INTERMITTENCY IN PARTICLE COLLISIONS

R. PESCHANSKI

*Theory Division, CERN
CH-1211 Geneva 23, Switzerland*

and

*Service de Physique Théorique, Cen-Saclay
91191 Gif-sur-Yvette, France*

ABSTRACT

Some methods and concepts which are at the origin of and are presently relevant to the study of intermittency in multi-particle reactions are reviewed in an introductory and pedagogical way.

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dedicated to the memory of Léon Van Hove*

SUMMARY

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1. INITIAL MOTIVATIONS

In order to introduce the rather new subject of Intermittency in Particle Physics, why not begin by the personal motivations which led A. Białas and the author to propose such a concept in the 1985 study [1] of rapidity density fluctuations in very high energy (cosmic-ray) particle collisions? Even if these motivations were only supported by the very few pieces of information which were available at that time compared with nowadays, the logics followed in the original work appear significant enough to provide one with a kind of "fil conducteur" to the study of intermittency in high energy collisions [2].

The initial study of Ref. [1] was directly motivated by a number of questions raised by the beautiful cosmic-ray JACEE event [3], which was, and perhaps remains, the World Champion for the total multiplicity of observed particles, their rapidity density and the c.m.s. energy, see Fig. 1a. "What is the (eventual) dynamical significance of the apparently large observed fluctuations in rapidity density?" "Can we find a quantitative and objective way to characterize these fluctuation patterns which are known to vary from one event to the other, or even in different pieces of phase-space inside the same event?" "Which physical information (if any!) can be obtained from considering only one event, even if (or because of) exceptional?" Such are the questions which arise from the mere consideration of Fig. 1a, and such are the questions which one wanted to address in Ref. [1]. Similar density fluctuations were noticed since a long time, but no definite approach had been found before for reasons which will soon become clear. In some sense, the exceptional character of the JACEE event was increasing the level of interrogations.

The first hint of an answer comes from the consideration of the large multiplicity and density of the JACEE event. With more than thousand particles with at least 10 particles for a minimum resolution of 0.1 rapidity unit, it was conceivable to consider the JACEE event as the random realization of a statistical (or complex dynamical) system. For instance, the repartition and statistical properties of the eddy structures in a turbulent fluid can be studied with appropriate statistical methods [4]. Among them, two main ones seemed to be interesting to apply to particle density fluctuations of the JACEE event: *the method of moments* and *the variation of structures with the binning scale* (see further, Section 2).

However, soon after the first trials, an important, in fact basic, drawback appeared, namely the role of the purely statistical fluctuations, which one may call the *statistical noise*. High enough as it may seem for particle physics standards, the multiplicity of the JACEE event is very small compared with the number of degrees of freedom in

a small volume of fluid. Indeed, even if a probabilistic law of rapidity fluctuations is supposed to exist, its even-by-event observation in terms of a finite multiplicity would be largely dominated by the fluctuations due to this finite multiplicity itself. Take, for instance, a "white-noise" distribution realized in terms of a finite number of trials (c.f. a finite number of coin tossings), you will generate fluctuations around the (asymptotic) probability. The problem is even worsened when considering the scale dependence of these statistical fluctuations which increase quickly with the resolution. Applied without care, the moment method and many other ones proposed to study particle fluctuations are essentially nothing else than a study of noise.

The proposal of Ref. [1] was to deal with the statistical noise by assuming a very simple law of statistical fluctuations. It is the Poisson law of independent realizations in each rapidity box, or equivalently a (multi-) Poissonian law, see Section 2. Converted in terms of moments, and in order to deconvolute the Poisson noise, it was proposed to study the *factorial moments* of the rapidity density distribution, and examine their binning scale dependence.

Interestingly enough, the application of this recipe to the JACEE event (see Fig. 1b) led to both a confirmation and a surprise. Confirmed by the study of a Gaussian noise and also simulations [5] of realistic models for Silicon-Silver collisions, the possibility of damping, and even erasing, statistical fluctuations was demonstrated. However, the JACEE event fluctuations gave evidence for a different behaviour, compatible (within large error bars) with a non-flat, power-law dependence on the bin size. Strikingly enough, this possible behaviour was reminiscent of a well-known [4,6,7], property of fluid turbulence, that is *intermittency*: fluctuations in the size or activity of turbulent eddies are present and superposed at all scales (in a given range) and not related to fixed-size clustering; a fluid cannot be entirely and homogeneously turbulent.

In fact, studying one event with a limited (for statistical study) number of particles gives at best a simple indication that something interesting could be present. It is only three years later that the possibility of applying the factorial moment method to data at accelerators [8] and the evidence for intermittent-like fluctuations in various reactions (see Section 3) asked for a more general and systematic study of fluctuation patterns in multiparticle collisions which is the subject of the present review. In fact, in the meantime, and independently, the experimental detection of an exceptional rapidity density fluctuation in hadron-hadron collisions by the NA22 collaboration [9], and a theoretical proposal to apply to particle physics some mathematical models of one-dimensional intermittent systems [10] help preparing the way to the systematic searches of nowadays. Other physicists and collaborations have had contact in the past with the problem of fluctuations in particle collisions [2,11] and credit should be given to

them [including those unknown to us and to which I apologize]. It is fair to quote in particular authors [12] which probably were the first, and during a long time almost alone, to pinpoint the importance of the subject and propose various methods to deal with fluctuations and chaotic structures in particle physics.

2. THE APPROACH TO INTERMITTENCY

Three main features can summarize the different aspects of the intermittency approach to particle production fluctuations, as they were proposed in Ref. [1], but largely developed in subsequent work on the subject; namely the "*factorial observable method*" (fighting against the statistical noise), the analysis of *resolution scale dependence* or multi-fractal properties, and the α -*model representations* of fluctuations. These are the mathematical model realizations of intermittency patterns (providing one with a kind of quantitative "toy model" for intermittent behaviour) in the framework of the relativistic collisions of particles. A word of caution: one will not consider in this section dynamical models and the different (if not controversial) interpretations of the phenomenon which will be discussed later on. In fact, the first approach to the problem of particle fluctuations is much in the spirit of the study of fractals in Nature [4] for which an important step is to express the random and fractal geometrical properties of fluctuations before their interpretation in terms of more specific and dynamical properties of the system under examination.

2.1 Damping the Statistical Noise

In terms of simple properties of the theory of probabilities, the problem of *statistical noise* can be reinterpreted as the classical distinction [13] between frequency and probability weight. Let us consider for example a series of K tossings of a coin; if K_+ is the number of times the coin is heads, the frequency $f_+ = K_+/K$ to get heads after a finite number of tossings can be different from the asymptotic probability weight $p_+ = 1/2$ (but converges to it if the coin is legal!). Even larger can be the difference between f_+ and p_+ , if the number of tossings is very small. A very good and general description of this effect is provided by the Bernoulli law which tells you that the probability of finding K_+ times the same side among K trials if you know p_+ is, up to a factor, $\binom{K}{K_+} p_+^{K_+} (1-p_+)^{K-K_+}$. In fact, this law comes from the minimal assumptions that all K_+ realizations of your event are i) independent of each other and of the K_- realizations, ii) indistinguishable, iii) submitted to no other constraints, such as systematic biases, conservation laws, etc. In fact, constraints can often be superimposed on the Poisson-law probability distributions by convolution with, for instance, δ -functions, etc.

In particle physics, the application of these simple ideas enters by considering that in each bin M of the variable where fluctuations have been observed, e.g., rapidity, the statistical noise can be described by the Poisson law. The only, but essential, difference with the previous "coin-model" is that one ignores the asymptotic probability weight or, in physical terms, the conveniently normalized *local rapidity density* ρ_m . One can write

$$Q(K_1, \dots, K_M, \dots, K_M) = \int \mathcal{P}(\rho_1, \dots, \rho_M) \prod_{m=1}^M \mathcal{B}_{(K_m, \rho_m)} d\rho_m \quad (1)$$

with $\mathcal{B}_{(K_m, \rho_m)} \propto \rho_m^{K_m} / K_m!$ up to the normalization; Q is the observed distribution of bin multiplicities, $\mathcal{P}$ the (unknown) asymptotic density distribution. Following the previous discussion, the ansatz (1) comes from the assumptions of a "Poisson statistical noise". Surprisingly enough, these assumptions turn out in most cases to be realized in particle production physics, provided one takes care of known constraints, (the first example of which being a fixed total multiplicity [1]). In fact, whenever possible, the basic assumptions on the noise have to be analyzed and confronted with realistic models where the "noise" is not explicitly Poisson or Bernoulli.

However, the final goal is to extract information on the non-statistical statistical properties, namely on the function $\mathcal{P}(\rho_1, \dots, \rho_M, \dots, \rho_M)$. This is done by disentangling the convolution integral of formula (1) by the appropriate moment method. Indeed, factorial moments (and other factorial correlators) allow one to factorize the Poisson factors. One gets, for instance, in the case of events with no fixed multiplicity [14]

$$\begin{aligned} F_q^{(m)} &= \frac{\langle K_m (K_m - 1) \dots (K_m - q + 1) \rangle_Q}{\langle K_m \rangle_Q^q} = \frac{\langle \rho_m^q \rangle_{\mathcal{P}}}{\langle \rho_m \rangle_{\mathcal{P}}^q} \\ C_{q,q'}^{(m,n)} &= \frac{\langle K_m \dots (K_m - q + 1) K_n \dots (K_n - q' + 1) \rangle_Q}{\langle K_m \rangle_Q^q \langle K_n \rangle_{Q'}^{q'}} = \frac{\langle \rho_m^q \rho_n^{q'} \rangle_{\mathcal{P}}}{\langle \rho_m \rangle_{\mathcal{P}}^q \langle \rho_n \rangle_{\mathcal{P}}^{q'}} \end{aligned} \quad (2)$$

where $\langle \rangle_Q$ (resp. $\langle \rangle_{\mathcal{P}}$) denote the averaging integration over the frequency distribution Q (resp. the probability weight distribution $\mathcal{P}$). In Eq. (2), the averaging is over events with different multiplicities but for *fixed* bins m and n . But similar relations can be obtained when varying the bins m and n for a given set of events of fixed multiplicity, if not a single one like the JACEE event as in Ref. [1]. In fact relations (2) can be most easily derived by the following property of φ , the generating functional of moments,

$$\varphi(\lambda_1, \dots, \lambda_M) \equiv \langle K_1^{\lambda_1} \dots K_M^{\lambda_M} \rangle_Q = \langle \exp \left\{ \sum_m (\lambda_m - 1) \rho_m \right\} \rangle_{\mathcal{P}} \quad (3)$$

by q -successive derivations with respect to λ_m and q' with λ_n , and adequate normalizations [14,15]. Indeed, the factorial moments and observables, whose first

examples are given in formulae (2), allow one to erase as much as possible the purely statistical fluctuations. In fact, it was shown by Léon Van Hove [16] that in the limit of very small bins, the multiplicity distribution in the considered small window is completely dominated by the small multiplicities or even empty bins, unless one uses the factorial moments. This is the virtue of the factorial numerator of Eqs. (2) to kill the lower part of the multiplicity distribution, up to $K_m = q$, where q is the rank of the moment, [see Eqs. (2)]. Indeed this procedure enhances the effect of rare configurations of events of large or exceptional density; this is just what one wants: looking for large fluctuations beyond statistics. However, low-rank moments also give a rich information on the structure of fluctuations.

