,

$$Z[J_m, J_a = 0; F, \eta] = \exp \left\{ -\frac{i}{2} J_m D_\alpha^m G_\alpha^\beta \delta \eta^\gamma G_\gamma^\delta J_n \right\} Z[J_m, J_a = 0; F, \eta = 0]. \quad (B.8)$$

Eqs. (B.6, B.8) have first been derived by Zumino [36], however within a formalism which from the start is applicable only to an abelian theory. These identities, valid only for $J_a = 0$, are in fact just reexpressing the fact derived in Sect. 2.2 by an extremely simple diagrammatic argument that vertex functions without external matter fields in an abelian theory (with linear gauge conditions) are gauge independent through all orders.

Still more remarkable is that one can also integrate the identities for the gauge dependence of Green functions with external matter lines [36]. In our approach these identities can be obtained by differentiating eq. (B.1) with respect to J_a 's, then setting $J_a = 0$ and using the analogous reasoning as above.

Doing this e.g. for the matter propagator, one finds first for a variation of δF at $\eta = 0$

$$\begin{aligned} &\frac{\delta^2 Z}{i\delta J_i i\delta J_d} [J_m, J_a = 0] \\ &= (iJ_m D_\alpha^m \delta_{ef}^d + D_{\alpha, \delta}^d \delta_{ef}^d + D_{\alpha, \delta}^d \delta_{ef}^c) \delta \left(G_\alpha^\beta F_n^\beta \right) \frac{\delta}{i\delta J_n i\delta J_i i\delta J_f}. \end{aligned} \quad (B.9)$$

Because $D_{\alpha,\beta}^c \propto \delta_{\alpha\beta}^c$ the matrix structure of eq. (B.9) is in fact trivial, so that a finite gauge variation again corresponds to a pure substitution of J_m .

The variation of $\eta^{\alpha\beta}$, on the other hand, gives

$$\begin{aligned} \delta \frac{\delta^2 Z}{i\delta J_c i\delta J_d} [J=0] = & -\frac{i}{2} G_{\alpha\beta}^c \delta\eta^{\alpha\beta} \gamma^c \gamma^d \left(2D_{\alpha,e}^c D_{\beta,f}^d \right. \\ & \left. + D_{\alpha,\beta}^c D_{\beta,e}^d + D_{\alpha,\beta}^d D_{\beta,e}^c \right) \frac{\delta^2 Z}{i\delta J_c i\delta J_f} [J=0], \end{aligned} \quad (B.10)$$

where the matrix structure is again trivial, and eq. (B.10) can be integrated immediately.

Leaving the abstract notation and specifying

$$G_{\alpha\beta}^c \Delta\eta^{\alpha\beta} \gamma^c \gamma^d \rightarrow M(x_{(\alpha)} - x_{(\beta)})$$

(for which the gauge conditions must not break translation invariance) and with (2.26) one finds for the complete η -dependence of the electron propagator the well-known formula [36]

$$G(x-y; \eta + \Delta\eta) = \exp \{ i e^2 [M(x-y) - M(0)] \} G(x-y; \eta), \quad (B.11)$$

which allows to derive exact results on the infra-red behaviour [37], or also that the regularized electron mass is gauge independent [38] (which, however, is already covered by the considerations of Sect. 4.2 which were based on infinitesimal gauge variations).

In the nonabelian case, the analogue of eq. (B.10)* is not as useful because there the G 's still depend on $\delta/\delta J$, and the right-hand side becomes an infinite series of Green functions.

*In Ref. 39 this generalization has been done incorrectly. There it has been assumed erroneously that the operator gauge transformations invoked by Landau and Khalatnikov [36] to derive the abelian version would carry over immediately to the nonabelian case. Actually this operator approach has been criticized by Zumino [36] as being questionable even in the case of QED, who has deemed a functional approach more reliable.

Appendix C. Generalization of the Gluon Propagator Gauge Dependence Identities to Gauges with 2 Fixed Vectors

In this Appendix we consider the case when the gluon propagator depends on two fixed vectors m_μ and n_μ , as well as on the four-momentum K_μ . At zero temperature (where an extra fixed vector enters e.g. with the LM-prescription for axial gauges [14]) these fixed vectors can have arbitrary directions, but at finite temperature one direction is determined by the velocity of the heat bath. We start with the four-momentum projection operator

$$P_{\mu\nu} = g_{\mu\nu} - \frac{K_\mu K_\nu}{K^2}, \quad (C.1)$$

and define the vectors $\tilde{m}_\mu = m_\nu P^\nu_\mu$ and $\tilde{n}_\mu = n_\nu P^\nu_\mu$, both being orthogonal to K_μ . From these we construct

