BETHE-SALPETER APPROACH TO THE TOP MODE STANDARD MODEL

by

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Abstract

We discuss a Bethe-Salpeter (BS) approach to the minimal dynamical symmetry breaking model where the electroweak symmetry is broken by a condensate of top quarks. We solve the Schwinger-Dyson (SD) gap equation for the top quark self energy function in the approximation of keeping only leading log QCD corrections. We then insert this solution into the BS amplitude for $t\bar{t} \rightarrow t\bar{t}$ scattering, calculated upto leading log QCD corrections and make a numerical search for poles in the pseudoscalar and scalar channels. Predictions for the Higgs boson mass in this approximation are given and compared with the results from renormalization group (RG) analysis. The two sets of results are found to be in reasonable agreement. A four family model is considered and results from both RG and BS analysis are compared. Our results show good agreement between the two approaches even in the low cut-off region $\Lambda = 1 - 10$ TeV where the RG predictions are least reliable.

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1 Introduction

Recent results have shown the top quark to be very massive, ($m_t > 90 \text{ GeV}$), much heavier than the other five quarks. As this bound increases so speculation grows that the $SU(2) \times U(1)$ symmetry breakdown may be a dynamical mechanism intimately involving the top quark [1]. A number of papers considering this possibility have recently been published all of which have the same basic starting point, the introduction of a new fundamental interaction associated with some high energy scale Λ . This interaction brings about the formation of $\bar{t}t$ condensates and thus triggers chiral symmetry breakdown. In these models there is no fundamental Higgs field, the Higgs boson is a composite particle, a scalar bound state of top quarks. The usual C-K-M structure and fermion mass spectrum can be readily accommodated, but predictions of fermion masses can not be made until the dynamics at the scale Λ is defined more precisely.

The idea is that well below the cut-off Λ , the effective low energy theory resembles the minimal standard model with a single complex scalar Higgs doublet Φ . The scalar and pseudoscalar components of Φ are constructed as bound states of top and bottom quarks. For example the Higgs boson H is a scalar $\bar{t}t$ bound state. Mass predictions for m_H and m_t as a function of Λ may be obtained using the usual renormalization group (RG) equations of the standard model, supplemented by the boundary conditions that the running top quark Yukawa coupling $g_t(\mu)$ and the running scalar self-coupling $\lambda_t(\mu)$ diverge as $\mu \rightarrow \Lambda$ [2]. These RG predictions include all radiative corrections such as Higgs exchange, gluon exchange and so on. To what extent are these predictions reliable? They may be criticised on two grounds. Firstly Suzuki [3] has shown that the presence of higher dimensional operators containing derivatives can alter the predictions dramatically, providing the coefficients of the additional operators are sufficiently large. Secondly the RG analysis is based on using effective standard model Higgs doublet all the way up to the cut-off Λ , whereas we know such a description gives a completely inadequate description of physics close to Λ . In the minimal model, where the top quark is used to break the symmetry, the cut-off is necessarily very high otherwise the top quark mass is too large. For example, $\Lambda = 10^{15} \text{ GeV}$ leads to $m_t = 230 \text{ GeV}$ [2]. However due to a hierarchy problem it is desirable to reduce this cut-off to a few TeV. This can be done by introducing a fourth family of quarks with a naturally heavy fourth neutrino [4]. In these models we have no reason to believe that RG results are at all reliable. The common feature of both the above criticisms is that the perturbative one loop RG analysis of BHL [2] ignores all non-linearities. By contrast the Schwinger-Dyson (SD) gap equation including QCD corrections in ladder approximation, which enables the top quark self-energy function

$\Sigma_t(p)$ to be determined directly, does contain some non-linear features of the dynamics and certainly provides a much more trustworthy description of physics close to the cut-off Λ . This approach has been pursued in ref [5].

In a recent paper [6] we proposed a Bethe-Salpeter (BS) approach for the bound states corresponding to the various components of the effective Higgs doublet, including gluonic corrections to the sums over bubble graphs. We were able to show analytically that the charged and neutral pseudoscalar components are massless, but were unable to estimate accurately the Higgs boson mass. In this paper we combine the SD and BS approaches to give predictions for the top mass and Higgs mass in the approximation of keeping leading log QCD corrections only. We perform the angular integrations analytically and make a numerical search for the positions of the scalar and pseudoscalar poles. These results are compared with results obtained from the RG equations at the same level of approximation. We also study a degenerate four family model at very low cut-offs of just a few TeV and again compare results from our SD/BS analysis to the RG predictions.

In Section 2 we briefly review the RG approach [2] and the SD gap equation analysis, first in the approximation where only fermion bubbles are kept, then with gluon exchange corrections added [5]. In Section 3 we consider the bound states of the model in our BS approach [6]. Iterative equations for $\Gamma_p(p, q)$ and $\Gamma_s(p, q)$, the propagators of the pseudoscalar and scalar $\bar{t}t$ bound states are considered and approximate analytic arguments are given which predict a pole at $p^2 = 0$ and $p^2 = 4m_t^2$ respectively. The massless pole in the pseudoscalar state does not depend on the approximate arguments given, but, the pole in the scalar propagator $\Gamma_s(p, q)$ is expected to shift when the equation is solved exactly without making any assumptions. The rest of the paper is dedicated to the numerical solutions of the equations for $\Gamma_s(p, q)$ and $\Gamma_p(p, q)$. Angular integration of the equations is carried out in Section 4 and the results of numerical solutions given in Section 5. The pole at $p^2 = 0$ for the pseudoscalar particle is shown to be correctly predicted and the pole in the scalar propagator shown to be shifted down somewhat. Values of m_t and m_H from this analysis and the RG analysis are compared and found to be in reasonable agreement. Results are also given for a degenerate fourth family model from the two approaches.

