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Quantum Mechanics on the moduli space from the Quantum Geometrodynamics of the Open Topological Membrane.

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Abstract

The Quantum Geometrodynamics of the Open Topological Membrane is described in terms of 2+1 Topologically Massive Gravity (TMG) where the inverse graviton mass is proportional to the 2d central charge and thus is the measure of the off-criticality. The hamiltonian quantization of TMG on Riemann surfaces is considered and the moduli space appears as the subspace of the quantum mechanical configuration space containing, besides the moduli, the first order time derivatives of half of the moduli. The appearance of the first order time derivatives as coordinates, not momenta, is due to the third order derivative in the TMG Lagrangian. The Hamiltonian for the latter leads us to the discrete levels picture which looks like the Topologically Massive Gauge Theory (TMGT) case, where we also get the Landau levels picture and the lowest Landau level corresponds to the Hilbert space of the Chern-Simons Theory (CST). The connection between the positivity of the energy and the complex structure on the moduli space is discussed.

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1. Introduction

This paper is devoted to the description of the moduli space in the Open Topological Membrane (OTM) approach [1],[2],[3] where the string world sheet is considered as boundary of the open membrane (really there are two boundaries, corresponding to the left- and right- moving sectors, which makes the heterotic construction quite natural), with topologically massive gauge fields and topologically massive gravity inside (and also may be with some additional matter fields, see [4]). The string degrees of freedom (2d Conformal Field Theory, CFT) arise on the boundary as pure gauge degrees of freedom of the 2+1 Topologically Massive Gauge Theory (TMGT). This very interesting physical theory was suggested in papers [5],[6],[7]. The membrane approach is based on the remarkable connection between 2+1 Chern - Simons gauge theory (CST), which is the low energy limit of the TMGT and 2d CFT found by Witten [8] and Moore and Seiberg [9]. The gravitational sector is described by the Topologically Massive Gravity (TMG) of Deser, Jackiw and Templeton [7], and its low energy limit is the 2+1 Einstein gravity. There are no physical degrees of freedom for the Einstein gravity in three dimensions [10], moreover it is the topological $ISO(1,2)$ CST and the physical phase space is ultimately connected to the moduli space as was shown in [11] (see also the papers [12],[13]). Let us note also, that this theory was considered by Leutwyler more then 20 years ago [14], where it was shown, that in spite of the absence of degrees of freedom, there is a nontrivial Hilbert space in the theory. The path integration over all three-dimensional metrics gives us the integration over moduli space, which is the necessary ingredient of the string theory.

The gravitational Chern-Simons term [7], which introduce the physical degrees of freedom into 2+1 gravity, plays a very important role in coding the 2d information - it describes the central charge (Weyl anomaly) of the 2d theory [8],[15],[2],[3]. But it is necessary to stress that TMG with gravitational Chern-Simons term is a different physical theory, than Witten topological 2+1 gravity [11]. Both the theories have at first sight the same actions:

$$S = \kappa \int_M \epsilon^{\mu\nu\lambda} \epsilon^{abc} e_{\mu a} R_{\nu b c} + \frac{\kappa'}{8\pi} \int_M \epsilon^{\mu\nu\lambda} (R_{\mu\nu ab} \omega_{\lambda}^{ab} + \frac{2}{3} \omega_{\mu a}^b \omega_{\nu b}^c \omega_{\lambda c}^a) \quad (1)$$

where the curvature

$$R_{\mu\nu ab} = \partial_{\mu} \omega_{\nu ab} - \partial_{\nu} \omega_{\mu ab} + \omega_{\mu a}^c \omega_{\nu cb} - \omega_{\nu a}^c \omega_{\mu cb} \quad (2)$$

Here we use the CS representation for the 2+1 Einstein action (the first term in (1), suggested by Witten in [11]). But there is a difference between TMG and Witten 2+1 topological Einstein + Chern-Simons gravity which is the following: the spin connection $\omega_{\mu ab}$ in Witten theory is an independent variable, and in TMG the only independent variables are the dreibeins $e_{\mu a}$, with the spin connection equal to

$$\omega_{\mu ab} = \frac{1}{2} \epsilon_a^c (\partial_{\mu} e_{c b} - \partial_b e_{\mu c}) + \frac{1}{4} \epsilon_a^c \epsilon_b^d (\partial_{\mu} e_{c d} - \partial_d e_{\mu c}) e_{\mu}^e - (a \leftrightarrow b) \quad (3)$$

In the case of Witten gravity, varying (1) with respect to $e_{\mu a}$ we get the zero curvature condition (differential form notation is used)

$$R = d\omega + \frac{1}{2} \omega \wedge \omega = 0,$$

and varying the action with respect to ω we get the relation between spin connection and dreibein:

$$de + \omega \wedge e + \frac{\kappa'}{\kappa} (d\omega + \frac{1}{2} \omega \wedge \omega) = 0,$$

but due to the zero curvature condition we get the Maurer-Cartan equation for the spin connection $de + \omega \wedge e = 0$, the solution of which is the definition of the spin connection in terms of the dreibein (3).