$$\begin{aligned} M_\mu &= \sqrt{\tilde{n}^2} \tilde{m}_\mu + \sqrt{\tilde{m}^2} \tilde{n}_\mu, \\ N_\mu &= \sqrt{\tilde{n}^2} \tilde{m}_\mu - \sqrt{\tilde{m}^2} \tilde{n}_\mu, \end{aligned} \quad (C.2)$$

which as well as being orthogonal to K_μ also satisfy $M \cdot N = 0$. We then define the projection operator

$$C_{\mu\nu} = P_{\mu\nu} - \frac{M_\mu M_\nu}{M^2} - \frac{N_\mu N_\nu}{N^2}, \quad (C.3)$$

which is orthogonal to K_μ , M_μ , and N_μ and normalized to $C_{\mu\nu} C^\nu_\sigma = C_{\mu\sigma}$.

With this, we assume the following form for the bare inverse propagator,

$$\begin{aligned} [\Delta_{(0)}^{-1}]_{\mu\nu} = & K^2 P_{\mu\nu} + B_{(0)} M_\mu M_\nu + C_{(0)} N_\mu N_\nu + D_{(0)} (M_\mu N_\nu + M_\nu N_\mu) \\ & + E_{(0)} (M_\mu K_\nu + M_\nu K_\mu) + F_{(0)} (N_\mu K_\nu + N_\nu K_\mu) + G_{(0)} K_\mu K_\nu, \end{aligned} \quad (C.4)$$

and for the self energy,

$$\begin{aligned} \Pi_{\mu\nu} = & \Pi P_{\mu\nu} + \chi M_\mu M_\nu + \sigma N_\mu N_\nu + \psi (M_\mu N_\nu + M_\nu N_\mu) \\ & + \phi (M_\mu K_\nu + M_\nu K_\mu) + \rho (N_\mu K_\nu + N_\nu K_\mu) + \lambda K_\mu K_\nu. \end{aligned} \quad (C.5)$$

Note that the gauge fixing parameters satisfy

$$D_{(0)}^2 = B_{(0)} C_{(0)}, \quad E_{(0)}^2 = B_{(0)} G_{(0)}, \quad F_{(0)}^2 = C_{(0)} G_{(0)}. \quad (C.6)$$

Using Dyson's equation gives the full propagator,

$$\Delta_{\mu\nu} = QC_{\mu\nu} + RM_{\mu}M_{\nu} + SN_{\mu}N_{\nu} + T(M_{\mu}N_{\nu} + M_{\nu}N_{\mu}) + U(M_{\mu}K_{\nu} + M_{\nu}K_{\mu}) + V(N_{\mu}K_{\nu} + N_{\nu}K_{\mu}) + WK_{\mu}K_{\nu}, \quad (C.7)$$

where

$$\begin{aligned} Q &= \frac{1}{A}, \quad R = \frac{1}{L} \frac{F^2 - CG}{M^4}, \quad S = \frac{1}{L} \frac{E^2 - BG}{N^4}, \\ T &= \frac{1}{L} \frac{DG - EF}{M^2 N^2}, \quad U = \frac{1}{L} \frac{CE - DF}{M^2 K^2}, \\ V &= \frac{1}{L} \frac{BF - DE}{N^2 K^2}, \quad W = \frac{1}{L} \frac{D^2 - BC}{K^4}, \end{aligned} \quad (C.8)$$

and

$$L = CE^2 + GD^2 + BF^2 - 2DEF - BCG, \quad (C.9)$$

in which we have defined

$$\begin{aligned} A &= K^2 + \Pi, \quad B = \frac{K^2 + \Pi}{M^2} + B_{(0)} + \chi, \quad C = \frac{K^2 + \Pi}{N^2} + C_{(0)} + \sigma, \\ D &= D_{(0)} + \psi, \quad E = E_{(0)} + \phi, \\ F &= F_{(0)} + \rho, \quad G = G_{(0)} + \lambda. \end{aligned} \quad (C.10)$$

With this, we now consider eq. (4.10) expressing the change in the propagator when the gauge condition is changed,

$$-\delta\Delta^{\mu\nu} = \Delta^{\mu\sigma}\delta X^{\nu}_{\sigma} + \Delta^{\nu\sigma}\delta X^{\mu}_{\sigma}. \quad (C.11)$$

Projecting this equation onto $C_{\mu\nu}$ gives

$$\delta A = 2A C^{\rho}_{\sigma} \delta X^{\sigma}_{\rho} \quad (C.12)$$

which leads to the conclusion that zeroes of $A = K^2 + \Pi$ are gauge independent except at locations where $C^{\rho}_{\sigma} \delta X^{\sigma}_{\rho}$ develops a pole.