2 Renormalization Group vs. Gap Equation

In Section 1 we briefly discussed the possibility that the electroweak symmetry is broken by a condensate of top quarks, $\langle \bar{t}t \rangle \neq 0$, and mentioned that two separate approaches can be adopted to obtain predictions from these models. In this section we give a brief

review of the basic ideas of these models and look at these two approaches.

All models which use the idea of the top quark condensate have the same basic starting point, the usual standard model Lagrangian with all the quark, lepton and gauge boson degrees of freedom, but with no fundamental scalar field. A four fermi operator of the form

$$G (\bar{\Psi}_L^c t_R^c) (\bar{t}_R^c \Psi_R^c) \quad (2.1)$$

is introduced, where $\Psi_L = (t_L, b_L)$, c, c' are colour indices and $G = g^2/\Lambda^2$, Λ being the mass scale of new physics which is responsible for the operator and g is a dimensionless coupling constant. If the dimensionless coupling exceeds a critical value $g^2 > 8\pi^2/3$ this operator leads to a non-zero solution of the top quark self energy function and thus triggers chiral symmetry breaking. Accompanying this breakdown are three massless Goldstone boson states. These can be shown to be composite pseudoscalar $\bar{t}\gamma_5 t$, $\bar{b}\gamma_5 t$ and $\bar{t}\gamma_5 b$ bound states. In the presence of electroweak interactions the pseudoscalar states are eaten by the Z , W^+ and W^- bosons to become their longitudinal components and thus the electroweak symmetry is broken. In these models there is also a scalar $\bar{t}t$ bound state which corresponds to the Higgs boson H in the standard model.

In the RG approach the Lagrangian is written in terms of an effective composite Higgs doublet Φ ,

$$\mathcal{L} = \mathcal{L}_{kinetic} + (g_t \bar{\Psi}_L t_R \Phi + \text{h.c.}) + |D_\mu \Phi|^2 - m_H^2 \Phi^\dagger \Phi - \frac{\lambda_t}{2} (\Phi^\dagger \Phi)^2 + \dots$$

The Higgs boson mass, m_H , and top quark mass, m_t , are determined by running the couplings from high values at the scale Λ , down to the mass-shell conditions

$$m_t^2 = v^2 g_t^2(m_t) \quad (2.2)$$

and

$$m_H^2 = 2v^2 \lambda_t(m_H) \quad (2.3)$$

where $v^2 = 1/2\sqrt{2}G_F$. G_F is the Fermi constant.

It can be shown that the boundary conditions for the running coupling constants are [2]

$$\begin{aligned} g_t(\mu) &\xrightarrow{\mu \rightarrow \Lambda} \infty \\ \lambda_t(\mu) &\xrightarrow{\mu \rightarrow \Lambda} \infty. \end{aligned} \quad (2.4)$$

The physical masses of the Higgs boson and the top quark can be determined by solving the RG equations for $g_t(\mu)$ and $\lambda_t(\mu)$ in whichever approximation we wish to work. The

simplest approximation where only fermion bubbles are kept gives the RG equations

$$\begin{aligned} \frac{dg_t}{dt} &= \frac{1}{16\pi^2} N_c g_t^3 \\ \frac{d\lambda_t}{dt} &= \frac{1}{16\pi^2} (4N_c g_t^2 \lambda - 4N_c g_t^4) \end{aligned} \quad (2.5)$$

Solving these equations (2.5) and applying the above boundary conditions (2.4) one gets

$$\frac{1}{g_t^2(\mu)} = \frac{N_c}{16\pi^2} \ln \left(\frac{\Lambda^2}{\mu^2} \right)$$

and the usual bare result $m_H = 2m_t$ [1,2].

In the next level of approximation one introduces planar QCD corrections in the RG equation. The equation for λ_t remains unchanged but the equation for g_t becomes

$$\frac{dg_t}{dt} = \frac{1}{16\pi^2} (N_c g_t^3 - (N_c^2 - 1) g_s^2 g_t) \quad (2.6)$$

This shifts the masses somewhat from their bare values. However, the largest corrections occur when Higgs boson exchange corrections are included [2]. This is due to the strong coupling between the Higgs and the top because of the large top mass. The Higgs mass is shifted a long way below its bare value of $2m_t$ to one with large binding energy [2]. The Higgs particle is therefore a strongly bound scalar $\bar{t}t$ state in this picture.

The alternative approach to that of the RG is one which utilises the gap equation. Again different levels of approximation can be considered. Although there is little comparison between the parameters of the two approaches, results from equivalent levels of approximation can be compared. The simplest level of approximation, that of keeping only fermion bubbles is considered in ref [2]. In this approximation the gap equation (see Fig. 1) is [2]

$$m_t = 6Gm_t \frac{i}{(2\pi)^4} \int \frac{d^4 l}{(l^2 - m_t^2)} \quad (2.7)$$

which can be integrated to give [2]

$$\frac{1}{G^2} = \frac{3}{8\pi^2} \left(\Lambda^2 - m_t^2 \ln \left(\frac{\Lambda^2}{m_t^2} \right) \right)$$

This equation has a non trivial solution providing $G \geq 8\pi^2/3\Lambda^2$ or defining $\lambda = 3\Lambda^2 G/8\pi^2$, providing $\lambda \geq 1$. We unfortunately see an obvious hierarchy problem here for large values of Λ . For example if the scale Λ is set to be $O(10^{15} \text{ GeV})$ in order to obtain a top mass $O(10^2 \text{ GeV})$ as required, λ must be tuned to a very high level of accuracy, in this case to twenty-six decimal places. However, this problem is limited to the gap equation. That is

once λ is tuned no other fine tuning is necessary. In this approximation the generators of the dynamically generated bound states are given by a sum of bubbles generated by the four fermi contact force shown in Fig. 2. By implementing the gap equation it can be shown that the pseudoscalar states are massless Goldstone modes while the scalar $\bar{t}t$ bound state has a pole at $p = 2m_t$. This means we again have the N-J-L result of a composite scalar particle with vanishing binding energy. However this prediction can not be taken seriously since no radiative corrections have been taken into account. As discussed in the next section.