In the TMG case when varying respect to $e_{\mu a}$ we also take into account the second term in (1) and get the dynamical equation [7]:

$$M_g G^{\mu\nu} + C^{\mu\nu} = 0, \quad (4)$$

where $M_g = 4\pi\kappa/\kappa'$ is the graviton mass and $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}R$. The parity odd tensor $C^{\mu\nu} = \frac{1}{\sqrt{g}} \epsilon^{\mu\nu\alpha\beta} D_{\alpha} (R_{\beta}^{\gamma} - \frac{1}{4} \delta_{\beta}^{\gamma} R)$, which is invariant under local conformal transformations of the metric and vanishes if and only if the metric is conformally flat is the 2+1 analog of the Weyl tensor. Besides the trivial solutions $R = 0$, giving us the same results as in Witten topological gravity, equation (4) has nontrivial solutions with non-zero curvature, describing topologically massive gravitons. Moreover, the classical equations of motion are of third derivative order since the Chern-Simons term has three derivatives, including three time derivatives, which leads to a rather unexpected phase space, because the first time derivative of the coordinates (moduli) will be again coordinates, not momenta (really only half of the first order time derivatives are coordinates, as will be demonstrated later). We also wish to note here, that it is likely, that the TMG may be renormalizable theory, without a loss of unitarity and/or causality. This possibility was considered in [7] and after it Deser and Yang proved in [16] that TMG is naive power counting renormalizable.

The main aim of the paper is to discuss the status of Topologically Massive Gravity in the Open Topological Membrane picture, to study the unusual phase space for these metric fluctuations which are connected with moduli space and then to consider the resulting quantum mechanics on the moduli space.

The organization of the paper is as follows. In the Section 2 we recall the connection between the gravitational Chern-Simons term and the central charge and discuss the possibility to compute the central charge for the nontrivial 2d CFT as well as the 2d Quantum Gravity renormalization of the central charge using the TMGT and TMG perturbative expansions. We consider the quantization of TMG on Riemann surfaces, the decoupling of the moduli from other metric excitations, the moduli phase space and the corresponding quantum mechanics in Section 3, where we also remember the analogous problem in TMGT where the Landau levels picture arises. Briefly we discuss the status of the string loop expansion in the OTM approach as some version of the Born-Oppenheimer approximation in the end of Section 3. Quantum mechanics for moduli will be considered in Section 4, as well as the connection between the positivity of the moduli Hamiltonian spectrum and the holomorphic dependence of wave functions on moduli (analytical anomaly cancellation). Section 5 summarizes basic conclusions and outlines a program for future investigations.

2. Topological graviton mass in TMG as a measure of the off-criticality

How it is possible to connect the coefficient k' , which defines the graviton mass $M_g = 4\pi k'/\kappa$ with the central charge c ? This connection is based on the Witten observation [15] that the three dimensional description of the central charge can be done in terms of a "global" interpretation of Virasoro anomaly [17],[18] as a central extension of the mapping class group of a Riemann surface. In three dimensions the action of the mapping class group can be described as a change of framing of three-dimensional manifold M . Under a change of framing, the partition function changes by a phase, proportional to the central charge of the associated conformal theory. But this change of framing is directly connected with the large local Lorentz transformation under which the gravitational Chern-Simons term (for analytically continued $SO(3)$ group) is shifted by some constant, proportional to the winding number of $\pi_3(SO(3))$ times k' . This is correct both for topological gravity and TMG. So we get some phase factor proportional to k' . Comparing it with the phase factor for a central extension of the mapping class group we get the connection between c and k' [8],[2],[3]

$$k' = \frac{c}{24} \quad (5)$$

We can compare (5) with the calculation of the induced gravitational CS term by the topologically massive gauge fields in perturbation theory [19] (see also [20], where the fermionic contribution to k' was calculated), or by path integral methods [21]. For a $G = U(1)^n$ TMGT, one gets $k' = n/24 = \dim G/24$, which gives the correct central charge $c = n$ for n copies of the free 2d bosons. For nonabelian TMGT with the gauge group G this is only the first term in the expansion of the well known formula for the central charge in the WZW model

$$c_{WZW} = \frac{k \dim G}{k + c_v} = \dim G (1 - \frac{c_v}{k} + \dots), \quad (6)$$

where k is the gauge CS coefficient (the central charge of the 2d Kac-Moody algebra) and c_v is the dual Coxeter number of G (N for $SU(N)$).

However, the gravitational Chern-Simons term in TMG is of third derivative order, not first, as the gauge CS term. This means that the Coleman-Hill theorem [22] is unapplicable and that there are higher loop corrections ("gluon" "gluon" scattering, see Fig.1), contrary to the TMGT case, where the one loop result [23] (see also [24],[25],[26]) is exact and gives us the famous shift $k \rightarrow k + c_v$. The perturbative expansion parameter in TMGT is nothing but c_v/k [23], so we get the same perturbative expansion as for c_{WZW} . It would be interesting to prove that 2+1 perturbation theory gives us the same geometrical progression as the c_v/k expansion of c_{WZW} . Let us consider the simplest two-loop contribution to the renormalization of k' , where only bare four-"gluon" vertex

$$\Gamma_{\mu\nu\rho\sigma}^{abcd} = \frac{1}{\gamma} [f^{abc} f^{cde} (g_{\mu\rho} g_{\nu\sigma} - (\mu \leftrightarrow \nu)) + f^{ace} f^{bde} (g_{\mu\sigma} g_{\nu\rho} - (\mu \leftrightarrow \rho)) + f^{ade} f^{bce} (g_{\mu\nu} g_{\sigma\rho} - (\mu \leftrightarrow \sigma))]$$

is taken into account (see Fig.2). Performing the colour index contraction we get $c_v \dim G$. Picking out the gravitational CS structure

$$h^{\alpha\beta}(q) h^{\gamma\delta}(-q) \epsilon_{\alpha\gamma\mu} q^\mu (q g_{\beta\delta} - g_{\beta\sigma} q^\sigma), \quad (7)$$