2.2 Varying the resolution scale

Even if the relations (2) on factorial observables such as F_q or $C_{q,q'}$ allow one to depress the statistical fluctuations, they represent only a limited piece of information on the probability distribution $\mathcal{P}(\{\rho_m\})$ itself. Indeed, the factorial moments distribution would be the same by an a priori redistribution of bins of same size among themselves, while this would correspond to an entirely different probability distribution! A way to solve this ambiguity and get a deeper relation to $\mathcal{P}(\{\rho_m\})$ is to vary the width δY of the bins (i.e., the resolution scale) and examine the subsequent variation of moments F_q as a function of δY . A practical way to do it is to consider the minimum accessible width $\delta_{\min}$ by experiment, measure $F_q(\delta_{\min})$ and then group the bins 2 by 2, measure $F_q(2\delta_{\min})$, group the bins 2 by 2, etc., up to the largest scale ΔY in which one wants to study the phenomenon. It is convenient to represent this partition process by a simple (Cayley) tree structure which is shown in Fig. 2a. With this procedure it is quite clear that two different probability weight distributions could eventually give the same average moment at a given rapidity scale but not at other ones; a large part of the ambiguity problem is solved (at least if the distribution is not too chaotic) [17].

This procedure was used in Ref. [1] to analyze the JACEE events, considering now the distributions of bin densities and using the statistical noise assumption (1), where Q is now the observed distribution of multiplicities for different bins and $\mathcal{P}$ is the density distribution which one tries to extract from its (unique) realization in the JACEE event. In Fig. 1b is shown the result of this analysis in Ref. [1] together with the moments for a simulation of a gaussian "white-noise". The result is compatible with a power-like behaviour, namely

$$F_q(\delta Y, \Delta Y) = \left(\frac{\Delta Y}{\delta Y} \right)^{\varphi_q}, \quad (4)$$

with exponent φ_q . The data were coming from one event and one could only see an evidence for the behaviour (4); but either the absence of selection of a given scale, or the difference with the statistical noise simulations were interesting enough to try and understand a behaviour like (4) which one interpreted in Ref. [1] as intermittency-like patterns. In fact, behaviours like (4) are now observed in different reactions with much less uncertainty than for the unique JACEE event (see Fig. 1c and Section 3).

An important point to be noticed, following Ref.[18], for particle physics, is that the scaling property (4), can be reanalyzed in terms of fractal properties of the particle density fluctuations. In particular, one can relate the indices φ_q , which will soon be called intermittency indices, to the generalized fractal dimensions called Renyi dimensions D_q [19]. More precisely, one may write:

$$\frac{\varphi_q}{q-1} = \bar{D}_q = 1 - D_q \quad (5)$$

where D_q is the Renyi fractal dimension of the fluctuation domain contributing to the moment of rank q and $\bar{D}_q$ the codimension. In terms of random geometry, Eq. (5) means that the intermittency dimension is measured by the *Renyi codimension* $\bar{D}_q$. As a by-product, a system determined by a single (mono)-fractal set of fluctuation patterns will be characterized by $\bar{D}_q = cste$, that is intermittency indices following the simple rule

$$\frac{\varphi_q}{q-1} = \varphi_2 = cste. \quad (6)$$

This is the first hint that the intermittency indices and their variation with the rank [20] could give an information on the inner structure of dynamical fluctuations.

2.3 Intermittency and its α -model representations

Interestingly enough, the power-law behaviour (4), and the existence of fractional dimensions (5), (6), is well known in the experimental and theoretical investigations of fully-developed fluid turbulence [4,6,7]. It is attributed to intermittency as a characterization of the random occurrence in a turbulent medium of pieces of laminar flow. Different models of hydrodynamical intermittency have been proposed, but no theory has yet emerged which could explain the intermittency phenomenon from the first principles, that is for instance the Euler or Navier-Stokes equations [21].

Because of the lack of basic justification for the use of hydrodynamical equations in the domain of relativistic collisions of particles, one has adopted in Ref. [1] a different point of view. Let one ask for a more abstract and mathematically better defined

question : is it possible to construct a large class of models, which would exactly verify the scaling law of moments (4), independently of the initial (ΔY) and final (δY) range of observation and for any value of the momentum rank (q) ? The answer can be found [4,6,7], and one called them [1] α -models of intermittency, since they are inspired by the same name given to the probabilistic models of turbulence which were the most interesting for the study of intermittency [7], even if proposed in a different context and different purposes (see Section 5 for a more detailed discussion of hydrodynamical turbulence).

Let one consider the partition of phase-space in terms of a Cayley tree (see Figs. 2). One event will be defined by a set of randomly chosen numbers W 's, one for each link of the tree (Fig. 2a) or, equivalently, for each box of the phase-space partition diagram (Fig. 2b). These random numbers are considered to be independent random realizations of a random function following a normalized but otherwise arbitrary probability distribution $r(W)$. Namely one has the only following constraints

$$\{W^0\} \equiv \int r(W) dW = 1; \quad \{W^1\} \equiv \int W r(W) dW = 1 \quad (7)$$

Note that with these conventions, a second "event" will require a new set of random choices for all the W 's in the tree structure, from the larger box to the smallest ones (and as small as the final resolution is requiring). The model for the event-by-event particle densities can be written as a function of the W 's. One writes

$$\frac{\rho_m}{\langle \rho_m \rangle} = \prod_{s=1}^{\nu} W(\alpha_s), \quad (8)$$

where the number of (link) generations is s running from 1 to ν and α_s denotes the *particular link* at the s^{th} generation which belongs to the *particular branch* leading to the bin m (see Fig. 2). With these conventions, one has the following obvious relations

$$M \equiv \frac{\Delta Y}{\delta Y} \equiv \lambda^{\nu}, \quad (9)$$

where M is the total number of bins and λ the branching number (which is, for instance, chosen to be two in Figs. 2). Note that the average density $\langle \rho_m \rangle$, appearing in relation (8) can as well be an average over bins for a given event ("horizontal" averaging) or an average over events for a fixed final bin (or "vertical" averaging).

Bringing the definition (8) in the formula for moments, Eq. (2), one gets

$$F_q \equiv \langle \prod_{i=1}^{\nu} W_{(\alpha_i)}^q \rangle = \{W^q\}^{\nu} = \left(\frac{\Delta Y}{\delta Y} \right)^{qn(W^q)/ln\lambda}, \quad (10)$$

where one uses successively the independence of the realizations W 's along the path on the tree of Fig. 2a and the relation (9). Denoting $\varphi_q = \ln\{W^q\}/\ln\lambda$, one recovers the property (4). Moreover, the derivation (10) does not in any way depend on the total generation number ν , or of the rank of the moment q . In some sense, the class of models here introduced give an exact realization of intermittency patterns. In fact, they will serve as a definition of intermittent patterns for our purpose. A more basic approach to intermittent fluctuations is not yet existing, and after all it is our final goal to find it for multiparticle production. So, as a starting point, one will consider the intermittency patterns generated by α -model as the reference definition of intermittent fluctuations.

Indeed, considering α -models to study intermittency in particle collisions leads to many useful applications. They give a kind of toy-model to discuss the observed fluctuation patterns in terms of (or by difference with) intermittency-like ones and allow one to generate useful simulations [8,22]. In fact, it was originally noticed in Ref. [1] that they meet a realistic interpretation in terms of random cascading processes of multiparticle production. Indeed, if one converts the abstract way of dividing phase-space into a genuine cascading process, one is able to propose a space-time interpretation of the formation of intermittent patterns, see Fig. 3. At a first (proper-) time τ_1 of the process, the relativistic system formed by the collision decays into 2 (or λ) pieces occupying adjacent parts of phase space. This division appears after the expansion goes over the conventional correlation-size of one fermi. Then the system evolves till a second (proper)-time τ_2 where the pieces separate again generating density fluctuations. Further on, the system is submitted to successive dividing steps, generating intermittency-like fluctuation patterns, observed at the final stage. This qualitative picture possesses the merit of showing a rather general implication of intermittent models, namely the existence of long (in fermi units) cascading processes in some events, or in parts of an event. Note that the *averaged* interaction time remains the known one (≈ 1 fermi).

As a final remark, it is worth noticing that the α -models lead to specific predictions for other observables, such as multi-bin factorial correlators [14] and bin-size correlations [23]. For instance, bin-bin factorial correlators, as defined in (2), can be expressed in terms of the random W 's. Distinguishing the contribution of common

ancestors (see Fig. 2a) from the two branches leading to two different bins, one gets [14] the interesting prediction:

$$\frac{C_{q,q'}^{(m,n)}}{F_q^{(m)} F_{q'}^{(n)}} = \left(\frac{\Delta Y}{DY} \right)^{\varphi_{q,q'} - \varphi_q - \varphi_{q'}}, \quad (11)$$

where DY represents the distance between bins m and n . The noticeable properties of relation (11) are its independence on the resolution δY and the expression of correlators in terms only of intermittency exponents. As such, the relation (11) exemplifies the structure of correlations in an intermittent model of fluctuations. Apart the (interesting) case where $\varphi_{q+q'} - (\varphi_q + \varphi_{q'}) = 0$, they are decreasing as a power law on the distance, rather than the usual Gaussian suppression of long-range correlations. This is a consequence of the scale invariant properties of random cascading by comparison with scale-dependent correlations of the Gaussian case. The property (11), which deserved to be confronted to data, would indicate an analogy with a critical system in statistical mechanics at a phase transition (see examples and references in Section 4).

3. A SELECTION OF EXPERIMENTAL AND PHENOMENOLOGICAL ASPECTS

As seen by experience now, the experimental study of dynamical fluctuations and intermittency effects using the method of factorial observables is not a comfortable task. Many systematic errors have to be avoided. For a (realistic) example, counting three tracks for a (broken) one creates a sizeable but fake intermittency effect; the same with losing tracks. So, caution must be taken and a great deal of self-criticism must be kept when attacking the problem of handling dozens or hundreds of tracks to analyze rather sophisticated structures of fluctuations, moreover superimposed to statistical ones !

In fact, due to the large development of experimental and phenomenological works on intermittency in multi-particle production, a exhaustive review on these aspects will not be done here. One would suggest to look for conference review talks [24], which give now a regular, and evolving account on the subject. In the following of this section, one will select, on a somewhat subjective basis, a few aspects related to the theoretical discussion. Let the numerous authors and collaborators contributing with hard work on the subject apologize for the restrictive point of view one is led to follow in this review.