In the next level of approximation QCD corrections should be added to the gap equation. The gap equation will now have two terms, the tadpole term of Eq (2.7) plus a new term involving gluon exchange (Fig. 3),

$$\Sigma_t(q) = 6G \frac{i}{(2\pi)^4} \int \frac{d^4 l \Sigma_t(l)}{(l^2 - \Sigma_t^2(l))} - 4g_s^2 \frac{i}{(2\pi)^4} \int \frac{d^4 l \Sigma_t(l)}{(l^2 - \Sigma_t^2(l)) (l - q)^2} \quad (2.8)$$

where g_s is the QCD coupling constant[†].

This is the equation for the momentum dependent top quark self energy. The top quark mass is identified as $m_t = \Sigma_t(m_t) \approx \Sigma_t(0)$. This equation was solved numerically by King and Mannan [5]. As mentioned above, in these pictures the Goldstone bosons which are eaten by the Z , W^+ and W^- gauge bosons to become their longitudinal components are pseudoscalar $\bar{t}\gamma_5 t$, $b\gamma_5 t$ and $\bar{t}\gamma_5 b$ bound states. The Z , W^+ and W^- masses may therefore be expressed in terms of the decay constants of these pseudoscalar particles which in turn are determined from the top quark self energy function, $\Sigma_t(q)$ [5]. There is also a scalar $\bar{t}t$ bound state. In order to obtain predictions for the masses of these bound states in our approximation we must add QCD corrections to the bound state propagators, Fig. 3. The equations one gets were first discussed by us in [6], however in that paper no attempt was made to solve these equations numerically. This is the problem to which we now turn our attention.

3 Generators of Dynamical Bound States

In this Section we consider the question of dynamically generated top quark bound states using the gap equation approach. As stated in the last Section in the fermionic determinant approximation the propagator of the dynamically generated bound state is given by a sum

[†]Note the misprint in the sign of the QCD term in ref [6].

of bubbles formed by the contact interaction. Leading log QCD corrections can be taken into account by considering an iterative equation (Fig. 4) for the propagator, Γ [7].

Let us first consider the pseudoscalar $\bar{t}\gamma_5 t$ bound state. In Landau gauge the diagrammatic representation Fig. 4 gives the following integral equation for $\Gamma_p(p, q)$

$$\begin{aligned} \Gamma_p(p, q) = & -\frac{G}{2} - \frac{3G}{2} \frac{i}{(2\pi)^4} \int d^4 l \frac{T\tau[(l + \Sigma_t(l))(\bar{t}\gamma_5)(l - p + \Sigma_t(l - p))(\bar{t}\gamma_5)]\Gamma_p(p, l)}{(l^2 - \Sigma_t^2(l))[(l - p)^2 - \Sigma_t^2(l - p)]} \\ & + g_s^2 \frac{i}{(2\pi)^4} \int d^4 l \frac{T\tau[(l + \Sigma_t(l))(\bar{t}\gamma_5)(l - p + \Sigma_t(l - p))(\bar{t}\gamma_5)]\Gamma_p(p, l)}{(l^2 - \Sigma_t^2(l))[(l - p)^2 - \Sigma_t^2(l - p)](l - q)^2} \end{aligned} \quad (3.1)$$

Performing the traces gives

$$\begin{aligned} \Gamma_p(p, q) = & -\frac{G}{2} - 6G \frac{i}{(2\pi)^4} \int d^4 l \frac{[l(l - p) - \Sigma_t(l)\Sigma_t(l - p)]\Gamma_p(p, l)}{(l^2 - \Sigma_t^2(l))[(l - p)^2 - \Sigma_t^2(l - p)]} \\ & + 4g_s^2 \frac{i}{(2\pi)^4} \int d^4 l \frac{[l(l - p) - \Sigma_t(l)\Sigma_t(l - p)]\Gamma_p(p, l)}{(l^2 - \Sigma_t^2(l))[(l - p)^2 - \Sigma_t^2(l - p)](l - q)^2} \end{aligned} \quad (3.2)$$

where $\Sigma_t(l)$ is the solution of the gap equation (2.8).

Comparing the equation for $\Sigma_t(l)$ with that for $\Gamma_p(p, q)$ we can see that the two are formally very similar, indeed as $p \rightarrow 0$, $\Gamma_p(p, q)$ is given by [6]

$$\begin{aligned} \Gamma_p(p \rightarrow 0, q) = & -\frac{G}{2} + 6G \frac{i}{(2\pi)^4} \int d^4 l \left(\frac{1}{(l^2 - \Sigma_t^2(l))} + O(p^2) \right) \Gamma_p(p, l) \\ & - 4g_s^2 \frac{i}{(2\pi)^4} \int d^4 l \left(\frac{1}{(l - q)^2(l^2 - \Sigma_t^2(l))} + O(p^2) \right) \Gamma_p(p, l) \end{aligned} \quad (3.3)$$

Comparison with the gap equation (2.8) shows that it is satisfied by a solution of the form $\Gamma_p(p, q) = A\Sigma_t(q)/p^2$ as p^2 approaches zero corresponding to the expected pole in $\Gamma_p(p, q)$ at $p^2 = 0$.