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where $h^{\alpha\beta}$ is the external gravitational field with momentum q_μ and considering the limit of zero q_μ we can see after all, that each loop contribution is factorizable. This means, that our two-loop result is different from the old one-loop by the simple one-loop integral

$$2i\gamma \int \frac{d^3 p}{(2\pi)^3} \frac{1}{(p^2 - M^2)^2}$$

where $M = \gamma k/4\pi$ is the topological gauge mass, k is the gauge Chern-Simons coefficient. Calculating this integral we get that each new loop contributes $(-1)\gamma/4\pi M = -1/k$. Remembering about the colour factor c_v we see, that the total two-loop contribution to the induced k' , is $-\dim G \frac{c_v}{k}$, which is the next-to-leading term in the c_{WZW} expansion (6) (the above mentioned factorization is extremely significant). It is possible to show, that each new loop contribution is proportional to the new $\frac{c_v}{k}$ factor. If we suppose that all higher corrections are also factorizable and give us the next terms of the geometrical progression we get the same expression for $c = 24k'$ (see eq.(5)) as for c_{WZW} . It would be extremely interesting to prove it and we see that our two-loop result confirms this suggestion. This result, if valid, has a natural generalization to coset and orbifold constructions. Coset construction in 2d CFT may be obtained by starting with the difference between two Chern-Simons (really two TMGT) actions [9]; both these topologically massive gauge fields make independent contributions to the total induced $k'_G/k'_H = k'_G - k'_H$. This gives us a natural explanation of the coset central charge $c_{G/H} = c_G - c_H$. The minus sign appears, because k' proportional to $\frac{1}{|k|}$ and if we have two TMGT with opposite signs of the CS terms (the above mentioned difference between the two Chern-Simons actions), the total induced k' will be the difference between the two. Orbifold constructions arise from semidirect product gauge groups $P \otimes G$, where P is a discrete automorphism group of G and this construction gives us the same central charge as for G .

Moreover, there are gravitational corrections to k' , and using the to Deser and Yang paper [16] we can calculate the renormalization of k' as a perturbative expansion. To find the expansion parameter we have to remember the Deser-Yang parametrization of the metric:

$$g_{\mu\nu} = \Phi^4 (\eta_{\mu\nu} + \frac{1}{\sqrt{k'}} h_{\mu\nu}) = \Phi^4 \bar{g}_{\mu\nu} \quad (8)$$

Due to the Weyl invariance of the CS term, Φ appears only in the Einstein term and the action(1) takes the form:

$$S = \frac{\kappa}{2} \int \sqrt{g} [8g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \Phi^2 R(\bar{g})] + \frac{k'}{8\pi} S_{CS}(\bar{g}) \quad (9)$$

Adding to the action (9) gauge-fixing and ghost terms, and expanding $\Phi = 1 + \frac{1}{2\sqrt{2\kappa}} \phi$ we can get the propagators; the large momentum behaviour of propagators is therefore

$$\langle \phi \phi \rangle \sim \frac{1}{p^2}, \quad \langle hh \rangle \sim \frac{1}{p^2}, \quad \langle bc \rangle \sim \frac{1}{p^2},$$

where b and c are the ghost fields. From the action (9) we next get the vertices (in schematic-form, p is the momentum):

$$\frac{1}{k'^m/2} p^2 \phi^2 h^n, \quad \sqrt{\kappa} \frac{1}{k'^m/2} p^2 \phi h^n, \quad \frac{1}{\kappa} \frac{1}{k'^m/2} p^2 h^n, \quad k' \frac{1}{k'^m/2} p^2 h^n, \quad \frac{1}{k'^m/2} p^2 b c h^n, \quad (10)$$

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and ϕ field does not interact with ghost fields b and c . Because dimensional constant κ enters in non-negative powers only, it is a superrenormalisable constant in contrast to usual Einstein gravity. The real expansion parameter now is $1/k'$. If we want to calculate the one-loop correction to k' we can see, that there are only convergent one-loop integrals in front of the gravitational CS structure (7) :

$$\frac{M_g}{k'} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{(p^2 - M_g^2)^2} \sim \frac{\text{const}}{k'},$$

where $M_g = 4\pi\kappa/k'$ is the topological graviton mass. Let us note, that there are no logarithmic divergences, because at one-loop order there are no integrals like

$$\int \frac{d^3 p}{(2\pi)^3} \frac{1}{p^3}$$

Possible UV divergences can arise at the two-loop order in principle, but perhaps there is no such a contribution to the CS structure (7). It is also possible to show, that at n -loop order we get the corrections of order $1/k'^n$ (if we neglect the possible log terms). Thus for k'_{ren} or for c_{ren} we get the $1/k'$ or $1/c$ expansion. We have the same expansion for c in 2d Quantum Gravity [27], [28], [29]. It will be extremely interesting to understand if both these expansions coincide. Of course, due to the extremely complicated nature of the TMG perturbative expansion we can only hope to compare the first terms in both the expansions [30]. May be TMG can also shed some light to the case of $1 < c < 25$, which can not be described by the 2d Quantum gravity, due to the famous square root in the Knizhnik-Polyakov-Zamolodchikov formula $\sqrt{(c-1)(c-25)}$.

If all these qualitative arguments will be confirmed by the future calculations, we really can interpret the topological graviton mass $M_g = 4\pi\kappa/k'$ as the measure of the off-criticality, and critical behaviour will be recovered in the case $k'_{ren} \rightarrow 0$, or $M_g \rightarrow \infty$, when all physical degrees of freedom decouple from the topological Einstein gravity. However, as we shall see below, even in that case there will be some new features in the description of the low-energy sector.