3.1 Horizontal versus vertical analyses

The interest in fluctuations and intermittency effects has grown when it appeared possible to study them at accelerators. Indeed this crucial extension was made realistic after a seminal study [8] in which the analytic predictions of an exactly intermittent model were confronted with numerical simulations including statistical noise. Decreasing arbitrarily the total number of simulated observed particles from infinity (cf. analytical results on densities), through 1000 (cf. JACEE events), to 30 or less (cf. Accelerator Physics), quite surprisingly, the method of factorial moments remained valid up to rather small multiplicities. One then replaces the statistical loss on total multiplicity by the gain on the number of events. The structure of the simulation model is given in Fig. 4 as an example [22].

In fact, this essential extension, if true not only for the simulation of a given model, but also for real events, reflects a rather curious property of the statistical distribution of particles in a collision : it is equivalent either to look for the distribution of particle densities among different bins for a given event with large total multiplicity (*horizontal analysis*) or to look for the distribution of particle densities in a given bin over a large enough number of events (*vertical analysis*). These two probabilistic distributions are a priori different, given their origin. However, for similar reactions, the behaviour of factorial moments F^V and F^H defined as follows

$$F_q^V = \frac{1}{N} \sum_{events} \frac{K_m(K_m-1)\dots(K_m-q+1)}{(normalization)_V}; \quad F_q^H = \frac{1}{M} \sum_{bins} \frac{K_m(K_m-1)\dots(K_m-q+1)}{(normalization)_H} \quad (12)$$

is the same, apart from the normalization [15,24]. It is in particular quite astonishing that the intermittency indices extracted from accelerator data for emulsions [25] are quite in the range obtained from JACEE, compare Fig. 1b and 1c. This is a kind of "ergodic property" which, in statistical systems, allow to identify (in probability) sampling in phase space, and sampling locally a great number of equivalent systems. It may reflect the self-similarity of the scattering process.

Concerning the technicalities, a few interesting problems appear and are not all solved. One of them concerns the effect of a non-constant averaged density distribution which may affect the normalization of the moments, at least for the "horizontal analysis". Indeed, this effect can be partly cured by adequate correction factors [26] and avoided by a suitable change of variable [27]. A more difficult and yet unsolved question is the evaluation of the "empty-bin" effect. A rapid fall-off of the moments F_q (in log-log plots) comes from the definition (2) in itself. For very thin binning, there is always a limit when all bins contain less than q particles, where q is the rank of the

moment, giving $F_q \equiv 0$. Though obvious in its principle, the effects of this limitation on actual measurements is very hard to estimate quantitatively, since it affects the tail of the multiplicity distribution in each bin, and thus both the normalization and the relative contribution of different moments [28]. Simulations can be used to evaluate these effects [22], but they may rely on the yet unknown distribution of high rank moments. Work is in progress in this direction [29], but the actual analysis avoids the "empty-bin" effects by limiting the resolution scale even if the statistics of events could allow to go further.

Note, however, that thin bins are very interesting to analyze, since they increase the range of fluctuation scales and allow one to look at the physical boundary of the phenomena in rapidity space. Fighting against the "empty-bin" effect and for the best resolution is thus a continual challenge in this domain of experimental investigation. Some interesting results come recently from heavy-ion data [30], showing a smooth bending of one-dimensional factorial moments when $.01 < \delta y < .1$.

3.2 Strength and dimensionality of Intermittency

Indeed, using the factorial moment analysis and despite the specific problems of fluctuation analysis, a behaviour compatible with the Intermittency behaviour (4), was found in various reactions [25, 31-35] obtained at accelerators, with a much better precision than the initial analysis of the JACEE events see Fig. 5. Taking as an example the nucleus-nucleus interactions, one can compare the Fig. 1b with the moments obtained in Fig. 1c from emulsion-nucleus collisions of the KLM collaboration [25] which were the first published data on the factorial moments. Indeed, these results lead to a few remarks.

- (i) The comparison with a prediction of random "noise" fluctuations for the same reaction clearly shows the presence of a strictly dynamical effect quite compatible with equation (4), at least in the $(.1 - 1)$ range of rapidity bin sizes. Existing simulation models, without dynamical fluctuations, fail to reproduce the observed trend [35].
- (ii) Comparison of Figs. 1b and 1c for similar rank of the moments shows that the strength of the effect, as being measured by the slope of the straight line, the so-called intermittency index, is similar in both types of analyses. This is a confirmation of the (at least approximative) validity of the "ergodic" property referred to previously. There is an equivalence between the structure of subprocesses of a large multiplicity event and the structure of fluctuations over many events in the same small piece of phase space.

- (iii) One should remark that the process is, all in all, rather weak in terms of a turbulence strength. A way to exhibit this property is to compare the typical values of Renyi dimensions $\bar{D}_q$ (see formula (5)), which are of order 0.02-0.03 for heavy ions [25, 30], with those of fully-developed turbulence in three dimensions, which are near 0.2. Indeed, the weakness of the effect in rapidity and the range limitation of its study is an argument to question the intermittency interpretation of a dynamical effect which, however, is undoubtedly present [36-39]. For instance, the intermittency signal seems to weaken for $\delta y < .1$ [30].

A few recent experimental pieces of information allow one to shed a new light on the problem. On the one hand, a thorough study of gold-nucleus interactions at very small bins by the EMU01 collaboration [30] has shown that the power-law behaviour was damped between 0.1 and 0.01 bin sizes. On the other hand, initiated by the study of self-similar hadronic cascading models [40], the phenomenological study of bi-dimensional [41], of tri-dimensional [42] fluctuations has revealed a quite stronger power-law rise of factorial moments when one considers a larger phase-space including, e.g., the azimuthal variable [43]. Factors 4 to 6 in the intermittency indices are found experimentally [43], and the model analysis [41, 42] reach comparable values. More precisely, it has been remarked [41], and verified in the framework of multi-dimensional α -models, that assuming intermittent patterns of fluctuations in a space of dimension d , they are not preserved by projection into dimension $d - 1$ (or less) except in some limited range. These analyses, which are currently developed, give some credit that the dynamical effect is indeed compatible with the relation (4) and the structure of intermittency in three-dimensions and its strength is of "normal" intensity for intermittency phenomena. At least some typical one-dimensional effects, such as the turn-over already mentioned [30], and the increase of intermittent structures at (small) transverse momentum found in hadron-hadron reactions [31, 44] can be interpreted as a projection effect from intermittent-like fluctuations in three-dimensions, with a P_T cut-off [42].

Note that the three-dimensional analysis of intermittency requires a definition of phase-space variables for which the average particle density remains constant [27, 45]. One is interested in the anomalous dimensions of fluctuations over a constant background and not, a priori, to the dimensions of the event shape. This is not the case in a recent experimental analysis [46] which results cannot be easily discussed in terms of intermittency.

All in all, the increase of intermittency effects with the dimensionality of the considered phase-space seems to confirm that the weak effect detected in Ref. [1] was only a small reflection of a stronger dynamical, scale-invariant pattern of fluctuations in momentum space.

3.3 Universality and non-universality aspects of Intermittency

In first attempts [1, 47] at an interpretation of the effect observed in the JACEE event, it was expected that a complex object - such as a drop of quark-gluon plasma - has been formed in the TeV-range JACEE event and subsequently decays through an intermittent cascade. The process is the same as schematized in Fig. 3, expressed in terms of fluctuations generated by the cooling process of the quark-gluon plasma. After some (proper-) time, the initially formed drop reaches a longitudinal distance larger than the standard one-fermi correlation length. It then splits into drops of plasma with various densities. But the expansion proceeds further since the energy in the field is not completely transformed into mass or kinetic energy. Further on, and with some random character, the new drops reach the critical size and split with new density distribution, etc. In this framework, the intermittency would signal the space-time expansion and decay of the quark-gluon plasma, see Fig. 3.

However, this interpretation of intermittency is far from being unique and characteristic of the plasma. We will soon have a clear-cut proof of this fact, but let us mention that this versatility is an already well-known property of many physical or "mathematical" turbulent systems. Intermittency appears in fully-developed turbulence in three-dimensions [4,6,7], but also in the onset of turbulence in dynamical systems, in diffusion limited aggregation models of clusters, etc. (see Section 5 for discussion and references). This universality of intermittency is an interesting property since it leads to a quite well-known geometrical description [4]. But this is also a problem for the search of its dynamical origin.

In particle physics, it was soon recognized [48] that intermittency patterns are not the privilege of heavy ions reactions and neither its possible description in terms of plasma formation. The case of e^+e^- annihilation into hadrons at high energies appeared particularly remarkable. Not only the intermittency structures are present but they are in general stronger [48, 33] (in terms of intermittency exponents) than in reactions involving nuclei.

Indeed, as shown recently by the DELPHI collaboration at LEP [49], intermittency patterns show up very clearly and can be described by the Lund Parton Shower Model (version 6.2.) in a satisfactory way. Even if this parametrization of intermittency surely needs an explanation, it is very difficult to interpret a perturbative QCD cascade of quarks and gluons followed by a string-like hadronization as a quark gluon plasma ! (see, however, the interesting remarks of reference [50] on the possibility of cold gluon plasma drops). At the stage of the QCD parton cascade, and if one simulates processes at very-high energies and/or long parton cascading down to very low virtualities

intermittency patterns show up [26, 51-53].

In fact, many different scale-invariant random models lead to Intermittency, as shown, e.g., in the review of Ref. [2]. The question then arises to look for features which would distinguish the different possible origins. Recently, a rather noticeable different dependence of intermittency indices with the rank of the moment was reported and commented [54]. The data on Renyi dimensions $\bar{D}_q$ (see definition (5)) are compatible with a unique dimension $\bar{D}$ for heavy-ion reactions [25, 30] in contrast with other reactions, such as $\mu - p$ reactions [34], as shown in Fig. 5d, where they depend on the rank q .

The interpretation of the heavy ion data in terms of a quark-gluon plasma is based on the remark (see for instance [20]), that a "thermal" phase transition of second order leads to a unique Renyi dimension, in contrast to "non-thermal" phase transitions or cascading within one phase, hadronic [40] or partonic [51-53]. This interesting relation is further discussed in the next section (note, however, that a monofractal cascading mode has been initially proposed for turbulence, without referring to a thermal phase transition [6]). The problem of quark-gluon plasma signatures remains ; however in intermittency signals can help, adding new pieces of information and comparison with predictions, such as those on correlators, see Eq. (11). These correlators are predicted [55] to behave very differently depending on the process, cascading or phase transition.