The pole in $\Gamma_p(p, q)$ at $p^2 = 0$ can also be made apparent by recourse to the following approximate argument. The solution for $\Sigma_t(q)$ is very flat indeed : for a cut-off of 10^{15} GeV $\Sigma_t(0) = 268$ GeV, while $\Sigma_t(10^{15}) = 112$ GeV. Let us therefore approximate $\Sigma_t(q)$ to be flat, independent of q . Furthermore because of the similarity between the equation for $\Sigma_t(q)$ and that for $\Gamma_p(p, q)$ we assume that $\Gamma_p(p, q)$ is also independent of q . In this approximation the gap equation (2.8) reduces to

$$1 \approx 6G \frac{i}{(2\pi)^4} \int d^4 l \frac{1}{(l^2 - \Sigma_t^2)} - 4g_s^2 \frac{i}{(2\pi)^4} \int d^4 l \frac{1}{(l^2 - \Sigma_t^2)(l - q)^2} \quad (3.4)$$

and the equation for $\Gamma_p(p, q)$, Eq (3.2), becomes

$$\Gamma_p(p) \approx -\frac{G/2}{\mathcal{D}_p(p)}$$

where

$$\mathcal{D}_p(p) = 1 - 6G \frac{i}{(2\pi)^4} \int d^4l \frac{l(l-p) - \Sigma_t^2}{(p^2 - \Sigma_t^2)[(l-p)^2 - \Sigma_t^2]} + 4g_s^2 \frac{i}{(2\pi)^4} \int d^4l \frac{l(l-p) - \Sigma_t^2}{(l-q)^2[(l-p)^2 - \Sigma_t^2](l-q)^2} \quad (3.5)$$

The equation for $\mathcal{D}_p(p)$ can be rearranged to give

$$\begin{aligned} \mathcal{D}_p(p) = 1 & - 6G \frac{i}{(2\pi)^4} \int \frac{d^4l}{(l^2 - \Sigma_t^2)} + 4g_s^2 \frac{i}{(2\pi)^4} \int \frac{d^4l}{(l^2 - \Sigma_t^2)(l-q)^2} \\ & + 3G \frac{i}{(2\pi)^4} p^2 \int \frac{d^4l}{(l^2 - \Sigma_t^2)[(l-p)^2 - \Sigma_t^2]} \\ & - 2g_s^2 \frac{i}{(2\pi)^4} p^2 \int \frac{d^4l}{(l^2 - \Sigma_t^2)[(l-p)^2 - \Sigma_t^2](l-q)^2} \end{aligned} \quad (3.6)$$

Thus upon implementing the gap equation (3.4) the required pole at $p^2 = 0$ is once more exposed.

Now let us turn our attention to the scalar $\bar{t}t$ dynamically generated bound state. The propagator for this state is given by an equation similar to (3.1) but with the $i\gamma_5$ factors replaced by the identity. Consequently the equivalent of Eq (3.2) is

$$\begin{aligned} \Gamma_s(p, q) = & -\frac{G}{2} - 6G \frac{i}{(2\pi)^4} \int d^4l \frac{[l(l-p) + \Sigma_t(l)\Sigma_t(l-p)]\Gamma_s(p, l)}{(l^2 - \Sigma_t^2)(l-p)^2 - \Sigma_t^2(l-p)} \\ & + 4g_s^2 \frac{i}{(2\pi)^4} \int d^4l \frac{[l(l-p) + \Sigma_t(l)\Sigma_t(l-p)]\Gamma_s(p, l)}{(l^2 - \Sigma_t^2)(l-p)^2 - \Sigma_t^2(l-p)(l-q)^2} \end{aligned} \quad (3.7)$$

In order to give an indication of the position of the pole in $\Gamma_s(p, q)$ let us use the same approximation as before, namely that $\Sigma_t(q)$ and $\Gamma_s(p, q)$ are both constant with respect to q . We get the equation

$$\Gamma_s(p) \approx -\frac{G/2}{\mathcal{D}_s(p)}$$

where

$$\begin{aligned} \mathcal{D}_s(p) = & 1 - 6G \frac{i}{(2\pi)^4} \int d^4l \frac{l(l-p) + \Sigma_t^2}{(l^2 - \Sigma_t^2)[(l-p)^2 - \Sigma_t^2]} \\ & + 4g_s^2 \frac{i}{(2\pi)^4} \int d^4l \frac{l(l-p) + \Sigma_t^2}{(l-q)^2[(l-p)^2 - \Sigma_t^2](l-q)^2} \end{aligned} \quad (3.8)$$

When the equation for $\mathcal{D}_s(p)$ is rearranged we get an equation similar to Eq (3.6), but with the factors of p^2 replaced by $p^2 - 4m_t^2$. So this time after enforcing the gap equation a pole in $\Gamma_s(p, q)$ at $p^2 = 4m_t^2$ emerges. So in this approximation, assuming $\Gamma_s(p, q)$ and $\Sigma_t(q)$ are flat in q , a scalar bound state with vanishing binding energy is predicted.

The position of the pole in the pseudoscalar channel at $p = 0$ is exact even without the assumptions of flat Σ_t and Γ_p and constant QCD coupling, but the pole in the scalar propagator at $p^2 = 4m_t^2$ is only true if Σ_t and Γ_s are assumed flat. In order to find the true position of the pole in Γ_s , Eq (3.7) must be solved numerically allowing the QCD coupling constant to run. The position of the pole in Γ_s does not give a correct physical mass of the scalar particle since it takes no account of Higgs exchange corrections which turn out to be the dominant corrections in these models. However results obtained from a solution of Eq (3.7) can be compared with results form the same level of approximation in RG analysis. An assessment can then be made as to the reliability of RG predictions. The solution of this equation forms the subject of the rest of this paper.