3. Hamiltonian quantization of Topologically Massive Gravity on Riemann surfaces.

We know, that TMG is the theory of topologically massive gravitons, as well as TMGT is the theory of topologically massive gauge particles. Does this mean, that in the limit of infinite mass we have nothing ? Of course not, because both low-energy limits - Chern-Simons theory and Einstein gravity are topological field theories with some degrees of freedom. But these degrees of freedom are very specific - the corresponding Hilbert spaces are finite-dimensional and appear only if we quantize the theory on a Riemann surface. How can we get such a strange Hilbert space from normal quantum field theory ? Let us remember the simplest $U(1)$ TMGT case.

We start from the action [5],[6],[7]:

$$S_{U(1)} = -\frac{1}{4\gamma} \int_M F_{\mu\nu}^2 + \frac{k}{8\pi} \int_M \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \quad (11)$$

Let us choose gauge $A_0 = 0$ and try to get free massive particles. We repeat here the arguments from [7] (see also [31]). If we forget for a moment about topology and consider fields on a plane, we can represent the vector-potential as:

$$A_i = \partial_i \xi + \epsilon_{ij} \partial_j \chi \quad (12)$$

Substituting this decomposition in the constraint

$$\frac{1}{\gamma} \partial_i \dot{A}_i + \frac{k}{4\pi} \epsilon_{ij} F_{ij} = 0, \quad (13)$$

we get

$$\partial^2 \xi = \frac{k\gamma}{2\pi} \partial^2 \chi$$

Neglecting all possible zero modes we can put $\xi = \frac{k\gamma}{2\pi} \chi = M/2 \chi$. Substituting this constraint into action (11) we get the action

$$S = \frac{1}{2\gamma} \int_M (\partial_i \dot{\chi})^2 - (\partial^2 \chi)^2 - M^2 \chi^2 \partial^2 \chi$$

which becomes the free massive particle action for the new field $\Phi = \sqrt{\partial^2/\gamma} \chi$:

$$S = \frac{1}{2} \int_M \dot{\Phi}^2 - (\partial_i \Phi)^2 - M^2 \Phi^2$$

To get this free action it was extremely significant that we used constraint (13). But this constraint is not a real constraint for the constant field $A_i(x, t) = A_i(t)$. For this field we get the quantum mechanical (no coordinate dependence) lagrangian:

$$L = \frac{1}{2\gamma} \dot{A}_i^2 - \frac{k}{8\pi} \epsilon_{ij} A_i \dot{A}_j \quad (14)$$

which describes a particle on a plane A_1, A_2 in a magnetic field (Landau problem); the canonical Landau Hamiltonian is:

$$H = -\frac{\gamma}{2} \left(\frac{\partial}{\partial A_i} - i \frac{k}{8\pi} \epsilon_{ij} A_j \right)^2 \quad (15)$$

$$E_n = (n + 1/2)M$$

where the mass gap $M = \gamma k/4\pi$ is nothing but the topological gauge mass. This picture of the Landau levels was suggested in [31], where it was proved, that due to the degeneracy at each level we can get a long-range part of the gauge propagator in TMGT (the propagator in pure CS theory), in spite of the fact that there are no massless particles in theory. In other words, the CS Hilbert space is nothing but the first Landau level. But can we consider the constant gauge fields as the correct variables or they are they gauge artefacts ? Yes, if we have a 2-dimensional Riemann surface different from the sphere. Each one-form A can be uniquely written according to the Hodge theorem as

$$A = d\xi + \delta\chi + \Lambda \quad (16)$$

$$d\Lambda = \delta A = 0$$

which generalizes the decomposition (12); the harmonic form A equals

$$A = \sum_{i=1}^g (A^i \alpha_i + B^i \beta_i) \quad (17)$$

where α_i and β_i are canonical harmonic forms on the Riemann surface with genus g . The gauge transformations act on the $2g$ quantum-mechanical coordinates as $A^i \rightarrow A^i + n^i$, $B^i \rightarrow B^i + m^i$, so $0 \leq A^i, B^i < 1$ are the gauge invariant degrees of freedom, which decouple from all other degrees of freedom and give us g copies of the Landau Hamiltonian (15) on the torus. The total Hilbert space of the theory

$$H = H_\Phi \otimes \prod_{i=1}^g H_A \quad (18)$$

is the product of the free massive particles Hilbert space and g copies of the Landau problem Hilbert space.

Let us now make the same analysis for gravity, where instead of gauge connections we deal with dreibeins (or metric tensor). Let us again choose M to have the topology $R \times \Sigma$, where Σ is the Riemann surface of genus g and write the metric in Arnowitt-Deser-Misner (ADM) form (signature $(- + +)$)

$$ds^2 = -(N dt)^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

$$G_{\mu\nu} = \begin{pmatrix} N_i N^i - N^2 & N_j \\ N_i & g_{ij} \end{pmatrix} \quad (19)$$

where g_{ij} is a fixed metric on Σ . The Hamiltonian (or ADM) quantization of the usual Einstein gravity is well known [32],[33]. The ADM representation of the Einstein action is:

$$\begin{aligned} S_E &= \int dt \int_\Sigma (\pi^{ij} g_{ij} - N \mathcal{H} - N^i \mathcal{H}_i) \\ \pi^{ij} &= \sqrt{g} (g^{ij} K^j_i - K^{(ij)}) \\ K_{ij} &= \frac{1}{2N} (N_{ij} + N_{ji} - \dot{g}_{ij}) \end{aligned} \quad (20)$$

with constraints

$$\begin{aligned} \mathcal{H} &= \frac{1}{\sqrt{g}} (\pi^{ij} \pi_{ij} - \pi^i_i \pi^j_j) - \sqrt{g} R = 0, \\ \mathcal{H}^i &= -2\pi^{ij}_j = 0 \end{aligned} \quad (21)$$

There are three constraints, corresponding for the three Lagrange multipliers $\mathcal{H}, \mathcal{H}^i$. For D -dimensional space-time we have $D(D+1)/2$ independent component of metric tensor. There are D general coordinate transformations (diffeomorphisms) and D constraints, so the number of physical degrees of freedom is $D(D+1)/2 - 2D = D(D-3)/2$. In three dimensions Einstein gravity has no degrees of freedom. However, TMG has one physical degree of freedom without violating general covariance. It means, that either the number

of constraints in TMG equals 2, not 3 as in Einstein gravity, or there are 4 coordinates, not 3. In this short paper we can not discuss the ADM formulation of TMG (this will be done in future publications), but only demonstrate how the CS term changes the number of coordinates.

