CERN-TH. 5826/90

THE UNIVERSAL LINK POLYNOMIAL

E. GUADAGNINI*

CERN - Geneva

Abstract

The solution of the non-Abelian $SU(N)$ quantum Chern-Simons field theory defined in R^3 is presented. It is shown how to compute the expectation values of the Wilson line operators, associated with oriented framed links, in closed form. The main properties of the universal link polynomial, defined by these expectation values, are derived in the case of a generic real simple Lie algebra. The resulting polynomials for some simple examples of links are reported.

* *On leave from Dipartimento di Fisica dell'Università di Pisa and INFN, Sezione di Pisa*

Address after January 1, 1991 : Dipartimento di Fisica, Università di Pisa , Piazza Torricelli 2, 56100 Pisa, Italy.

CERN-TH. 5826/90

November 1990

TABLE OF CONTENTS

1. Introduction
2. Basic Notions of Knot Theory
3. Framing in Field Theory
4. Non-Abelian Chern-Simons Theory
5. Wilson Line Properties
6. Braid Group
7. R -Matrix and Braids
8. Chern-Simons Monodromies
9. Defining Relations
10. The Extended Jones Polynomial
11. General Properties
12. Group $SU(N)$
13. Conclusions

1. INTRODUCTION

Examples of soluble non-trivial quantum field theories in more than two space-time dimensions are not easy to find. In the present paper, a particular class of non-trivial three-dimensional gauge theories defined by the Chern-Simons (CS) action are considered and their explicit solution is presented.

The CS action is

$$S_{CS} = \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} \text{Tr} \left(A_\mu \partial_\nu A_\rho + i \frac{2}{3} A_\mu A_\nu A_\rho \right), \quad (1.1)$$

where $A_\mu = A_\mu^a T^a$, $\{T^a\}$ being the Hermitian generators of a compact simple group G in its defining representation. We consider the case in which the integral appearing in (1.1) is performed on a three-manifold of the same topological type of R^3 . In what follows, the ambient space is taken for simplicity to be precisely R^3 . Local gauge invariance and general covariance are the fundamental symmetries of the model. The basic observables of the CS theory are the expectation values $\langle W(L) \rangle$ of Wilson line operators associated with oriented links $\{L\}$. Given an oriented knot C and an irreducible representation ρ of G , the associated Wilson line operator is

$$W_\rho(C) = \text{Tr} \text{P exp} \left(i \oint_C A_\mu^a(x) T_\rho^a dx^\mu \right), \quad (1.2)$$

where the path ordering is performed along C and $\{T_\rho^a\}$ are the generators of G in the ρ representation. Consider now an oriented link L in R^3 with m components $\{C_1, \dots, C_m\}$ and let ρ_i be the irreducible representation of G associated to the i -th component C_i of L . The vacuum expectation values

$$\langle W(L) \rangle \equiv \frac{\langle 0 | W_\rho(C_1) \dots W_{\rho_m}(C_m) | 0 \rangle}{\langle 0 | 0 \rangle} \quad (1.3)$$

obtained for generic links and arbitrary representations $\{\rho_i\}$ are the gauge-invariant observables in which we are interested. Finding the solution of the CS quantum field theory in R^3 means finding the explicit expressions of $\langle W(L) \rangle$. Given a generic

oriented (framed) link L , whose components are associated with arbitrary representations of the gauge group $G = SU(N)$, the rules which permit the direct computation of the associated expectation value $\langle W(L) \rangle$ in closed form are derived here from the properties of the CS field theory.

New peculiar phenomena have been observed [1] in the physical systems described by an action in which a CS term is added to the ordinary gauge-invariant Lagrangians with matter. These models are expected to describe, for instance, the behaviour of certain solid state systems [2,3]. A completely different kind of physics is obtained when the pure CS action (1.1) is considered. In this case, in addition to the ordinary gauge symmetry, one has a quantum realization of general covariance (the action does not depend on the metric defined on the three-manifold).