3.4 Alternatives to Intermittency

The existence of dynamical fluctuations, which can be isolated from the "statistical noise" has now been well established using models and simulations. However, different sources of dynamical fluctuations (which have often been partly forgotten in the meantime) have been re-examined in the light of the factorial moment analysis. Namely, the role of resonances [56], of long and short-range order correlations in rapidity [37-39], the Bose-Einstein [56] and other quantum correlation effects [36]. The question is to estimate whether some, or a superposition of these conventional dynamical effects, can describe the experimental results on fluctuation effects as they are seen in the factorial observables. In fact, the discussion is made easier [37, 38] when considering the two-particle correlation function. In particular, considering the one-dimensional projection on the rapidity axis, there exist useful relations implying the factorial moments of the rapidity distributions in a given interval δy . For instance, one gets:

$$F_2(\delta y) = 1 + \frac{1}{M} \sum_{[m]} \int_{Y_1, Y_2 \in [m]} c(Y_1, Y_2) dY_1 dY_2, \quad (13)$$

with

$$C(Y_1, Y_2) = \frac{d^2\sigma}{dY_1 dY_2} \frac{d\sigma}{dY_1 dY_2} - 1. \quad (14)$$

Short-range rapidity correlations have been known since a long time, with a characteristic rapidity extension of order 1. It is thus conceivable that, through relation (13), their effect should be present in factorial moments for bin size of order 1 or so. However, it has been noticed in various analyses [36-39] that correlations of this kind could give rise to dynamical effects at smaller scales. To be more specific, one should compare scale-dependent functions of exponential [36-38] ($e^{-|Y_1 - Y_2|/\Delta}$) or gaussian [39] type ($e^{-(Y_1 - Y_2)^2/\Delta^2}$), with scale-independent distributions which are those [42] giving rise to the intermittent behaviour of formula (4), namely:

$$C(Y_1, Y_2) \sim |Y_1 - Y_2|^{-\alpha} \quad (15)$$

where the exponent α of the singular behaviour (15) is related to the fractal dimension $\bar{D}_2$ (see Eqs. (5), (6)). Note, however [57], that this relation is only exact for asymptotically small scale δY and the additive constant in (13) can play a role in masking the singularity.

Other "classical" effects can play a role in dynamical fluctuations, such as the production of long-lived resonances and the Bose-Einstein interference [56]. For instance, considering the production and subsequent decay of $\rho : \omega : \varphi$ with suitable ratios of branching fractions can describe a hadronization cascade in space-time, due to a hierarchy of life-times and production rates. The Bose-Einstein enhancement effect, which only depends on the proximity in phase-space of two or more identical bosons, is also to be considered, even if this purely quantum effect is difficult to parametrize. Note however that these two last processes could lead to actual scale-invariant patterns by a really cascading process in space-time, on contrary to the short-range correlations which have a fixed and limited range. Note also that Bose-Einstein correlations are expected to be (twice) larger for identically charged particles, on contrary to short-range correlations which are expected to be weaker in this case at least in the Muller-Regge framework [37]. Since these proposals, some experimental results came in the discussion. For instance, the Bose-Einstein mechanism is weak or absent in e^+e^- , $\mu\mu$ and hadron-hadron reactions [58].

One important conclusion of ref. [37] remains to be confronted with data. If intermittency patterns are to be explained by the above-mentioned "classical" sources of dynamical fluctuations, different dominant sources have to be attributed to each

reaction, e.g. short-range correlatives for e^+e^- and $\mu\mu$ scattering into hadrons, short-range and long-range for hadron-hadron scattering and the Bose-Einstein mechanism for nucleus-nucleus reactions. This proposal [37] came from the remark that an independent juxtaposition of nucleon-nucleon diffusions gives an insignificant contribution to intermittency, of order $1/AB$ if A and B are the atomic numbers of both incident nuclei. On the other hand, the Bose-Einstein effect is predicted [37] to be strong and independent of the atomic numbers. The natural prediction of this approach is a sizeable enhancement of the intermittency effect for equal charged produced particles (assumed to be identical pions). The measurement does not exist yet. The comparison of second moments F_2 and their intermittency slopes φ_2 has been made for different reactions involving nuclei [59]. By plotting f_2 as a function of the particle density dN/dy , one may check the independent juxtaposition of sources (for p -nucleus reactions) while some evidence of a collective effect has been found in nucleus-nucleus reaction [60]. However, this collective effect is itself dN/dY -dependent, casting some doubt on the possibility of a Bose-Einstein mechanism.

3.5 Intermittency and the Lund model(s)

In the phenomenological analysis of intermittency, it is of particular interest to confront the different versions of the Lund Model and, for those different reactions with the existing data on fluctuations. The popularity of the Lund model(s) among the experimental groups justifies a special analysis. Moreover, the fact that in e^+e^- annihilation at LEP, one version of this model gives the yet unique example of hadronization model describing the factorial moments in e^+e^- annihilation into hadrons [49] is teasing the curiosity of theoreticians, if not of experimentalists.

Before coming to this case, it is worth noticing that no Lund model, nor any other Monte-Carlo simulation for reactions initiated by hadrons [44], $\mu\mu$ scattering [34], or nucleus-nucleus reactions [35] allows one to describe the behaviour of factorial moments. It would be of interest to understand which features of existing Monte-Carlo simulations should (or could) be changed to describe the intermittency patterns. This study deserves some future work.

Coming to e^+e^- annihilation into hadrons, the data remain unexplained for lower energy data [48,33]. However, the Delphi analysis shows that one version of the Lund model gives a quite satisfactory description of moments [49] at LEP energy. However, the origin of this agreement is not yet understood. Probably, the fact that the version in question is based on QCD parton cascading followed by (small) string fragmentation into particles and resonances helps describing intermittency patterns [61]. It has effectively been shown by different authors that the perturbative QCD branching of partons

is naturally intermittent [51-53]. However, the fact that the intermittency properties survive the hadronization steps is not trivial [25,51,61], even if it can be favoured by the dominance of parton cascading fluctuations at LEP energies [26]. It is important to note that, in this application of the Lund model, the intermittency structure is indeed present in the model through the perturbative self-similar cascade, on contrary to alternatives based on short-range correlations [36-39]; it is thus highly premature to interpret the Lund Model description as an absence of intermittency. It better reveals an intermittent feature of this model!

Another remark to be quoted is that string fragmentations itself, which is at the basis of the Lund Model(s) philosophy can present an intermittent structuration. As an example, the Schwinger mechanism of particle creation by tunnelling in a strong electromagnetic field has been extended to QCD string fragmentation [62]. In this framework, it can be shown in simulations of e^+e^- annihilations into $q\bar{q}$ strings [63], that intermittency patterns can appear as a result of a mechanism of non-saturation of the field energy. At each step of $q\bar{q}$ pair production, the rate of production does not exhaust the available energy, nor does the kinetic motion of the pairs. Thanks to the ultra-relativistic expansion of the process, new strings are created at each step, leading to intermittency patterns, see Fig. 7. As a by-product, interesting non-linear effects appear, especially visible in the space-time representation (Fig. 7b).

It is worth noticing that this mechanism is nothing else than a non-perturbative realization of the one advocated by Léon Van Hove [50]. Due to their long life-time as partons, some of them do not easily find partners to form the colour singlets of hadronic asymptotic states. As a result, they would form drops of cold quark-gluon plasma which extend very far in volume and time, and give rise to fluctuations in very small rapidity bins. Other yet unexplained phenomena, such as the abundance of very soft photons could be related to this phenomenon. In all these descriptions, we are very far from the ordinary assumptions on a uniform formation of hadrons at a fixed scale of 1 fermi ! All this justifies to have a critical look at our theoretical knowledge in order to re-evaluate the common understanding on hadronization and reaction mechanisms. This is done in the next section.

4. THEORETICAL INVESTIGATIONS

Ultimately, the properties of density fluctuations of particles should result from a complete formulation of the theory of Strong Interactions, whose best candidate is Quantum Chromo Dynamics (QCD). Thus, if intermittency is actually present, it should come as a scale invariant consequence of QCD. However, our knowledge of this theory is poor in a regime where one observes fluctuations in the distribution of hadrons after a very long (proper-) time (see Section 2.3). The intermittency phenomena involve the non-perturbative "far infra-red" regime of QCD, which is not understood. Even in cases when intermittency already appears at the level of the perturbative "ultra-violet" QCD regime, it is not obvious why the intermittency patterns expected for partons are not washed out by subsequent hadronization, see for instance Ref. [51]. Summing up the situation, one does not know the exact relation between a (renormalizable) gauge quantum field theory and the structure of fluctuations in the final states of a collision, at least in the (non-perturbative) strong coupling regime.

However, one knows another situation in modern physics where theoretical investigations on complex systems (even mathematical ones) can be performed even if one does not know the intimate relation of the observed structures with a microscopic theory; this is the case of Chaos under different forms, starting from deterministic chaotic systems to Statistical Mechanics of ordered/disordered systems. The complexity of the systems helps in defining statistical observables which are relevant, e.g., the order-parameters for a phase transition [64], or the multifractal spectrum of a set of singularities in a medium [64], etc. More specifically, one should ask the following question: can one relate the *local* fluctuations encoded in the intermittency pattern with the *global* behaviour of the overall system ? By *global*, one means the overall mechanism leading to particle production in the whole phase-space. One will do so by introducing a formulation borrowed from Statistical Mechanics [20] and defining a partition function on the system and some other thermodynamics quantities such as a "local entropy", a "local free energy", together with a substitute for temperature. This kind of extension of the thermodynamical formalism has also been introduced for dynamical chaotic systems [65], and thus should not be considered as a mere artifact. It is just a way to relate local to global excitations of a random process such as multiparticle production appears to be.

4.1 A statistical-mechanic formulation of intermittency

Let us consider an event-by-event realization of density fluctuations, as schematized in Fig. 8. In each bin m of given size $\delta = \frac{\Delta}{M}$, (with Δ total size and M the total number of bins), one registers a random density ρ_m . One then defines [20] a partition function $Z(q)$ on the system as follows:

$$Z(q) = \sum_{m=1}^M \rho_m^q, \quad (16)$$

where p_m is defined in such a way as to represent a normalized probability weight on the ensemble of bins. For a uniform density of particles, it is suitable to choose:

$$p_m = \frac{\rho_m}{M < \rho_m >} ; \sum_{m=1}^M p_m = M \langle p_m \rangle = 1 \quad (17)$$

where one considers (horizontal) averaging $<>$ over the sample. Note that with definition (17), the partition function $Z(q)$ is defined in terms of the particle densities and not as a function of the observed multiplicities in order to avoid the interference with the (non-dynamical) statistical noise. In fact, it is interesting to remark that $Z(q)$ being itself a random variable, one can define (vertical) averaging (over events) quantities, such as the moments $< Z^p(q) >$ which can in turn be expressed in terms of actual multiplicity distributions through the factorial moment method.