This drastic change of picture is due to the third order time derivative part of the CS term, which can be rewritten as:

$$\frac{k'}{4\pi} \int dt \int_\Sigma \epsilon^{ij} \Gamma_{ib}^a \partial_b \Gamma_{ja}^c \quad (22)$$

where we rewrite spin connection in terms of metric connection $\Gamma_{\mu\nu}^\sigma$

$$\omega_{\mu\alpha\beta} = e_\alpha^\nu \partial_\mu e_{\nu\beta} - \Gamma_{\mu\nu}^\sigma e_{\sigma\alpha} e_\beta^\nu$$

and neglect all irrelevant terms. Substituting to (22) the $\Gamma_{\mu\nu}^\sigma$:

$$\Gamma_{ij}^0 = -\frac{1}{N} K_{ij}, \quad \Gamma_{im}^i = \frac{G^{0l}}{G^{00}} \Gamma_{im}^0 + \dots, \quad \Gamma_{i0}^i = -\frac{G^{ml}}{G^{00}} \Gamma_{im}^0 + \dots, \quad \Gamma_{i0}^0 = \frac{G^{0m}}{G^{00}} \Gamma_{im}^0 + \dots;$$

where ... describes terms without time derivative and $G^{\mu\nu}$ is the contravariant metric tensor, inverse to (19), we get

$$\frac{k'}{2\pi} \frac{G^{0l} G^{0m} - G^{00} G^{lm}}{(G^{00})^2} \epsilon^{ij} \Gamma_{im}^0 \partial_0 \Gamma_{jl}^0 + \dots = \frac{k'}{2\pi} \epsilon^{ij} g^{lm} K_{im} \partial_0 K_{jl} + \dots \quad (23)$$

It is convenient to use the complex notation for the two-dimensional metric

$$g_{ij} dx^i dx^j = g_{z\bar{z}} dz d\bar{z} + g_{\bar{z}z} d\bar{z} dz + g_{zz} dz dz + g_{\bar{z}\bar{z}} d\bar{z} d\bar{z}$$

and it is easy to see, that $K_{z\bar{z}}$ disappears from (23) and we get structure proportional to $K_{z\bar{z}} \bar{K}_{\bar{z}z}$. Since K_{ij} is proportional to the conjugate momentum $\pi^{ij} = \sqrt{g} g^{ij} K^j_i - K^{(ij)}$ (see (21)), equation (23) gives us the time derivative terms of π^{ij} in the action. This means, that besides the metric g_{ij} one of the π^{ij} components ($\pi^{z\bar{z}}$ for example) becomes a coordinate, so we have now 4, not 3, coordinates and 3 constraints leaving one degree of freedom. In other words we can solve one constraint, to eliminate this new coordinate, then we get the three usual components of the two-dimensional metric and only two constraints.

To study the phase space of the theory, we consider the metric (19) $G_{\mu\nu} = \eta_{\mu\nu} + (\kappa)^{-1/2} h_{\mu\nu}$, where the deviation from a flat metric is small, this means, that the lapse function $N = 1 + h_{00} = 1 + n$. The two-dimensional metric can be written as:

$$g_{ij} dx^i dx^j = \exp(\phi) |dz + \mu d\bar{z}|^2 \quad (24)$$

where $\mu = \partial\epsilon + \bar{\epsilon}$, $\epsilon = \epsilon(z, \bar{z})$ is the parameter of the two-dimensional diffeomorphisms, and the holomorphic quadratic differential ϵ , $\partial\bar{\epsilon} = 0$ describes those variations of metric which are orthogonal to the group of two-dimensional diffeomorphisms, i.e. the moduli of the Riemann surface. The complex dimension of this moduli space is 1 for $g = 1$ and $3g - 3$ for higher genus $g > 1$. Moduli here are the precise analogs of the harmonic one-forms A in the gauge field decomposition (16). Substituting (24) to the TMG action (1) and

assuming all deviations from flat metric to be small we get the action (the case without moduli was considered in [7], we use here slightly different notation)

$$S = -\frac{1}{2} \int (\partial_\mu \phi \partial_\mu \phi + \lambda \phi + \sigma^2 + \frac{1}{M_g} \sigma \lambda) \quad (25)$$

$$\lambda = \partial^2(n + \phi) + \partial(2\dot{N} + \dot{\epsilon} + \dot{\sigma}^{-1}\dot{f}) + \partial(2\dot{N} + \dot{\epsilon} + \dot{\sigma}^{-1}\dot{f})$$

$$\sigma = i[\partial(2N + \dot{\epsilon} + \dot{\sigma}^{-1}\dot{f}) - \partial(2\dot{N} + \dot{\epsilon} + \dot{\sigma}^{-1}\dot{f})]$$