4 Angular Integration of Bound State Generators

In the last section we found that by making certain assumptions about the behaviour of $\Sigma_t(q)$, $\Gamma_p(p, q)$ and $\Gamma_s(p, q)$ we could exhibit poles in Γ_p and Γ_s at $p = 0$ and $p^2 = 4m_t^2$ respectively. The position of the pole at $p = 0$ in the pseudoscalar channel is exact even when one does not make the assumption that $\Gamma_p(p, q)$ is independent of q . This argument is also valid for a running QCD coupling rather than the constant coupling assumed in the last section. However, for the scalar channel the position of the pole is expected to be altered when the equation for $\Gamma_s(p, q)$, Eq (3.7) is solved exactly without making these assumptions. Thus, in order to obtain a reliable value of the scalar mass at this level of approximation we follow the procedure detailed below.

We work in the dressed ladder approximation in which the QCD coupling is allowed to run. The equations for $\Gamma_s(p, q)$ and $\Sigma_t(q)$ must be Wick rotated to Euclidean space and the angular part of the integration performed. An ultraviolet cut-off, Λ , together with an ansatz for the behaviour of the running QCD coupling must be introduced. The equation for $\Sigma_t(q)$ can now be solved numerically and the solution may then be used as input to solve the equation for $\Gamma_s(p, q)$. A precise position of the scalar pole can thereby be determined. In order to give us confidence in this strategy we will follow it through for the pseudoscalar as well as the scalar propagator.

Let us start with the gap equation since the solution for $\Sigma_t(q)$ is needed as input in the equation for $\Gamma_s(p, q)$. In Landau gauge in Euclidean space, after performing the angular integrations and introducing an ultra-violet cut-off we have

$$\Sigma_t(q) = \frac{\lambda}{\Lambda^2} \int_0^{\Lambda^2} \frac{\Sigma_t(l)^2 dl^2}{(l^2 + \Sigma_t^2(l))} + \frac{1}{\pi} \int_0^{\Lambda^2} \frac{\alpha_s(\max(q^2, l^2))}{\max(q^2, l^2)} \frac{\Sigma_t(l)^2 dl^2}{(l^2 + \Sigma_t^2(l))} \quad (4.1)$$

This is solved numerically using the following ansatz for the running of the QCD coupling. Above the scale $\mu_x = 2\Sigma_t(0)$ the QCD coupling runs with six quark flavours, whereas below μ_x , when the top quarks have condensed, it runs with only five flavours. Confinement is approximated by introducing a confinement scale Λ_{QCD} below which it takes on a constant value α_0 . Refining this ansatz by introducing other heavy quark thresholds only changed the value of $\Sigma_t(0)$ by 0.1 GeV and so this same ansatz will be used in all that follows.

Transforming the equations for $\Gamma_p(p, q)$ and $\Gamma_s(p, q)$ into Euclidean space and writing $G = 8\pi^2/3\Lambda^2$, gives us

$$\begin{aligned} \Gamma_{p(s)}(p, q) = & -\frac{G}{2} + \frac{\lambda}{\pi^2\Lambda^2} \int \frac{d^4 l [(l-p) \pm \Sigma_t(l)\Sigma_t(l-p)]}{(l^2 + \Sigma_t^2(l))(l-p)^2 + \Sigma_t^2(l-p)} \Gamma_{p(s)}(p, l) \\ & - \frac{1}{\pi^2} \int \frac{d^4 l \alpha_s [(l-p) \pm \Sigma_t(l)\Sigma_t(l-p)]}{(l^2 + \Sigma_t^2(l))(l-p)^2 + \Sigma_t^2(l-p)} \Gamma_{p(s)}(p, l) \end{aligned} \quad (4.2)$$

To perform the angular integration we make the small p approximation that $\Sigma_t(l) \approx \Sigma_t(l-p)$.

After carrying out the angular integration (see Appendix) we find that

$$\begin{aligned} \Gamma_{p(s)}(p, q) = & -\frac{G}{2} + \frac{\lambda}{\Lambda^2} \int_0^{\Lambda^2} \frac{d^2 l^2 \Gamma_{p(s)}(p, l)}{(l^2 + \Sigma_t^2(l))} \left(\frac{1}{2} + \frac{A_p(l)}{U} \right) \\ & + \frac{1}{2\pi} \int_0^{\Lambda^2} \frac{d^2 l^2 \alpha_s (\max(l^2, q^2)) \Gamma_{p(s)}}{(l^2 - \Sigma_t^2(l))} \left(\frac{A_p(l)}{pq} \tan^{-1} \left(\frac{2pq l^2}{\max(l^2, q^2)U} \right) + \frac{l^2}{\max(l^2, q^2)} \right) \end{aligned} \quad (4.3)$$

where

$$\begin{aligned} U = & l^2 + p^2 + \Sigma_t^2(l) + \sqrt{(l^2 + p^2 + \Sigma_t^2(l))^2 - 4l^2 p^2} \\ A_p = & l^2 - p^2 + \Sigma_t^2(l) \\ A_s = & l^2 - p^2 - 3\Sigma_t^2(l) \end{aligned}$$

Eq.(4.3) is now be solved numerically, results are presented in the next section.