The importance of the action (1.1) for knot theory and the study of three-manifolds was pointed out by Schwarz [4] and Atiyah [5]. Further developments on the subject, together with a large number of possible connections of the CS theory with different areas of mathematics and physics, have been discussed by Witten in a beautiful paper [6] in which several conjectures have been formulated. One of the open problems of the program proposed in [4-6] was to give a precise meaning to the expectation values $\langle W(L) \rangle$ in R^3 and produce them. The solution of this problem is contained in the material presented in this article, which represents the continuation of the work done in [7-11].

Because of general covariance, the expressions obtained for $\langle W(L) \rangle$ describe link invariants of the same type considered in knot theory. In fact, the expectation values take the form of polynomials in a certain variable which is a function of the coupling constant k of the model. After $\langle W(L) \rangle$ have been found, the main issue is to identify these polynomials. It turns out that the link invariants obtained in the CS theory are those associated with the braid group representations described by the quasi-tensor category of quasi-triangular quasi-Hopf (QTQH) algebras [12] associated to the quantum deformations of ordinary Lie algebras. In this sense, the obtained link polynomials are not completely unknown.

In quite general terms, the link invariants described by the quasi-tensor category of QTQH algebras are called by Drinfeld the universal link (or knot) invariants [12]. For this reason, the polynomial $E(L)$, associated to the expectation value $\langle W(L) \rangle$, is called the universal link polynomial.

From the field theory point of view, the possibility of producing the explicit and exact solution of a non-trivial gauge theory is quite remarkable. On the other hand, the link polynomial $E(L)$ is of particular interest for knot theory also. For many reasons,

$E(L)$ is the natural generalization of the Jones polynomial [13]. The most important aspect of this generalization is based on the existence of a Lie algebra structure underlying the construction of the universal link polynomial. The expectation values $\{< W(L) >\}$, computed in the case in which all the components of the links are associated with the fundamental representation of the gauge group $G = SU(2)$, essentially give the Jones polynomial (see sect.10). The first generalization consists of associating arbitrary representations of $G = SU(2)$ to the different components of the links. In this case, one obtains a new polynomial, which can be called the extended Jones polynomial because the $SU(2)$ Lie algebra remains the same. A further step consists of taking a generic real simple Lie algebra as the Lie algebra of the gauge group G and computing $\{< W(L) >\}$ when the different components of the links are associated with arbitrary representations of G . This is the case discussed in this paper. The general properties of the universal link polynomial associated with an arbitrary simple Lie algebra are obtained and the proof of the complete reconstruction of $E(L)$ in the case of $\{A_n\}$ algebras is presented.

It should be noted that CS-like actions based on complex Lie algebras also exist [14,15]. In fact, a large number of topological models can be constructed by adding new appropriate terms to the action (1.1); some relevant examples of this type will be analysed elsewhere.

The invariants described by $E(L)$ could also be obtained by means of the algebraic method based on the use of the universal R -matrix. The new features of the construction presented here are due to the existence of a consistent quantum field theory interpretation of the link invariants. With respect to the standard "quantum group" approach, the field theory method presents some advantages in the construction of the link polynomials. First, the well-known quantum group problems, encountered when the deformation parameter is a root of unity, do not appear in the CS theory. Second, the technical difficulties related to the representation problem of the universal R matrix and to the explicit construction and evaluation of the Markov trace, are absent in the field theory context. Finally, the most important feature of this new method is that the fundamental role of the tensor structure of ordinary simple Lie algebras is fully recognized.

As we shall see, the properties of the CS theory naturally suggest how to identify and extract the minimal information which is necessary for the construction of the expectation values $\{< W(L) >\}$. This information can be organized [16] in a set of simple defining relations which uniquely determine the universal link polynomial. Quite remarkably, $E(L)$ can be expressed entirely in terms of the values that the quadratic Casimir operator of the gauge group G takes in the different representations. The main

purpose of the present paper is to show first how these defining relations follow from the properties of the CS field theory and then how to use them for constructing $E(L)$.

A certain number of definitions and known results concerning knot theory are particularly useful for our discussion. So, in order to make this article essentially self-contained, a few basic notions of knot theory are recalled in sect.2. In sects.3 and 4 several quantum CS field theory aspects are discussed; in particular, the precise definition of the framing procedure for Wilson line operators is given, the CS renormalization properties are recalled and some explicit results obtained in perturbation theory are reviewed.