where we use complex notation $z(\bar{z}) = x \pm iy$, $\partial(\bar{\partial}) = 1/2(\partial_x \mp i\partial_y)$, $A(\bar{A}) = (A_x \mp iA_y)$. It is easy to see, that moduli f decouple from all other degrees of freedom, due to the conditions $\partial f = \bar{\partial} f = 0$ and the only non-zero contribution arises from the terms $\int \partial \bar{\partial}^{-1} f \partial \bar{\partial}^{-1} f = \int f f$ (we omit here possible time derivatives of f). Resolving the constraints we get finally the action, which is the sum of the old DJT action for the topologically massive graviton ϕ , plus the quantum mechanical action for moduli $f = y_i \bar{f}^i$, $i = 1, \dots, 3g - 3$ (1 for $g = 1$):

$$S = S_\phi + S_f = -\frac{1}{2} \int_M (\partial_\mu \phi \partial_\mu \phi + M_g^2 \phi^2) + \int dt [\dot{y}_i \dot{\bar{y}}_j + \frac{i}{2M_g} (\dot{y}_i \dot{\bar{y}}_j - \dot{y}_j \dot{\bar{y}}_i)] N_2^{\bar{i}j} \quad (26)$$

where matrix $N_2^{\bar{i}j} = \int_\Sigma \bar{f}^i f^j$ is the metric on moduli space (in our quadratic approximation). In the quadratic approximation we get thus $3g - 3$ (1 for $g = 1$) copies of the same quantum system, whose Lagrangian $\mathcal{L}(Q, \bar{Q}, \dot{Q})$ contains a second order time derivative. The conjugate momenta for such a Lagrangians are defined as follows (we omit all indices for simplicity):

$$P_Q = \frac{\partial \mathcal{L}}{\partial \dot{Q}}, \quad P_{\bar{Q}} = \frac{\partial \mathcal{L}}{\partial \dot{\bar{Q}}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\bar{Q}}} \quad (27)$$

and Hamiltonian $\mathcal{H}$ is

$$\mathcal{H} = P_Q \dot{Q} + P_{\bar{Q}} \dot{\bar{Q}} - \mathcal{L} \quad (28)$$

It is convenient to rewrite S_f in real notation, $y(\bar{y}) = (y_1 \mp i y_2)$,

$$S_f = \int dt [-\dot{y}_1^2 - \dot{y}_2^2 + \frac{1}{M_g} (\dot{y}_1 \dot{y}_2 - \dot{y}_2 \dot{y}_1)] \quad (29)$$

From (27) and (29) we get the conjugate momenta:

$$P_{y_1} = -\frac{1}{M_g} \dot{y}_2, \quad P_{y_2} = \frac{1}{M_g} \dot{y}_1, \quad P_{y_1} = -2\dot{y}_1 + \frac{1}{M_g} \dot{y}_2, \quad P_{y_2} = -2\dot{y}_2 - \frac{1}{M_g} \dot{y}_1. \quad (30)$$

and we see that besides y_1 and y_2 one of the first order time derivative, $\dot{y}_2$, for example is also a coordinate and $\dot{y}_1$ becomes the momentum conjugate to $\dot{y}_2$. Substituting all this into (28) we get the quantum mechanical Hamiltonian

$$\mathcal{H}_f = M_g^2 P_{y_2}^2 + \dot{y}_2^2 + M_g P_{y_2} P_{y_1} + \dot{y}_2 P_{y_1} \quad (31)$$

Thus the Hilbert space of TMG is

$$H_{TMG} = H_\phi \otimes \prod_{i=1}^{3g-3} H_f \quad (32)$$

where wave functions of H_f depend on y_1, y_2 and $\dot{y}_2$. We consider this quantum mechanics in the next section and now let us briefly discuss the general picture of the interacting TMGT and TMG. Roughly speaking we have two interaction systems with two mass parameters $M = k\gamma/4\pi$ and $M_g = 4\pi\kappa/k'$. The usual string loop expansion corresponds to a situation, where we first fix some geometry and consider quantum matter in the background of classical geometry, and only after it integrate over geometry (including sum over topology). This is nothing but the Born-Oppenheimer approximation where we consider matter (TMGT) as fast variables (electrons) and geometry (TMG) as slow variables (nuclei). This can be done only in the case when the energy gap for fast variables is much larger than the energy gap for slow ones, i.e. $M \gg M_g$, which gives us the inequality $kk' \gg 1$. However in the case of small k and k' , which is the extreme quantum case (small scales of compactified dimensions for example, the critical situation, when $k' \rightarrow 0$) we get the opposite inequality $M < M_g$. In this case the geometry becomes the fast variable and matter the slow one. Then we are to integrate over quantum geometry with fixed classical matter first, and only then integrate over matter. It will be interesting to understand what will be the analog of the loop (genus) expansion in this case. This picture certainly incorporates the summation over topology before the integration over matter field and must gives the correct description in the strong coupling case.

4. Quantum mechanics of moduli

Let us rewrite the Hamiltonian (31) as

$$\mathcal{H}_f = M_g \left(-\frac{\partial^2}{\partial x^2} + x^2 \right) - \sqrt{M_g} \left(\frac{\partial}{\partial x} \frac{\partial}{\partial y_1} + ix \frac{\partial}{\partial y_2} \right) \quad (33)$$

where we define x as $y_2 = \sqrt{M_g} x$. Introducing standard creation and destruction operators $a^\pm(a) = (p \pm ix)/\sqrt{2}$ we get

$$\mathcal{H}_f = M_g (aa^\dagger + a^\dagger a) + \sqrt{M_g} \left(ia \frac{\partial}{\partial y} + ia^\dagger \frac{\partial}{\partial \bar{y}} \right) \quad (34)$$