5 Results

The numerical solutions for $\Gamma(p, q)$ depends upon using a very precise solution of $\Sigma_t(q)$ as input. The reason for this can be seen if we reconsider the approximate arguments given in Section 3. Consider Eqns (3.8) and (3.6) the equations for $\mathcal{D}_s(p)$ and $\mathcal{D}_p(p)$ the denominators of $\Gamma_s(p, q)$ and $\Gamma_p(p, q)$ respectively. Implementing the gap equation 3.4 cancels the quadratically divergent terms in $\mathcal{D}(p)$ and the poles in $\Gamma(p, q)$ appear. The appearance of these poles is therefore dependent upon the exact cancellation of the quadratically divergent terms in $\mathcal{D}(p)$ due to the gap equation. When no assumption is made about the

behaviour of $\Gamma(p, q)$ we can not explicitly make this cancellation before looking for a numerical solution. So this cancellation must occur implicitly during numerical calculation of $\Gamma(p, q)$ and therefore depends on a highly exact numerical solution for $\Sigma_t(q)$ as input.

Unfortunately, this makes it impossible to solve Eq (4.3) numerically for cut-offs as high as 10^{15} GeV. The hierarchy amplifies the rounding errors making numerical calculations unstable (In essence this amounts to trying to do the gauge hierarchy problem on a computer). Because of this the equation has been solved for values of the cut-off $\Lambda \leq 10^7$ GeV only. Fortunately this is the interesting range since RG predictions are least trustworthy for low cut-offs.

Fig. 5 shows numerical solutions for the pseudoscalar propagator, $\Gamma_p(p, q)$, with a cut-off $\Lambda = 10^4$ GeV for $q = 10^{-3}$ GeV and $q = 10^4$ GeV. It demonstrates the expected behaviour $\Gamma_p(p, q) \propto 1/p$ and gives us confidence in our procedure. In Euclidean space we expect the scalar propagator to have a pole for some $p^2 = -x^2$, the mass of the scalar particle is then given by $m_H = x$. Around the pole $\Gamma_s(p, q)$ should look like $1/(p^2 + m_H^2)$. Fig. 6 shows the value of $\Gamma_s(p, 10^{-3})$ and $\Gamma_s(p, 10^4)$ as Minkowski p varies, again with a cut-off $\Lambda = 10^4$ GeV. It reveals a pole at $p = 1062$ GeV the position of which is not affected by the value of q . The expected $1/(p^2 - m_H^2)$ behaviour for Minkowski p^2 near $m_H^2 = (1062 \text{ GeV})^2$ is demonstrated by Fig. 7.

Table 1 displays results for the top quark mass m_t and the Higgs boson mass m_H as a function of the ultra-violet cut-off Λ . These results are not physically relevant since such low values of Λ leads to values of m_t higher than experimentally permissible. Nevertheless the results may be usefully compared to the RG results in the low Λ region. For comparison we also calculate the quantities $m_t(\text{bare})$ and $m_H(\text{bare})$ which correspond to pure Nambu-Jona-Lassino predictions corresponding to the calculations in the first part of ref [2]. We also calculate the quantities $m_t(\text{full})$ which and $m_H(\text{full})$ correspond to the fully corrected predictions, including gluon exchange, electroweak exchange and propagating Higgs exchange using the full RG equations discussed in the second part of ref [2]. For our purposes the most interesting quantities are those denoted by $m_t^{GAP}(\text{QCD})$ and $m_H^{GAP}(\text{QCD})$ which are the main results of this paper. These quantities have been obtained from the Schwinger-Dyson Bethe-Salpeter equations including the leading log QCD corrections, but neglecting electroweak exchange and propagating Higgs exchange. The details of how $m_t^{GAP}(\text{QCD})$ is obtained has been discussed elsewhere [5]. The value of λ is adjusted so that the solution leads to the correct value of the pion decay constant $F_\pi = 246.1$ GeV, using the Pagels-Stoker formula [5]. The results for $m_t^{GAP}(\text{QCD})$ and $m_H^{GAP}(\text{QCD})$ may be compared to $m_t^{RG}(\text{QCD})$ and $m_H^{RG}(\text{QCD})$. In order to calculate $m_t^{RG}(\text{QCD})$ and

$m_H^{RG}(\text{QCD})$ we have integrated the RG equations (2.6). This includes only QCD corrections but drops electroweak and Higgs exchange corrections. We have used the same QCD coupling ansatz here as used in solving Eq.(4.1) ($\alpha_{QCD}(m_Z) = 0.138$).

Table 1 shows that $m_t^{GAP}(\text{QCD})$ and $m_t^{RG}(\text{QCD})$ differ by 1 GeV at $\Lambda = 10^7$ GeV and 17 GeV at $\Lambda = 10^4$ GeV showing the good agreement first observed by Barrios and Mahanta [7]. Our results show a less spectacular agreement between $m_H^{GAP}(\text{QCD})$ and $m_H^{RG}(\text{QCD})$ however, but still the two values are within the errors quoted in ref [2]. Notice that our gap equation approach leads to an approximate relation

$$\frac{m_H^{GAP}(\text{QCD})}{2m_t^{GAP}(\text{QCD})} \approx 0.9$$

which is approximately independent of Λ for low values of Λ . By contrast

$$\frac{m_H^{RG}(\text{QCD})}{2m_t^{RG}(\text{QCD})} \approx 1.0 - 0.8$$

for $\Lambda = 10^4$ GeV – 10^7 GeV, suggesting that the Higgs boson becomes increasingly tightly bound due to gluon exchange as Λ is increased.