As defined by (16), the partition function Z is a global observable on the system and can be expected to test the existence of large domains of dynamical fluctuations. However, if the system is large enough, the summation (16) may lead to self-averaging, e.g., to a deterministic function $Z(q) \equiv < Z(q) >$. Within this assumption, one may write, using relations (16), (17) :

$$Z(q) \approx M \langle p_m^q \rangle = M^{1-q} \frac{\langle \rho_m^q \rangle}{\langle \rho_m \rangle^q} = M^{1-q+\varphi_q}, \quad (18)$$

where the last equality holds for an intermittent distribution, following relations (4) or (10). In fact, the relation (18) is the proper way of deriving the connection between intermittency indices φ_q and the Renyi codimension $\bar{D}_q$ (see Eq. (5)): by definition, the Renyi dimension D_q defines the fractal dimension of the fragments of the system which contribute to the partition function Z near a given value of q [65]; thus, writing

$Z(q)$ under two different forms, one of which is given by (18), one is brought back to Eq. (5), namely:

$$\frac{\varphi(q)}{q-1} = 1 - D_q = \bar{D}_q \quad (19)$$

where $\bar{D}_q$ is the Renyi codimension. The relation (19) is, within self-averaging conditions, the expression linking *global* properties of the system (expressed through Renyi dimensions), and the *local* ones (expressed by the intermittency indices).

It is quite interesting to derive the expression of the partition function when the densities are given by the random cascading models defined in Section 2.3. Inverting the expression (8) in the partition function (16), one gets:

$$Z(q) = \sum_{m=1}^M \prod_{s=1}^p \left(\frac{W(\alpha_s)}{\lambda} \right)^q, \quad (20)$$

where one used (8) and (17) to relate the probability weights p_m to the random numbers W defined on a tree structure of connectivity number λ , see Fig. 2. The expression (20) can be put in a different and more familiar form by a change of random variable. One writes:

$$W(\alpha_s) = \lambda e^{-\beta \epsilon_s}, \quad r(W) dW = \pi(\epsilon) d\epsilon, \quad (21)$$

where $\pi(\epsilon)$ is obtained from the normalized distribution of W , and β_0 is arbitrary but fixed. With this transformation, expression (20) takes a new form:

$$Z(q) = \sum_{[m]} \exp(-q \beta_0 \epsilon), \quad (22)$$

where $[m]$ denotes a given path in the tree structure of Fig. 2 and $\epsilon = \sum \epsilon_s$ is the sum of the "link energies" ϵ_s along the path $[m]$. In Eq. (22) one formally recognizes the well-known thermodynamical partition function with Boltzman factors $e^{-\beta E}$, where, by consequence,

$$q = \frac{\beta}{\beta_0} = \frac{T_0}{T}, \quad (23)$$

and T is the analogue of the temperature. Taking the thermodynamical analogy further, $Z(q)$ is thus the partition function of a system of links on a tree structure at temperature T , and "link energies" ϵ_s . These "link energies" are defined randomly on the tree structure with the probability distribution $\pi(\epsilon)$, Eq. (21).

The formal equivalence between intermittent random models and Boltzmann thermodynamics on a Cayley tree structure has quite interesting consequences which will be studied in more detail in the next subsections. It is to be noted that the same kind of thermodynamic models appear in many cases in the Statistical Mechanics of disordered systems. It appears in a large class of solvable models of spin-glass interactions, the generalized random energy models [66], as well as in the problem of directed polymers on a tree structure [67], the random breaking problem [68], the Kauffman automaton problem [69] etc. The phase structure of these systems is quite rich and, at least, different from the usual order-disorder transitions. In particular, one can reach phases where the function $Z(q)$ is no longer self-averaging, leading to modified relations between global and local excitations, not expressed by the Renyi dimensions.

The more general relation between local and global properties of a system of fluctuations can be expressed in a different way, in which one aims at a more precise relation between the moments of the distribution and the actual singularity structure in phase-space [70, 65]. In our case, one may reconsider [71] the partition function Z of definition (16) by reorganizing the summation in the following way: let us reorder the probability weights p_m as a function of their dependence in the resolution scale $\Delta/\delta = M$. One may write the definitions:

$$p_m \propto M^{-\alpha_m}, \quad N_m \propto M^{f_m}, \quad (24)$$

where α_m characterizes the singular behaviour as $M \rightarrow \infty$ and N_m is the number of degenerate cells (e.g. for which the probability weight follows the same scaling behaviour). As such in def. (24), f_m is nothing other [4] than the fractal dimension of the set of cells with equivalent singular behaviour. Note that $f_m = 1$ is the maximal dimension (dimension of the support) and $f_m = 0$ corresponds to point-like singularities.

The fractal dimensions f_m , which are defined in the real space of fluctuations have to be distinguished from the Renyi dimensions $\bar{D}_q$, which are defined for moments. Indeed there is a relation between f_m 's and $\bar{D}_q$'s, which is true for the so-called multifractal systems in general. Coming back to the expression (16) of $Z(q)$ and using definitions (24), one rewrites it as follows:

$$Z(q) = \sum_m N_m M^{-q\alpha_m} \approx \int_{\alpha-}^{\alpha+} M^{f(\alpha)-q\alpha} d\alpha. \quad (25)$$

Note the substitution of summation by an integration, assuming a continuous limit, and the consequent substitution of the discrete variable m by the singularity index α .

The integral (25) can be [70, 71] evaluated around a saddle-point $\bar{\alpha}$, leading to the following set of extremum conditions

$$q = \frac{df}{d\alpha} \Big|_{\bar{\alpha}}, \quad \frac{d^2 f}{d\alpha^2} \Big|_{\bar{\alpha}} < 0 \quad (26)$$

and, comparing the expressions (18) and (25), to the following relation:

$$1 - q + \varphi_q = f(\bar{\alpha}) - \bar{\alpha} \frac{df}{d\alpha} \Big|_{\bar{\alpha}}. \quad (27)$$

The relation (27) expresses the connection between the rank dependence $\varphi(q)$ of the *intermittency indices* with the (singularity index) $\bar{\alpha}$ dependence of the *multifractal spectrum* $f(\bar{\alpha})$. This relation leads to the following useful identities derived from (26) and (27):

$$\frac{d\varphi(q)}{dq} = 1 - \bar{\alpha}(q); \quad \frac{d^2 \varphi(q)}{dq^2} = - \frac{d\bar{\alpha}(q)}{dq} = \left[\frac{d^2 f}{d\alpha^2} \Big|_{\bar{\alpha}} \right]^{-1} \quad (28)$$

which are typical of the Legendre transformation between $\varphi(q)$ and $f(\alpha)$. It is also possible to express directly $f(\alpha)$ as a function of $\varphi(q)$. By a suitable combination of (27) and (28), one finds [71] the interesting relation:

$$f(\bar{\alpha}) = -q^2 \frac{d}{dq} \left[\frac{\varphi(q) + 1}{q} \right], \quad (29)$$

which has an appealing thermodynamical interpretation. Indeed, $f(\bar{\alpha})$ which, by definition (24), is the exponent for the exponential number of cells with singularity index $\bar{\alpha}$, can be interpreted as the "q-local" entropy associated with this set of cells. The terminology "q-local" refers to the subset of cells which contributes to $Z(q)$ at q-fixed and determines the corresponding saddle-point $\bar{\alpha}(q)$. Keeping in mind from Eq. (20) that $-q^2 d/dq = d(1/q) = d(T/T_0)$, one finds that, up to an additive factor, the quantity

$$\lambda(q) = (\varphi(q) + 1)/q \quad (30)$$

is nothing other than the "q-local" free-energy density. In fact, relation (29) represents the "q-local" generalization of the thermodynamic relation between entropy S and free-energy E , namely $S = -dE/dT$. It is to be noted that the multifractal description allows one to extend the thermodynamical formalism to non-equilibrium systems. This description fits with the Boltzmann expression (22). Moreover, in this case, the "q-local" relations can be explicitly expressed [54] in terms of the probability distribution $\pi(\epsilon)$ or $r(W)$ and the connectivity of the tree structure, leading to an analytic derivation of the multifractal spectrum $f(\alpha)$.

Another noticeable interest of relation (29) is the connection it establishes between the existence of singularities α_c with fractal dimension $f(\alpha_c) = 0$, and the existence of a minimum of the free-energy. In a thermodynamic system, this is the expected signal for a *phase transition*. Thus, leaving for future analysis the nature of this specific phase transition, one can propose [20] a criterion for a phase transition based on the properties of intermittency indices, through the equation for $\lambda(q)$ of Eq. (30). This fact, also emphasized in a direct study of the α -models [72], is another illustration of the deep relations which may exist between the *local* structure of fluctuations and the *global* structuration of a system, particularly when the system meets a phase transition. This is quite clear when one applies the intermittency analysis to a typical second-order phase transition, as will be shown in Section 4.2. However, much more general types of transition or phase mixtures [73] may appear, even during random cascading, as discussed in Section 4.3. The comparison between "ordinary" phase-transitions and random cascading will be developed in Section 4.4.

4.2 Intermittency in phase transitions at equilibrium

Originally, the word "intermittency" was proposed to characterize the presence in a (turbulent) medium of islands of a laminar phase, and their presence at all scales for which the turbulence is observed. It is thus natural to pose the same problem in a typical situation where two phases are involved, namely a second-order phase-transition. Indeed, the problem of the existence of clustering at a transition from one phase to the other is old and well-known, but the first application of the factorial moment method to a second-order phase-transition and the search for intermittency patterns is rather recent [74]. Besides its interest for Statistical Mechanics problems, the possibility of extending the factorial observable method to order-disorder transitions, including those reproducible on computers with high accuracy, allows one to discuss in detail some left-over questions, such as

- (i) Do intermittent patterns appear, and if yes, do they appear exactly at the transition ?
- (ii) Can one relate the intermittency indices to critical indices ?
- (iii) Are the thermal fluctuations e.g. of a system of Ising spins, compatible with Bernoulli or Poissonian noise ?

In the case of Particle Physics, one would think of applying this discussion to the QCD phase-transition between a hadronic phase and a quark-gluon plasma, which is the subject of great efforts in the study of heavy-ion collisions.