It is easy to see, that the Hamiltonian we obtained is not positively defined and equals the positively defined oscillator Hamiltonian plus the Laplacian on moduli space, which is negatively defined:

$$\mathcal{H}_f = M_g (bb^\dagger + b^\dagger b) + \frac{M_g}{2} \frac{\partial^2}{\partial y \partial \bar{y}}$$

$$b = a + \frac{i}{2} \frac{\partial}{\partial \bar{y}}, \quad b^\dagger = a^\dagger + \frac{i}{2} \frac{\partial}{\partial y} \quad (35)$$

where we make a rescaling $y \rightarrow y/\sqrt{M_g}$ to make the moduli dimensionless. To get the positively defined spectrum we must restrict ourselves to the subspace of H_f on which Laplacian $\frac{M_g}{2} \frac{\partial^2}{\partial y \partial \bar{y}}$ gives zero. Thus we are to work either with holomorphic or antiholomorphic functions only, and we see that the positivity of energy leads to holomorphic or antiholomorphic dependence of wave function on moduli. The choice can be made by the requirement that the lowest level wave function is annihilated by the destruction operator

a. Then from (35) we get the holomorphic dependence. Thus we get the ground state wave functions

$$\Psi_0(x, y, \bar{y}) = \Psi_0(x) \Phi(y) = \text{const} e^{-x^2/2} \Phi(y)$$

It is interesting to note, that the wave functions of the excited levels are not simply the corresponding excited oscillator wave functions, but some superposition of the wave functions of all lowest levels. The wave functions at level n are generated by the operator $(b^+)^n$. From (35) we get

$$\Psi_n(x, y) = (b^+)^n \Psi_0(x) \Phi(y) = (a^+ + \frac{i}{2} \frac{\partial}{\partial y})^n \Psi_0(x) \Phi(y) = \sum_{i=0}^n C_n^i (\frac{i}{2} \frac{\partial}{\partial y})^{n-i} \Phi(y) \Psi_i(x) \quad (36)$$

Let us note, that all these functions holomorphically depend on the moduli and we get the whole Hilbert space H_{hol} which is the subspace of the total Hilbert space H_f . We can also get the antiholomorphic Hilbert space $H_{antihol}$. Then ground state wave functions are annihilated by operator b , but not a and from (35) we get

$$a \Psi_0(x) \Phi(\bar{y}) = -i \Psi_0(x) \partial_{\bar{y}} \Phi(\bar{y}) \quad (37)$$

Contrary to the holomorphic case we get definite $\Phi_\lambda(\bar{y})$ now, $\partial_{\bar{y}} \Phi_\lambda(\bar{y}) = \lambda \Phi_\lambda(\bar{y})$, and the oscillator wave function $\Psi_0(x)$ is now a coherent state wave function $a \Psi_0(x) = \lambda \Psi_0(x)$. All excited states in $H_{antihol}$ are

$$\Psi_n(x, \bar{y}) = (b^+)^n \Psi_0(x) \Phi(\bar{y}) = \Phi(\bar{y}) (a^+)^n \Psi_0(x) \quad (38)$$

However, if we wish to restrict moduli to the fundamental domain of the modular group, the antiholomorphic case may be incorrect, because function $\Phi_\lambda(\bar{y})$ may not have good modular properties. In the holomorphic case we may use any holomorphic function, so this problem does not arise.

Thus restriction to the positive energy sector leads to the disappearance of the half of the moduli space. In other words we can say, that in the restricted picture only half of the moduli are the coordinates, which has the analog in geometrical quantization of moduli space [15] based on the natural symplectic structure on the moduli space (really on the Teichmüller space) in terms of length-twist coordinates τ_i, θ_i [34]

$$\omega = \sum_{i=1}^{2g-3} d\theta_i \wedge d\tau_i$$

The canonical commutation relations are therefore

$$[\theta_i, \theta_j] = [\tau_i, \tau_j] = 0, \quad [\theta_i, \tau_j] = -i\delta_{ij}$$

Considering lengths τ_i as coordinates and twists $\theta_i = -id/d\tau_i$ as momenta one gets that the quantum Hilbert space can be identified with $L^2(X)$. However it is impossible to get all H_{hol} from geometrical quantization, because the coordinate $x \sim y$ arises only in TMG.

Let us stress, that not only in the critical situation as we get only (anti)holomorphic dependence on moduli. In some sense this is more than simply the three dimensional analog of the Belavin-Knizhnik theorem [35], where the cancellation of the analytical anomaly

was connected with the criticality. It will be interesting to understand if something additional appears in the three dimensional formulation in the case of the critical situation.

Finally let us briefly discuss the modular properties of our Hamiltonian. Due to our quadratic approximation we can consider only those modular transformations which are "small". Let us consider the case $g = 1$, for example. The modular group $SL(2, Z)$ acts on moduli τ as $\tau \rightarrow (a\tau + b)/(c\tau + d)$, $ad - bc = 1$. There are two transformations generating the whole group: $T: \tau \rightarrow \tau + 1$ and $S: \tau \rightarrow -1/\tau$. Let us consider $\tau = y_1 + i(1 + y_2)$, where y_i are small. Then we get for $T: y_1 \rightarrow y_1 + 1$ and for $S: y_i \rightarrow -y_i$. It is easy to see that (33) invariant under these transformations (in the S case we also change sign of the $\bar{y}_2$). However, to prove that we get quantum mechanics on moduli space with correct modular properties we need the total Hamiltonian beyond the quadratic approximation.