The main properties of $\{< W(L) >\}$ are derived in sect.5. All the results obtained in this section are non-perturbative and are consequences of the symmetry properties of the quantum CS theory, of the structure of the Wilson line operators and of the properties of the ambient space. The derivation of a satellite formula for the expectation values $\{< W(L) >\}$ is the most important result of this section.

Some elementary properties of the braid group are recalled in sect.6. The material presented in sect.7 is useful for identifying the $E(L)$ polynomial and concerns braid group representations obtained by means of the R -matrix method [17,18] and of the monodromy type considered by Kohn [19]. Turaev and Kirillov-Reshetikhin constructions [20-23] of link invariants and Drinfeld's general framework of quasi-triangular quasi-Hopf algebras [12] are also mentioned. A first aspect of the connection between these issues and the CS theory is shown in sect.8.

The $E(L)$ defining relations following from the properties of the three-dimensional field theory are formulated in sect.9. Sect.10 is devoted to enumerating the main features of the extended Jones polynomial and several examples are discussed. The general properties of the universal link polynomial associated to a generic simple Lie algebra are produced in sect.11. Sect.12 contains the proof of the reconstruction of the $E(L)$ polynomial by means of the defining relations in the case of $SU(N)$ groups. The results obtained in some simple examples are also reported. Finally, the conclusions are contained in sect.13, where some open problems are also listed.

2. BASIC NOTIONS OF KNOT THEORY

The full program of classifying and studying the properties of knots and links that one can construct in R^3 was formulated on the basis of physical motivations in the second half of the nineteenth century. The interest in this subject mainly originated from the vortex-atoms model proposed by J.C. Maxwell, P.G. Tait and W. Thomson around 1867.

Several beautiful results have been obtained in knot theory and important developments, connecting different fields of mathematics, have been performed. In this section, a few definitions and results are briefly recalled; a more detailed and complete exposition can be found, for instance, in [24,25] and in the references quoted there.

A smooth non-intersecting closed path C in R^3 is called a knot. Since the definition of the holonomy requires an orientation for the path, we will always consider oriented knots. An oriented link L with m components is the union of m oriented non-intersecting closed paths. The m components of L will be denoted by $\{C_1, C_2, \dots, C_m\}$. Smooth deformations performed in the ambient space do not modify the "topological" properties of links. Two links L_1 and L_2 in R^3 are called ambient isotopic if L_1 can be smoothly connected with L_2 in R^3 . If one is interested in the topological properties of links, only the equivalence classes of ambient isotopic links are relevant, of course.

A convenient description of links is given in terms of diagrams obtained by projecting the links on a plane. In order to avoid all ambiguities, one usually considers link diagrams containing only simple crossing points; at each crossing point the choice of over/under crossing is specified. Given two link diagrams D_1 and D_2 , the associated links L_1 and L_2 are ambient isotopic iff there exists a finite sequence of Reidemeister moves, shown in Fig.1, which transforms D_1 into D_2 .

Reidemeister moves (RM) are very important in knot theory because they encode the symmetry structure which is relevant for the link classification problem. Indeed, constructing link invariants of ambient isotopy precisely means finding invariants of the symmetry group generated by the RM. Several different methods of constructing link invariants have been discovered and some of them will be mentioned here. Before describing some explicit examples, let us analyse the RM a little bit more carefully.

Reidemeister moves of type I are very special. In fact, one can eliminate them from the list of admissible moves. In this way, one can define an interesting structure

which plays an important role in the construction of the universal link polynomial.