Let us consider first, as in Refs. [74-77], the simpler problem of a two-dimensional lattice of Ising spins, taking $\uparrow$ (up) and $\downarrow$ (down) values with a fixed interaction between nearest neighbours, see Fig. 9. As a first attempt, one could choose the local magnetization in a box m of size δ , as being the analogue of the number of particles in a rapidity bin. For symmetry preserving reasons one defines [76, 74]:

$$K_m = \sum_{\{\text{all spins in } m\}} \frac{1}{2} (1 + \text{sign}(s) \sigma_i), \quad (31)$$

where σ_i is $+1$ (resp. -1) for spins up (resp. down), and $\text{sign}(s)$ is the sign of the total spin of the block in the considered random configuration. One then uses the definition (31) in the formula (2) for factorial moments. The numerical and analytical results [74-76] can be as follows:

- (i) Fluctuation patterns compatible with intermittency appear at the transient temperature; e.g. there is a *critical intermittency*.
- (ii) The intermittency indices can all be related to the critical indices, and correspond to a fixed Renyi dimension. In the 2-D Ising case, one expects $\bar{D} = 1 - D = 1 - \frac{7}{8} = \frac{1}{8}$.
- (iii) The statistical noise, which is purely thermal in this problem, is quite consistent with Poissonian noise in the (low-temperature) ordered phase; however, some finite-size effects remain after the factorial moment method has been applied, showing the existence of non-Poissonian, non-critical fluctuations. However, their effect is lowered enough to reveal maximal intermittency at the critical temperature. In fact, quite recently, the noise was demonstrated to be of Bernoulli-type*.
- (iv) The intermittent fluctuations are due to the large (infinite) extension of dynamical correlations at critical temperature.

The fact that intermittent fluctuations are generated at the critical temperature can be shown [75] to be a universal feature of second-order phase transitions and is ultimately related to their scale invariance properties. However the relation with the dynamical correlations and the value of the critical Renyi dimension have been found [77] to be non-universal and dependent on the definition of the intermittent phase in terms of clusters. For instance, let one replace the definition (31) by the summation

* See I. Novak, preprint in Ref. [76].

over spins of the largest connected cluster in the box $[m]$, see Fig. 9, namely:

$$K_m = \sum_{\text{largest connected cluster in } [m]} \frac{1}{2}(1 + \text{sign}(s)\sigma_i) \quad (32)$$

In this case, the solution derives [61] from a Potts-model Hamiltonian, with a phase transition at the same temperature as the original Ising system, but with a different Renyi dimension, $D = 91/96$, leading to intermittency indices $\varphi_q = 5/96(q - 1)$. The origin of this somewhat puzzling property is that definition (32) corresponds to a typical *percolation* problem. Going from the high-temperature phase through the critical point, clusters of ordered spins are formed at different scales and aggregate at the critical temperature to form the infinite (up to the lattice finite-size) cluster of percolation. In this critical percolation model, dynamical factors due to the Ising correlations and geometrical ones related to percolation are mixed (note that percolation exists also outside of criticality!). It is a beautiful property of two-dimensional systems [78] that both dynamical and geometrical aspects of critical percolation can be described - and investigated analytically - by the Potts Hamiltonian. One has to conclude that, if a second-order phase transition determines the monofractal nature of the intermittency patterns ($\bar{D} = \text{cste}$), it does not determine unambiguously their geometrical properties and the values of the intermittency indices ($\varphi_q = (1 - \bar{D})(q - 1)$).

It is to be noted from the literature [79, 80] that percolation processes favour the appearance of intermittency patterns, even when they are not associated with a phase transition. For instance, the study of the one-dimensional non-critical Ising spin system with an external magnetic field reveals [79] the existence of intermittency patterns for certain values of dynamical couplings. They appear by studying the clusters of spins with opposite sign to the magnetic field. Interestingly enough, intermittency patterns appear also in another physical system, low-energy nuclear multi-fragmentation [80], for which a non-critical percolation model gives sensible predictions. One could show that the statistical mass repartition of fragments in the breaking of a gold nucleus exhibits the power-law of factorial moments. Moreover, a percolation model on a small lattice where each site is occupied by a nucleon reproduces the observed patterns.

In the ultrarelativistic regime of elementary particle physics, one is tempted to introduce the notion of a dynamical percolation between the perturbative QCD vacuum (of quarks and gluons) towards the vacuum phase of hadrons. However no such percolation model yet exists for collisions. Phenomenological models based on the hypothesis of a Feynman-Wilson gas of particles [81] with a second-order phase transition lead to a monofractal intermittency behaviour. However, one would like to establish a bridge

between the intermittency phenomena and the existence of a specific phase transition in Quantum Chromo Dynamics. It would thus be urgent to treat, at least numerically, the geometrical properties of fluctuation patterns at the QCD phase transition at finite temperature. Since the transition is probably (weakly) first-order [82], the previous discussion does not fully apply and the problem deserves attention in the near future.

4.3 Random cascading and "non-equilibrium" phase transitions or mixtures

The fact that intermittency patterns are related to scale-invariant processes of random cascading has been checked in hadronic cascading models [40], and in the perturbative QCD description of parton cascading [51-53]. Note that in these two cases, one has examples of "monophasic" intermittent cascades, without mixing or transition between partons and hadrons (indeed the hadronization of the parton cascade is shown to lead to a weakening of the fluctuation structures [51], except perhaps at LEP energies [26].) A general argument for intermittency based on scale invariance can be presented [2, 24] as follows: let $P(L)$ be a density of objects (particles) in a box of dimension L . Then suppose that there is a scale-invariant random relation between this density and the one inside a box of reduced dimension α with α fixed between 0 and 1, see Fig. 10. One would write:

$$P(L) = P(L\alpha_1)W(\alpha_1) = P(L\alpha_1\alpha_2)W(\alpha_1)W(\alpha_2) = \dots = P(L)W(\alpha_1)W(\alpha_2)\dots W(\alpha_n), \quad (33)$$

where $\frac{L}{L'} = \alpha_1\alpha_2\dots\alpha_n$ is the length reduction factor after n steps in the cascade. One recognizes in (33) the same formula as the relation (8) of α -models. Scale invariance is here ensured by the important property that random weights W 's do not depend on the scale at which they operate, namely: $P(L\alpha_1\dots\alpha_{n-1})/P(L\alpha_1\dots\alpha_n) = W(\alpha_n)$, with W , depending only on α_n and not on the other scale factors α_i , $i \neq n$.

However, even if each branch of the cascade can be described by the stochastic product of random numbers of Eq. (33), the geometry of the random cascading process may induce strong correlations between densities in different bins. Thus, in the case of random cascading models, and not only for ordinary phase transitions, one expects a relation between the local excitations, described by (33), and the global properties of the system. The problem is to exhibit this specific relation and to understand its meaning.

Let us consider an explicit example consistent with the geometry of branching which is typical of the α -models of intermittency (see Fig. 2 and Section 2.3). In this

case, one has three free parameters, namely the branching fraction λ (all $\alpha'_i = \lambda^{-1}$ in Eq. (33)), and the two independent parameters describing the probability distribution of the random weights $W(\alpha_n)$. Namely, one has:

$$\begin{aligned} \tau(W) &= p\delta(W - a) + (1 - p)\delta(W - b); \\ 0 \leq a \leq 1 \leq b &: pa + (1 - p)b = 1, \end{aligned} \quad (34)$$

the last relation being due to the normalization conditions (7). The interpretation of this distribution is straightforward. At each step of the cascade (33), there is a probability $(1-p)$ of a density enhancement $b > 1$. This enhancement has to be compensated by a damping factor $a < 1$ with probability p . The relevant quantities, such as the intermittency indices φ_q , Eqs. (4), (10), the Renyi codimensions (Eqs. (5), (19)), the factorial correlators (Eq. (11)), the multifractal spectrum (Eq. (29)), can be explicitly computed [71] or simulated [22] numerically. One recovers all the intermittency properties, and exhibits the global properties of this class of random cascading models as a function of the parameters.

Interestingly enough, a non-trivial classification of α -models is found [71] as a function of the parameters of the model. Depending essentially on only two parameters, p and λ , one finds four main subdivisions (with two bi-critical lines separating them and one quadri-critical crossing point). In part IV of the parameter diagram (see Fig. 11), one finds random cascading α -models without phase transition. In fact, they can precisely model percolation systems without transition. A novel aspect of random cascading models appears for the range of parameters I, II, III, (see Fig. 11). They can be interpreted [71] as random cascading with phase mixing (or transition) during the cascade. There are two types of such transitions: a "peak transition" in region I, a "dip transition" in region II and a mixture of both features in region III. The "peak transition" consists of strongly correlated peaks in the density over a fluctuating background. Correlated dips appear for the "dip transition". In general, these transitions cannot be interpreted as order-disorder phase transition since they do not correspond to monofractal properties ($\bar{D} \neq \text{cte}$). Indeed the thorough study of similar models for spin-glass systems [66-69] allows one [83] to investigate the properties of these different phases. In particular, it is not required that all the system be correlated like in a phase transition at equilibrium. In the so-called "spin-glass phase", a collection of quasi-degenerate ground states emerge from the random interactions of a regular lattice [84], or of a tree structure [66-69].

In Particle Physics, the relevance of such a spin-glass type phase is a challenge [83]. Interpretations of the observed NA32 "anomalous" event [9] and the morphological

difference between density fluctuations [83, 24] could come from the existence of a phase transition. More generally, the structure of the correlations in the distribution of the observed particles can serve as a test of the existence of hidden phases. For instance, it has been noticed that relation (11) between two-particle factorial correlators and the factorial moments is modified in a well-defined way when reaching a transition [83]. In the intermittent phase, one may write:

$$\frac{C_{pq}^\delta}{F_p^\delta F_q^\delta} = \left(\frac{\Delta Y}{DY} \right)^{\varphi_{p+q} - \varphi_p - \varphi_q} = \frac{F_{p+q}^D}{F_p^D F_q^D} \quad (35)$$

where m, n are two different bins separated by the distance D , with bin size δ . But relation (35) no longer holds in the spin-glass phase. Indeed, the difference between the two expressions can define [83] an order parameter function, which is the analogue of the overlap function $Q(x)$ of spin-glass calculations. Here $x = \frac{DY}{\Delta Y}$ denotes the fraction of highly correlated domain in rapidity space (in Statistical Mechanics x is the fraction of correlated configurations [84]).

4.4 Application to hidden phases in particle collisions

The calculations based on α -models are very preliminary, as far as the determination of a phase diagram of random cascading processes is concerned. However, they could be of fundamental and practical interest in the search for hidden states of hadronic matter in various collisions. For instance, the actual quest for a quark-gluon plasma phase in heavy-ion collisions at ultra-relativistic energies is asking for clear signals of the plasma formation in head-on collisions. Signals such as the detection of large interaction radii by Bose-Einstein interferometry [85] and, more promisingly, the relative suppression of ψ -production with respect of the di-muon pair background [85] are actively studied. However, there exist alternative interpretations of the observed effects based on non-plasma mechanisms [85]. One is thus interested in discussing the presence of hidden phases through the structure of fluctuation patterns. This idea is not new [86], but the studies of factorial observables allow a more systematic approach of the relation between intermittency patterns and an eventual QCD phase transition in the collision of heavy ions.

Indeed, the discussion [54,55] is based on the difference [20] between intermittent pattern at a second-order phase transition (cf. Section 4.2) and those coming from random cascading (cf. Section 4.3). The comparison with data [87] is in favour of a quark-gluon plasma, after a phenomenological analysis of Renyi dimensions. They are expected to be constant in this case (monofractal intermittency) while they should be

approximately linear as a function of the rank for multifractal intermittency related to random cascading.