To examine the effects of a change in gauge condition on the zeroes of L , first note that since M_{μ} , N_{ν} , K_{ν} and the matrix $C_{\mu\nu}$ represent an orthogonal decomposition of the vector space on which the propagator acts, the (ordinary matrix) determinant of the propagator can easily be calculated. It is simply

$$\det(\Delta_{\mu\nu}) = (AL)^{-1}. \quad (C.13)$$

Moreover, by contracting (C.11) with the inverse propagator, $[\Delta^{-1}]_{\mu\nu}$, one obtains

$$\delta \det(\Delta_{\mu\nu}) = 2 \det(\Delta_{\mu\nu}) \delta X^{\nu}_{\nu}. \quad (C.14)$$

Using (C.12) and (C.13), this yields

$$\begin{aligned} \delta L &= 2L \left(C_{\mu}^{\sigma} - \delta_{\mu}^{\sigma} \right) \delta X^{\mu}_{\sigma} \\ &= -2L \left(\frac{N_{\mu} N^{\sigma}}{N^2} + \frac{M_{\mu} M^{\sigma}}{M^2} + \frac{K_{\mu} K^{\sigma}}{K^2} \right) \delta X^{\mu}_{\sigma} \end{aligned} \quad (C.15)$$

which again leads to the result that zeroes of L are gauge independent, except at the poles of the projection of δX_{μ}^{σ} on the r.h.s. of (C.15).

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FIGURE CAPTIONS

- 1. Expansion of the effective action into primary diagrams
- 2. Additional vertices for the expansion of $\delta X^i[\varphi]$ into primary diagrams
- 3. Expansion of $\delta X^i[\varphi]$ into primary diagrams
- 4. Expansion of δX^i in the case of linear gauges up to 2-loop order
- 5. Gauge dependence of a general 1-loop 2-point vertex function
- 6. Gauge dependence of the fermion self-energy.
- 7. 1-loop contributions to $\delta X^i_{j\dots}$, where wavy lines represent the gauge fields,
i.e. either gluons or gravitons

$$\Gamma[\bar{\varphi}] = S_{\text{gr.}}[\bar{\varphi}] - \frac{i}{2} \text{ (circle) } + i \text{ (circle with cross) } \\ - \frac{1}{12} \text{ (circle with horizontal line) } + \frac{1}{2} \text{ (circle with horizontal line and arrow) } - \frac{1}{8} \text{ (two circles) } + \frac{1}{2} \text{ (two circles with arrow) } + \dots$$

Fig. 1

$$\begin{array}{ll} i \swarrow \text{---} \alpha \text{---} & D_{\alpha,i}^1 \\ i \swarrow \text{---} \alpha \text{---} & D_{\alpha}^1[\bar{\varphi}] \\ \text{---} \alpha \text{---} & \delta F_{,i}^{\alpha}[\bar{\varphi}] \\ \text{---} \alpha \text{---} & \delta F^{\alpha}[\bar{\varphi}] \end{array}$$

Fig. 2

$$\delta X^i = -i \text{ (circle with horizontal line) } + i \left\{ \begin{array}{l} \text{circle with horizontal line and arrow} \\ \text{circle with horizontal line and arrow} \\ \text{circle with horizontal line and arrow} \end{array} \right\} + \dots$$

Fig. 3

$$\delta X^i = i \text{ (circle with horizontal line) } + i \text{ (circle with horizontal line and arrow) } + \dots \\ + i \text{ (circle with horizontal line and arrow) } + i \text{ (circle with horizontal line and arrow) } + \dots \\ + \frac{1}{2} i \text{ (circle with horizontal line and arrow) } + \frac{1}{2} i \text{ (circle with horizontal line and arrow) } - \dots \\ + i \text{ (circle with horizontal line and arrow) } + i \text{ (circle with horizontal line and arrow) } + \dots \\ + \frac{1}{2} i \text{ (circle with horizontal line and arrow) } + \frac{1}{2} i \text{ (circle with horizontal line and arrow) } - \dots$$

Fig. 4

$$\delta \Gamma_{,ij}^{1\text{-loop}}[0] = i S_{,im}[0] \left\{ \begin{array}{l} \text{circle with horizontal line and arrow} \\ \text{circle with horizontal line and arrow} \\ \text{circle with horizontal line and arrow} \end{array} \right\} + i S_{,jm}[0] \left\{ \begin{array}{l} \text{circle with horizontal line and arrow} \\ \text{circle with horizontal line and arrow} \\ \text{circle with horizontal line and arrow} \end{array} \right\}$$

Fig. 5

Fig. 6

Fig. 7