We now consider a fourth family of quarks (h,l) [†] so that the four fermi interaction introduced to trigger symmetry breakdown becomes

$$G_h \left(\bar{h}_L \bar{l}_L \right) h_R \bar{h}_R \left(\begin{smallmatrix} h_L \\ l_L \end{smallmatrix} \right) + G_l \left(\bar{h}_L \bar{l}_L \right) l_R \bar{l}_R \left(\begin{smallmatrix} h_L \\ l_L \end{smallmatrix} \right) \\ + G_M \left(\bar{h}_L h_R \bar{l}_L l_R - \bar{l}_L h_R \bar{h}_L l_R + \text{h.c.} \right)$$

Assuming the high and low quarks are degenerate, that is $m_h = m_l = m_q$ we get $G_h = G_l = G_M$ and the gap equation (2.8) is modified by addition of an extra quark loop. Eq (2.8) thus becomes

$$\Sigma_4(q) = 2 \times 6G \frac{i}{(2\pi)^4} \int \frac{d^4 l \Sigma_4(l)}{(l^2 - \Sigma_4^2(l))} - 4g_s^2 \frac{i}{(2\pi)^4} \int \frac{d^4 l \Sigma_4(l)}{(l^2 - \Sigma_4^2(l)) (l - q)^2} \quad (5.1)$$

We now have three equations for the bound state propagators arising from the three separate scattering processes $hh \rightarrow hh$, $hh \rightarrow ll$ and $ll \rightarrow ll$. However the degeneracy of h and l means these three processes are equivalent. The equations for the propagators are thus similar to (3.2) and (3.7) with factors of two in front of the second terms again due to extra quark loops.

[†]In this fourth family the neutrino is assumed to be heavy, a fourth family with a light neutrino having been excluded at LEP [9]

Table 2 gives values of m_4 and m_H for a degenerate fourth quark family. RG values of m_4 and m_H in both the QCD approximation and full analysis have been given for comparison. Good agreement is again found between the two approaches. The values of m_4^{GAP} and m_4^{RG} in the QCD approximation are within 8% of each other for all cut-offs while the corresponding values of m_H differ by less than 14%. Interestingly the Higgs exchange corrections do not appear as important in this analysis, as shown by the fact that the values of $m_4^{RG}(\text{QCD})$ and $m_4^{RG}(\text{full})$ are much closer than before.

Table 3 shows values of m_4 and m_H for cut-offs of only a few TeV. For such low cut-offs we have no reason to believe the results of RG analysis. Indeed for a cut-off of $\Lambda = 2$ TeV the quark mass is close to the cut-off and RG equations have no solution since m_H wants to exceed the cut-off in this case. However for the other values of the cut-off the results of the two approaches are in good agreement, being within 14% of each other for all Λ . Interestingly the predicted values of m_H from both approaches are in better agreement than the corresponding values of the quark mass. Again the effect of Higgs exchange does not appear as important, as seen for example by comparing the RG predictions for $m_4(\text{QCD})$ and $m_4(\text{full})$ which only differ by 4 GeV. The RG and GAP predictions for $m_H(\text{QCD})$ agree surprisingly well for all cut-offs and are consistently 25% higher than $m_H(\text{full})$. The RG and GAP predictions for m_4 , however, diverge from each other more and more as the cut-off is reduced.

6 Conclusion

We have developed a BS approach to the top mode standard model, which may be useful for a number of reasons. To begin with, the approach lends support to the RG approach since the predictions of the two approaches show reasonable agreement. This is perhaps a little surprising since we would expect that close to the cut-off the RG approach should break down, since the coupling constants become infinite at the cut-off, and perturbation theory should therefore become unreliable. However we have shown that at the level of approximation where leading log QCD corrections are included, but Higgs and electroweak exchange is ignored, the SD/BS approach gives results for m_t and m_H in good agreement with those of the RG approach. This agreement is also maintained in the fourth family variant of the model, even when the cut-off is pushed to very low values, $\Lambda = 1 - 10$ TeV, however the quark mass predictions disagree as the cut-off is reduced, and for the lowest cut-offs the RG approach breaks down completely.

Another application our BS approach might be for dealing with higher dimensional

operators which may play an important role in these models [3]. In a region close to the cut-off such operators may play an important role and significantly affect the boundary conditions used in the RG approach. Although these operators are not felt in the RG equations, their presence would directly modify our BS calculations. The BS approach is therefore the most natural framework for dealing with such effects.

Although the agreement between the RG and BS approaches is encouraging it does nothing to address certain basic questions such as whether any of these results survive beyond the ladder or $1/N$ approximations. However recent work on the application of the ladder BS equation to meson properties in QCD [11] shows that the ladder approximation works surprisingly well, where it can be checked against data. This provides some support for using these techniques in theories beyond the standard model, such as the BS approach to the top mode standard model discussed here.

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Appendix

Let us write Eq. (4.2) [†] as

$$\Gamma_p(p, q) = -\frac{G}{2} + \frac{\lambda}{\pi^2 \Lambda^2} I_p - \alpha_s \frac{1}{\pi^3} J_p \quad (\text{A.1})$$

where I_p can be written as

$$I_p = \int_0^{\Lambda^2} dl^2 \left(\frac{l^2}{2(l^2 + \Sigma_l^2(l))} \int d\Omega_l \frac{(l^2 - lp \cos \theta + \Sigma_l^2(l))}{(l^2 + p^2 + \Sigma_l^2(l) - 2lp \cos \theta)} \right) \quad (\text{A.2})$$

and J_p is given by

$$J_p = - \int_0^{\Lambda^2} dl^2 \left(\frac{l^2}{2(l^2 + \Sigma_l^2(l))} \int d\Omega_l \frac{(l^2 - lp \cos \theta_p + \Sigma_l^2(l))}{(l^2 + p^2 + \Sigma_l^2(l) - 2lp \cos \theta_p)(l^2 + q^2 - 2lq \cos \theta_q)} \right) \quad (\text{A.3})$$

where θ_{xy} is the angle between the vectors x and y and $d\Omega_l$ is the angular part of the integration over l in Euclidean space.