In order to give a more quantitative basis to this analysis, let us review and generalize some approximations of random cascading processes when the number of steps is large [14]. Coming back to expressions (8), (33) for local fluctuations generated by random cascading, and using the same change of variables as defined in relations (21), (22), one may write:

$$\prod_{s=1}^{\nu} W^q(\alpha_s) = \exp(-\beta_0 q \sum_{s=1}^{\nu} \varepsilon_s) \sim \exp(-\beta_0 q \varepsilon), \quad (36)$$

where the random number ε is governed by a Gaussian law, $\pi(\varepsilon) \propto \exp(-\frac{\varepsilon^2}{2\sigma})$. This is a consequence [14] of the central limit theorem applied to the "link energy" variables, cf. Section 4.1. For the normalized factorial moments and correlators, one finds [14]:

$$F_q = \left(\frac{\Delta}{\delta}\right)^{\frac{q(q-1)}{2}}, \quad C_{qp}/F_q \bar{F}_p = \left(\frac{\Delta}{D}\right)^{qp f_2}, \quad (37)$$

from the application of the Gaussian law to expressions (10) and (11), (with the same definitions of the binsize (δ), the distance between bins (D) and the total range (Δ). One thus gets for the Renyi codimensions (see Eq. (5))

$$\bar{D}_q = \frac{q}{2} f_2, \quad (38)$$

indicating a multifractal property ($\bar{D}_q \neq \text{cste}$) of random cascading.

However, the validity of the central limit theorem for the logarithm of a multiplicative random distribution is to be questioned [55,88]. Indeed, the α -models themselves, though belonging to the class of random cascading processes, do not lead to the equation (38), even asymptotically. Even more, there is a particular type of model, considered for $a = 0$ in Eq. (34), which leads to a monofractal distribution, as first proposed for hydrodynamic turbulence (the β -models [6]). In order to improve the results of the approximation of a large number of steps, one may use diverse generalisations of the central limit theorem [55,88]. For instance [55], it is possible to find other stable limits of the sum $\varepsilon = \sum_{s=1}^{\nu} \varepsilon_s$, leading to Levy-stable probability laws $L_{\mu}(\varepsilon)d(\varepsilon)$. In fact, Levy-stable laws generalize the well-known fixed-point property of Gaussian distributions which remain Gaussian by summation. For the Levy-law approximation, one

finds [55]:

$$F_q = \left(\frac{\Delta}{\delta}\right)^{f_2 \frac{q^{\mu}-1}{\mu-1}}; \quad C_{qp}/(F_q)^2 = \left(\frac{\Delta}{D}\right)^{qp f_2}, \quad (39)$$

with the Levy index μ being an adjustable parameter. Compared to the log-normal case (37), the Levy index μ allows a natural interpolation between the monofractal case ($\mu = 0$; $\bar{D} = f_2 = \text{cste}$) and the log-normal approximation ($\mu = 2$; $\bar{D} = q f_2/2$). In fact, the known results of Renyi codimensions in μp reactions [34] and heavy-ion reactions [30], parametrized by Levy-law expressions [55], lead to a difference between the two types of reactions, see Fig. 12. However, it appears better to study the factorial correlators, which are predicted to be drastically different in the two above-mentioned reactions, (cf. Fig. 12). Note that some amount of cascading in heavy-ion reactions could be present even in the presence of a quark-gluon phase. This possibility could be indicated by a moderate value of the Levy index μ ($0 \leq \mu < 1$).

5. IN FINE: ASPECTS OF INTERMITTENCY IN "CLASSICAL" SYSTEMS AND PARTICLE PHYSICS

There are several physical systems in which one interprets fluctuating patterns in terms of intermittency [89]. Besides the well-known arhythmic irregularities of heart beats - arhythmia means intermittency in Serbian [90] -, the origin and the main developments on intermittency are to be found in hydrodynamic turbulence. Starting with the basic Kolmogorov theory of fully-developed turbulence [91] based on the assumption of a full turbulent medium, it was soon recognized, both from observational [92] and theoretical [93] reasons, that a turbulent medium cannot always be turbulent everywhere, leaving room for small random proportions of laminar flow at different scales, i.e. intermittent patterns. Approximately, the fractal Renyi codimension of intermittency is of order 0.2 in three-dimensional space, but the distribution is in fact multifractal and not compatible with a log-normal distribution for high moments [94].

Studying the properties of a turbulent flow, B. Mandelbrot proposed, in 1974, the general structure of multiplicative random weights, following a three-dimensional tree structure in real space [4]. As an illustration of intermittency, a model with full turbulence mixed with spots of fully laminar flow was proposed [6], leading to a monofractal structure describing the turbulent flow. However, both the experimental inadequacy of the monofractal description and the problem of the divergence of moments (when the number of steps goes to infinity) led to the proposal of α -models with multifractal behaviour [7].

By comparison, the physical situation in hydrodynamical turbulence seems quite different than in particle physics (even if similar techniques appear to be relevant). In particular, the energy density flow goes from high spatial scales to very small eddies, contrary to particle physics where it goes from short to large scales. Indeed, small rapidity intervals (in momentum space) correspond via quantum mechanical relations to large objects (in position space). So, the source of ultrarelativistic intermittency is different from the classical turbulence.

A different intermittent behaviour is obtained by driving systems from a regular, periodic behaviour to a chaotic regime, i.e., the onset of turbulence. There are different routes to chaos in various dynamical systems, such as the well-known period-doubling bifurcations of orbits [95]. This behaviour, with spatial chaotic behaviour, has been met in physical systems with tight boundary conditions, such as Rayleigh-Bernard rolls in small cells. However, it was conjectured from a theoretical analysis [96] and verified experimentally that a different route to chaos, with spatio-temporal intermittency, can be found. In this case, the boundary conditions are rather loose and a larger number of degrees of freedom are involved. Few application of these properties to particle physics has yet been made [10]. Note also the simulations of a Schwinger mechanism for intermittency [63] (see Section 3.5), which seem to possess some non-linear properties analogous to dynamical chaotic systems.

After all, if one believes that Quantum Chromo Dynamics is the best candidate for a theory of Strong Interactions. It should provide a suitable framework for intermittency. However, there is a long way towards the solution of the problem, which involves the highly non-perturbative regime in QCD; let us list a few theoretical questions addressed to a quantum field theory like QCD which could pave the way to a satisfactory solution.

- (i) Is it possible to exhibit solutions of a quantum field theory non-Poissonian statistics ? This problem could perhaps be first analysed in the following framework of two-dimensional field theory. One already knows that QCD₂ with non-massive quarks gives a Poissonian distribution of $q\bar{q}$ pairs [97], corresponding to coherent states [98] and the nature of confinement in this case. Could it be different for massive quarks ? And for QCD in four dimensions ?
- (ii) It has already been noticed in Section 4.3 that spin-glass models have solutions with intermittent structure. Since the typical model of spin-glasses corresponds to the solution of a Hamiltonian of random interactions on a lattice [84], could it be possible to build a spin-glass model for QCD ? What is the origin of randomness in this framework ? One interesting possibility could be that the perturbative QCD stage of parton cascading gives stochastic initial conditions to QCD lattice

calculations.

- (iii) In the phase diagram revealed by the structure of α -models [83] there appears three different phases, cf. Fig. 13. One phase (A) corresponds to the (ordinary) intermittency patterns random cascading. Let us call it the "hadronic phase". The second phase (B) does not correspond properly to a frequent state of the system, but is related to the rare occurrence of big spots or spikes in the distribution [73]. One would consider these "anomalous events" as good candidates for exceptional events encountered in some reactions [9]. The nature of the third phase (C) corresponds really to a more frequent situation of the system under certain conditions. One can speculate that this phase could indeed correspond to asymptotically free quarks and gluons and thus represent the perturbative QCD vacuum. Following such a picture, the intermittency patterns would arise from a percolation from this vacuum (C) to the vacuum of asymptotically confined states (A).

However, as difficult as it may be, the problem of connecting quantum field theory to a manifestation of chaos is new and exciting enough to justify efforts. It is a chance that particle collisions could be a laboratory for this conceptual challenge. At least, the dynamics of particle interactions at strong coupling can reveal new properties.

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FIGURE CAPTIONS

Fig. 1. Intermittency: first evidence.

- a) *Particle number distribution* observed by the JACEE Collaboration for a 5 TeV/nucleon, Si + Emulsion collision, from Ref. [3]. The minimal bin size is .1 in pseudorapidity unit;
- b) *Factorial moment of rank 5*. The factorial moments obtained from the data shown in Fig. 1a (black dots) are compared with a simulation with Gaussian statistical noise (crosses). The straightline is a typical prediction for intermittent-like fluctuations. The figure is from Ref. [1];
- c) *Intermittency at accelerators*. The factorial moments of rank 2 and 4 are shown (black squares) and compared with a simulation with only statistical fluctuations (white squares) for 16O-emulsion experiment at CERN. The straight lines correspond to the prediction of intermittent fluctuations. The figure is from Ref. [25].

Fig. 2. α -model representation of Intermittency.

In Figs. 2 a,b,c, one finds a schematization of the random cascading models leading to the α -model definition of intermittency.

a) *Cayley tree structure of intermittency*

The Cayley tree gives a one-to-one correspondence between a given bin $[m]$ and a path corresponding to successive partitions of the initial phase-space interval. The total number of bins M and thus the resolution is related to the number ν of considered cascading steps, namely $M = \lambda^\nu$. In the figure, one has shown (half of) the Cayley tree structure with $\nu = 4, \lambda = 2$. At each node, an integer α_s ($\alpha_s = 1$ or 2 ; $s = 1$ to ν) indexes the chosen branch. The hatched path is for the bin $[m]$.

b) *Box diagrams of intermittency*

The initial phase space is subdivided in boxes following the Cayley tree scheme of Fig. 2a. For each phase-space box, one randomly chooses a factor $W(\alpha_s)$. Thus, for each bin $[m]$ one builds the corresponding probability weight $P_m = M^{-1} \prod_{s=1}^{\nu} W(\alpha_s)$. In the figure, for simplicity, the random factors are simple enhancement ($W_+ > 1$) or damping ($0 < W_- < 1$) factors, denoted respectively by randomly chosen indices $+$ and $-$. The hatched boxes correspond to the bin $[m]$.

c) *Intermittency pattern (example of...)*

Using the Cayley tree structure of Fig. 2a and the randomly chosen factors of Fig. 2b, are displayed the resulting weights fluctuating from bin to bin. The hatched region is for the bin $[m]$ with probability weight $P_m = \frac{1}{M} W_+^2 W_-$. (Note that a

simulation of the statistical noise would relate P_m to an observed bin multiplicity K_m by a randomly chosen Bernoulli factors; see Fig. 4).