5. Discussion and Conclusion

Now let us briefly mention the results which have been obtained:

- 1 The quantum geodynamicals of the open topological membrane is described by topologically massive gravity where the gravitational Chern-Simons coefficient k' is proportional to central charge of the corresponding 2d conformal field theory. Some arguments were given in favour of the possibility, that the loop expansion in TMG reproduces (and maybe generalizes) the $1/c$ expansion in 2d quantum gravity.
- 2 We prove in quadratic approximation, that the Hamiltonian quantization of TMG on Riemann surfaces leads to decoupling of moduli from all other degrees of freedom and that the total Hilbert space $H_{TMG} = H_g \otimes \prod_{i=1}^{2g-3} H_f$, where H_f is the quantum mechanical Hilbert space with wave functions depending on moduli and first order time derivatives of half of the moduli.
- 3 the quantum mechanical Hamiltonian acting on the space H_f is not positive definite and gives us a system of discrete levels with gaps equal to the topological graviton mass M_g . Restriction to the positive energy subspace of Hilbert space H_f leads to the (anti)holomorphic dependence of the wave functions on moduli. This restriction may be some analog of the 2d analytical anomaly cancellation (Belavin-Knizhnik theorem).

Finally, let us mention some problems which are necessary to solve in the future. First of all, it will be amusing to find the gravitational renormalization of k' from the analysis of the TMG loop expansion and compare it with the predictions of 2d quantum gravity. Also it will be interesting to get the action of 2d quantum gravity from TMG in the same way as it is possible to derive the WZW action from TMGT. The second problem is to understand the Hamiltonian quantization of TMG (the ADM approach) beyond the quadratic approximation and obtain the total quantum mechanical Hamiltonian. Then we can study the properties of this Hamiltonian under the action of modular group, which is extremely significant for the correct description of the moduli quantum mechanics. These and related questions will be considered in future publications.

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REFERENCES

- [1] Ya.I.Kogan, Phys.Lett. **B231**, 1989, 377.
- [2] S.Carlip and Ya.I.Kogan, Phys.Rev.Lett. **64**, 1990, 148.
- [3] S.Carlip and Ya.I.Kogan, preprint UCDPHYS-PUB-31-90; CERN-TH-5919/90.
- [4] Ya.I.Kogan, preprints CERN-TH-5874/90, CERN-TH-5904/90, 1990.
- [5] W.Siegel, Nucl.Phys. **B156**, 1979, 135.
- [6] J.Shonfeld, Nucl.Phys. **B185**, 1981, 157.
- [7] S.Deser, R.Jackiw and S.Templeton, Phys.Rev.Lett. **48**, 1982, 975; Ann.Phys.(N.Y.) **140**, 1982, 372.
- [8] E.Witten, Commun.Math.Phys. **121**, 1989, 35.
- [9] G.Moore and N.Seiberg, Phys.Lett. **B220**, 1989, 422.
- [10] S.Deser, R.Jackiw and G. t'Hooft, Ann.Phys.(N.Y.) **152**, 1984, 220.
- [11] E.Witten, Nucl.Phys. **B311**, 1988, 46.
- [12] V.Moncrief, J.Math.Phys. **30**, 1989, 2907.
- [13] S.Carlip, preprint IASSNS/HEP-90/46, 1990 (to appear in Class.Quant.Gravity).
- [14] H.Leutwyler, Il Nuov.Cim., **XLII A**, 1, 1966, 1083.
- [15] E.Witten, in "Physics and Mathematics of Strings", Memorial Volume for Vadim Knizhnik, eds.L.Brink et al, World Scientific, Singapore, 1990.
- [16] S.Deser and Z.Yang, Class.Quant.Gravity **7**, 1990, 1603.
- [17] D.Friedan and S.Shenker, Nucl.Phys. **B281**, 1987, 509.
- [18] G.Moore and N.Seiberg, Commun.Math.Phys. **123**, 1989, 177.
- [19] J.J.Van der Bij, R.D.Pisarski and S.Rao, Phys.Lett. **B179**, 1986, 87.
- [20] I.Vuorio Phys.Lett. **B175**, 1986, 176.
- [21] S.Ojima, Prog.Theor.Phys. **81**, 1989, 512.
- [22] S.Coleman and B.Hill, Phys.Lett. **B159**, 1985, 184.
- [23] R.D.Pisarski and S.Rao, Phys.Rev. **D32**, 1985, 2081.
- [24] G.W.Semenoff, P.Sodano and Y.-S.Wu, Phys.Rev.Lett. **62**, 1989, 715.
- [25] L.Alvarez-Gaumé, J.M.F.Labastida and A.V.Ramallo, Nucl.Phys. **B334**, 1990, 103.
- [26] Ya.Kogan, preprint ITEP-163,1989 (to appear in Comm.Nucl.Part.Phys.).
- [27] V.G.Knizhnik, A.M.Polyakov and A.B.Zamolodchikov, Mod.Phys.Lett. **A3**, 1988, 819.
- [28] F.David, Mod.Phys.Lett. **A3**, 1988, 1651.
- [29] J.Distler and H.Kawai, Nucl.Phys. **B321**, 1989, 509.
- [30] S.Carlip, Ya.Kogan, work in progress.
- [31] Ya.I.Kogan and A.Yu.Morozov, Sov.Phys.JETP, **61**, 1985, 1.
- [32] R.Arnowitt, S.Deser and C.W.Misner, in "Gravitation: An Introduction to Current Research.", 1962, 227, ed. L.Witten, Wiley(N.Y.).
- [33] C.W.Misner, K.S.Thorne and J.A.Wheeler, *Gravitation*, W.H.Freeman and Company, San-Francisco, 1975.
- [34] S.Wolpert, Ann.Math. **117**, 1983, 207.
- [35] A.A.Belavin and V.G.Knizhnik, Phys.Lett. **B168**, 1986, 201; JETP **91**, 1986, 247.

FIGURE 1

FIGURE 2