The angular integrations in Eq.(A.2) are straightforward. For J_p we employ the technique of expansion in Chebyshev polynomials developed by Rosner [10]. These polynomials are generated by the relation

$$\frac{1}{1 + 2xt + t^2} = \sum_{n=0}^{\infty} C_n(x) t^n \quad (\text{A.4})$$

where $C_n(x)$ is the n^{th} order Chebyshev polynomial. These polynomials obey the orthonormality relation

$$\int d\Omega_l C_n(\cos \theta_p) C_m(\cos \theta_q) = \frac{2\pi^2}{n+1} \delta_{mn} C_n(\cos \theta_{pq})$$

and

$$C_0(x) = 1$$

After some algebraic manipulations we arrive at the result

$$\int d\Omega_l \frac{1}{(l^2 + p^2 + \Sigma_l^2(l) - 2lp \cos \theta_p)(l^2 + q^2 - 2lq \cos \theta_q)} = \frac{-2\pi^2}{pq l^2 \sin \theta_{pq}} \left(\tan^{-1} \left(\frac{\cos \theta_{pq} - st}{\sin \theta_{pq}} \right) + \theta_{pq} - \frac{\pi}{2} \right) \quad (\text{A.5})$$

[†]We consider only the pseudoscalar propagator since the scalar propagator involves no extra work.

where

$$s = \min(q/l, l/q)$$

and

$$t = \frac{(l^2 + p^2 + \Sigma_i^2(l)) - \sqrt{(l^2 + p^2 + \Sigma_i^2(l))^2 - 4p^2l^2}}{2lp}$$

Since the position of the pole in the scattering amplitude should not depend on the scattering angle we are at liberty to set $\theta_{pq} = \pi/2$ and the R.H.S. of Eq.(A.5) reduces to

$$\frac{2\pi^2}{pq l^2} \tan^{-1}(st) \quad (\text{A.6})$$

Substituting everything into Eq. (A.1) gives us the required result Eq. (4.3).

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Figure Captions

Figure 1: Diagrammatic representation of the gap equation in the fermionic loop approximation.

Figure 2: Bubble sum generated by the four-fermi interaction.

Figure 3: Diagrammatic representation of the gap equation with planar QCD corrections.

Figure 4: Iterative equation for the bound state propagators in planar QCD approximation.

Figure 5: Numerical solution of $\Gamma_p(p, q)$ with $q = 10^{-3}$ GeV and $q = 10^4$ GeV for a cut-off of 10^4 GeV.

Figure 6: Numerical solution of $\Gamma_s(p, q)$ with $q = 10^{-3}$ GeV and $q = 10^4$ GeV for a cut-off of 10^4 GeV.

Figure 7: Demonstration of the expected behaviour $\Gamma_s(p, q) \approx 1/(p^2 + m^2)$ where m is the pole in $\Gamma_s(p, q)$ shown in Fig. 6. $q = 10^{-3}$ GeV and the cut-off is 10^4 GeV.

$\Lambda(\text{GeV})$	10^4	10^5	10^6	10^7
$m_t(\text{bare})$	522	380	316	277
$m_t(\text{full})$	462	362	326	301
$m_t^{GAP}(\text{QCD})$	594	448	391	357
$m_t^{RG}(\text{QCD})$	577	446	391	358
$m_H(\text{bare})$	1044	760	632	554
$m_H(\text{full})$	618	464	399	360
$m_H^{GAP}(\text{QCD})$	1062	769	634	617
$m_H^{RG}(\text{QCD})$	1246	827	676	592
λ	0.96440	0.95366	0.95999	0.96182

Table 1. Predicted mass of the top quark and scalar $\bar{t}t$ bound state in varying levels of approximation. $m_{t(H)}(\text{bare})$, $m_{t(H)}^{RG}(\text{QCD})$ and $m_{t(H)}(\text{full})$ all come from solutions of equations (2.2) and (2.3). The bare values use (2.5) as input the QCD values use (2.6) and the equations for the full values are taken from ref [2]. $m_t^{GAP}(\text{QCD}) = \Sigma_t(0)$ where $\Sigma_t(q)$ is given by (4.1) and $m_H^{GAP}(\text{QCD})$ is the position of the pole in the solution to Eq.(4.3).

$\Lambda(\text{GeV})$	10^4	10^5	10^6	10^7
$m_4(full)$	393	317	284	263
$m_{4}^{GAP}(QCD)$	419	321	282	259
$m_{4}^{RG}(QCD)$	389	311	277	256
$m_H(full)$	555	419	360	326
$m_H^{GAP}(QCD)$	742	552	485	361
$m_H^{RG}(QCD)$	722	510	421	370
λ	0.47644	0.47503	0.478098	0.480495

Table 2. Predictions for the mass of a degenerate fourth family quark and scalar quark bound state in a four family model. RG equations for this model are taken from ref [8].

$\Lambda(\text{GeV})$	2000	4000	6000	8000
$m_4(full)$	-	466	427	406
$m_{4}^{GAP}(QCD)$	997	533	468	438
$m_{4}^{RG}(QCD)$	-	462	423	402
$m_H(full)$	-	706	622	581
$m_H^{GAP}(QCD)$	1781	958	838	778
$m_H^{RG}(QCD)$	-	975	829	762
λ	0.797	0.507	0.4856	0.47915

Table 3. Predictions for quark mass and scalar bound state mass in the four family model at very low cut-offs.

Fig.1

Fig.3

Fig.2

Fig.4

Fig.6

Fig.5

Fig.7