Fig. 3. Space-time interpretation of intermittent fluctuations

In Fig. 3a-b, one shows the space-time frame in which a collision takes place, at least projected on the (t, z) plane, namely (time, longitudinal distance). The causal cone ($t \pm z \geq 0, t \geq 0$) is displayed, together with causal hyperbolae, $\tau \equiv (t^2 - z^2)^{1/2} = cste$, where τ is the proper time. The figures are extracted from Refs. [2].

a) *In-out picture of hadron production*

The figure shows the conventional picture of hadron production after a high-energy collision. At the origin (O) an interacting "string" of partonic or hadronic matter (hatched region) is found. After a proper time τ (of order 1 fermi) of relativistic expansion, the "string" breaks into pieces measured (in average) by the conventional correlation length ξ , giving rise to hadrons.

b) *Random cascading in space-time*

In the $(1+1)$ space-time frame, Fig. 4b shows the (random) generation of intermittent fluctuations following the logical structure of the α -models, Fig. 2, namely a random cascading model of hadronization. At each step of the cascade, one iterates the process shown in Fig. 3a. A system (or string) is stretched by the relativistic expansion to a length larger than the correlation length ξ , and breaks into λ pieces with fluctuating density ρ . The fluctuations appear at successive proper-time values τ_s leading to structures at different values of rapidity size $\delta y = \xi/\tau_s$. In the figure, one has chosen $\lambda = 2, \tau_s = 2^s$, and random factors W_+ and W_- as in Figs. 2.

Fig. 4. Simulation scheme for intermittent fluctuations

The figure represents an "upside-down" mechanism, able to generate intermittent-like fluctuations (from ref. [22]).

a) In a first stage, one generates a realization of the random cascading model sketched in Fig. 2. At the end of the cascade of ν steps which is shown in the left part of the figure with 4 last steps, the probability weight P_m are obtained for the smallest chosen bin size.

b) At this stage, for each obtained value of P_m one uses the Poisson-law $Q(P_m, K_m; < K_m >)$ to get a random realization of particle multiplicity K_m in the same bin, with known given values of the average $< K_m >$, and the probability weight P_m .

Then, one sums up the multiplicity numbers upward, in order to calculate the factorial observables at the desired scale.

Note that the effects of partial intermittency (ν small) and truncation of the multiplicity per bin ($K_m < K_{max}$) can be studied by this type of simulation, see ref. [22].

Fig. 5. Some experimental results on intermittency

The figures shows typical factorial moment data obtained recently in hadron-hadron, nucleus-nucleus, $\mu^- p$ inelastic and $e^+ e^-$ annihilations at high energies. The figure is from ref. [24] where one can find detailed references.

a) *Intermittency study in $p\bar{p}$ collisions from Ref. [31]*

The data on the third factorial moments is compared with three different models through Monte-Carlo simulations: the Lund model PYTHIA, the same with Bose Einstein Correlations and the GENCL simulation with clusters of short-range correlations. Note that this last one simulates the growth of moments till $\delta\eta = .3$ but fails to reproduce the observed linear trend in the log-log plot, compatible with an Intermittency signal (the references on Monte-Carlo simulations can be found in Ref. [31])

b) *Intermittency study in two dimensions, Nucleus - Nucleus reaction, from [77]*
The moments F_2 to F_3 are given for a two-dimensional projection of the total phase space for a nucleus-nucleus collision, namely in squared boxes of pseudo-rapidity $\times$ azimuth: $\delta_\eta \delta_\phi$. Note the larger variation of moments with respect to one-dimensional data (compare e.g. with Fig. 1c).

c) *Intermittency study in μp deep inelastic scattering, from ref. [34]*

d) *Anomalous Renyi Codimensions (μp and $e^+ e^-$ reactions), from [24]*

The figure displays the Renyi codimensions $\bar{D}_q = \varphi_q/(q-1)$, for $q = 1$ to 5, see text, formula (5).

Fig. 6. Intermittency studies at LEP (DELPHI coll.).

The figure shows recent data on factorial moments together with their approach by Lund model simulations, from Ref. [49]. In all cases M denotes (in abscissa) the total number of bins, at each bin size.

a) *One-dimensional study (rapidity)*

The data on factorial moments of rank 2 to 5 are compared with different Lund model simulations; Lund model with QCD parton cascading (dark line) and QCD matrix elements (dashed lines) (References to Monte-Carlo simulations can be found in [49].)

b) *Two-dimensional study (rapidity $\times$ azimuth)*

The data (white circles) are compared to the Lund model simulations with QCD parton cascading (black crosses).

Fig. 7. Intermittency patterns from Schwinger pair-tunnelling

The figure displays one simulated event from Ref. [62]. A chromoelectric field produces $q\bar{q}$ pairs through Schwinger tunnelling mechanism extended to QCD in a $(1+1)$ dimensional semi-classical approximation (see Refs. in Ref. [62]). The pairs created follow a random cascading process of successive tunnellings. Note that one takes into account the feedback effect of pair production on the field: the dipole field generated by the $q\bar{q}$ pairs is (algebraically) added to the initial configuration.

a) Cascading in rapidity space

From bottom to top, the proper-time (τ) development of the cascade is shown in the figure for the produced $q\bar{q}$ pairs as a function of the rapidity of the pairs. The pairs have been called $yo-yo$ because of their movement in space-time (see Fig. 7b). Note the obtained pattern of a *randomized* Cayley tree similar to Fig. 2a.

b) Cascading in space-time

The figure shows the space-time development of the quarks and antiquarks for the same simulated event as previously. *Quasi-rapidity* defined as the analogous of rapidity in space-time (t, z); namely

$$\text{rapidity} = \frac{1}{2} \log \frac{E + P_L}{E - P_L}; \quad \text{quasi-rapidity} = \frac{1}{2} \log \frac{t + z}{t - z}.$$

Note the $yo-yo$ moment of the pairs, when they are not interacting. Interacting $yo-yo$'s give raise to interesting *non-linear* effects with aggregation zones of pairs separated by voids. They are non-linear since they depend on the "history" of the event and of the relative location of previous pair creations by tunnelling. (See Ref. [63] for a discussion of these effects). These patterns could be candidates for a non-linear process of pre-confinement in QCD.

Fig. 8. Partition function of a complex system

The figure shows the example of one random realization of a set of fluctuations of a two-dimensional complex system. The phase space binning is (here) 7×7 . For each of the 49 bins $[m]$, one considers the value P_m of the fluctuation, here represented by black dots. Two expressions for the partition function are given, following the text, cf. formulae (16) and (24). In the figure, M is the total number of bins, N_{α_m} the number of bins for which the fluctuating P_m verify the same scaling law $P_m \sim M^{-\alpha_m}$.

Fig. 9. Intermittency from two-Dimensional Ising-clusters

The figure represents a domain $[m]$ of 7×7 sites inside a two-dimensional Ising system at the transition. The configuration of spins up ($\uparrow$) and down ($\downarrow$) is typical of a random realisation of the model.

As an example, one gets a contribution of $K_m = 24$ to the cluster definition (31) and $K_m = 7$ to the definition (32). (32) corresponds to the largest connected percolation cluster in the subset $[m]$ shown in the figure by the hatched domain while (31) corresponds to the sum over all spin-ups.

Fig. 10. Intermittency generated by a scale-invariant random process

The figure represents a scale invariant multiplicative random process corresponding to formula (33). Starting with random realizations in the boxes (1) and (1'), one generates two independent random cascades of boxes from (2) to (4) and (2') to (4') with corresponding weights $P_4 = W_4 W_2 P_1$ and $P_4 = W_4 W_2 P_1$. Averaging over the random realizations of similar cascades, one can compute the q -moment, $\langle P_4^q \rangle$ and the (qq') -correlator, $\langle P_4^q P_4'^q \rangle$.

Fig. 11. Multifractal analysis of α -models

The figure shows the various types of multifractal spectrum obtained from the α -models, defined by formula (34), in the text. The figures are from Ref. [71].

a) Typical multifractal spectrum $f(\alpha)$

In abscissa, the index α of singular behaviour, namely $P_m \sim M^{-\alpha}$, $\alpha \in \bar{\alpha}_+, \bar{\alpha}_-$, with $N_m \sim M^{f(\alpha)}$ (see formula (25)). Note that the values q of the momentum rank can be followed along the curve from $-\infty$ to $+\infty$. The dotted line corresponds to the analytic continuation $f(\alpha)$ after the phase transition while the actual free energy is fixed at $f(\alpha_c) = 0$.

b) Multifractal classification of α -models

Depending on the value of the parameters λ and p , (cf. formula (34)), one finds four classes of multifractal spectra: spectra with a "peak transition" (region I), a "dip transition" (region II), both of them (region III), or none of them (region IV). The central (quadratic) case is added for completeness.

Fig. 12. Intermittency anomalous dimensions and Levy stable laws

In the main figure, one shows the Levy parametrization for the anomalous Renyi dimensions in two different cases (from Ref. [55]).

i) Nucleus-nucleus reactions

The Renyi codimension is almost constant, corresponding to a Levy index μ with $0 \leq \mu \leq 0.55$. This behaviour is typical of intermittency at a phase transition and thus has been advocated [54] in favour of the formation of a quark-gluon plasma.

ii) $\mu - p$ Deep inelastic scattering

The Renyi dimension is substantially growing with the rank, which is an indication of a random cascading process. The Levy index is large, $\mu \sim 3.2 - 3.5$. In the inserts, one shows the typical prediction to the behaviour of correlators.

Namely, $C_{q,q} \propto (M)^{\varphi(2q)-2\varphi(q)}$, see formula (11), with M the total number of bins and $\varphi(q)$ the intermittency index of order q . Note the important dependence of the exponent $\varphi(2q) - 2\varphi(q)$ on the Levy index μ , opening the possibility of a precise determination of the Levy index.

Fig. 13. Phase diagram of random cascading

In the figure one represents the three possible phases of a random cascading system (α -model). The coordinates (p, q) correspond to following definition:

$$\langle Z_{(q)}^p \rangle = \left(\left\langle \sum_m p_m^q \right\rangle^p \right).$$

Phase A corresponds to the intermittent regime of the α -model.

Phase B corresponds to anomalous configurations of very rare events with large excitations.

Phase C is the "spin-glass" type phase, for which the partition function is not self-averaging.

Each of these phases is characterized by given values of the free-energy $\lambda(q)$, see Eq. (30). These values are given in inserts.

FIG. 1

Random cascade in an expanding
ultra-relativistic system

FIG.3

FIG. 2

a)

b)

c)

d)

FIG. 5

FIG 4

FIG. 7

b)

FIG. 6

$$Z(q) = \sum_{(m)} P_m^q = \sum_m \mathcal{N}_{\alpha_m} M^{-\alpha_m q}$$

FIG. 8

Ising magnetisation: $K = 24$
Percolation cluster: $K = 7$

FIG. 9

FIG. 11

FIG. 10

FIG. 12

FIG. 13