Two link diagrams D_1 and D_2 related by RM of types II and III only are called regular isotopic. Consider now the equivalence classes of regular isotopic link diagrams. A useful invariant of regular isotopy is the writhe number $w(D_L)$ which is defined for any link diagram D_L by

$$w(D_L) = \sum_p \epsilon(p) \quad (2.1)$$

The sum in eq.(2.1) is performed over all the crossing points of the link diagram D_L and

$$\epsilon(L_{\pm}) = \pm 1 \quad (5.2)$$

where $L_{\pm}$ are shown in Fig.2. The configuration L_+ (L_-) will be called overcrossing (undercrossing). The proof that $w(D_L)$ is a regular isotopy invariant is very simple; indeed, it is immediately verified that $w(D_L)$ is invariant under RM of types II and III.

The concept of regular isotopy is useful because, by eliminating the RM of type I, one does not lose any information concerning the topology of links; on the contrary, one gains a free variable for each component of the link.

In fact, each equivalence class of ambient isotopy contains, by definition, all the equivalence classes of regular isotopy which are connected by RM of type I. But each RM of type I acts only on a single line of the diagrams; therefore, it can modify only the writhe number of a single component of the link diagrams. This being the case, each equivalence class of ambient isotopy corresponding to a link in R^3 contains infinitely many equivalence classes of regular isotopy, which are labelled by the writhe numbers $\{w(C_i)\}$ of the different components $\{C_i\}$ of the link. The crucial point is that one can give [25] the following interpretation of the above conclusion. The equivalence classes of regular isotopic link diagrams just describe ambient isotopy classes of links in R^3 in which each component is characterized by an integer number which, in turn, is an ambient isotopy invariant.

Now, suppose that one replaces links made, say, of strings with links made of oriented bands. The topology of the links is not modified; the only change is that for each component C_i of the link we now have an extra variable $T(C_i)$ telling us how many times the oriented band is twisted. The twist T of the band is an ambient isotopy invariant and therefore we are precisely in the same conditions mentioned before. In conclusion, one can represent the equivalence classes of ambient isotopic links made of bands with the equivalence classes of regular isotopic link diagrams. The only thing

which remains to be fixed is the connection between the writhe number $w(C_i)$ and the twist variable $T(C_i)$ of each component C_i of the links. The simplest choice is just

$$T(C_i) = w(C_i) . \quad (2.3)$$

The importance of regular isotopy for the CS theory is due to the fact that, in studying the properties of the expectation values $\{< W(L) >\}$, one has to consider framed links. This means that for each component C of the links one has to introduce another closed and oriented path C_f called the framing of C . A detailed discussion of this point will be performed in sect.3. For the moment, you can imagine that C_f lies within an infinitesimal neighbourhood of C with the condition that C and C_f never intersect. Since C and C_f can be considered as the two components of the boundary of an oriented band, framed links can be interpreted as links made of bands. Consequently, we will represent oriented framed links in R^3 with the regular isotopy classes of link diagrams with the identification (2.3); this representation will be called vertical framing. In order to simplify the notations, framed links in R^3 and their corresponding link diagrams in vertical framing will be indicated by the same symbol.

RM of type I are very peculiar also for another reason; as we shall see, they are the only moves in which the partial closure of braids is involved.

As far as RM of types II and III are concerned, one should note that they essentially determine the algebraic structure of the Artin braid group B_n . For open braids, the invariance under RM of type II is quite trivial because this invariance is automatically satisfied in terms of the B_n generators. However, in considering the closure of braids, RM of type II have the important effect of associating the conjugacy classes of B_n to the links. Finally, let us consider the RM of type III ; they represent the main feature of the braid group and enter directly the defining relations satisfied by the B_n generators. Finding a complete classification of the inequivalent realizations of the RM of type III is still an open problem. In considering matrix representations of B_n and with an appropriate choice of the form of the generators, RM of type III give origin to the famous quantum Yang-Baxter equation. Some relevant properties of the braid group will be considered in sect.6.

In the remaining part of this section some examples of link invariants are reported. Consider a two-component oriented link L in R^3 with components C_1 and C_2 and let D_L , D_1 and D_2 be the associated diagrams. As mentioned before, $w(D_L)$ is a regular isotopy invariant; it is easy to see that $w(D_1)$ and $w(D_2)$ also are separately regular isotopy invariants. Let us try now to construct an invariant of ambient isotopy by

combining these three writhe numbers. Under a RM of type I , one has

$$\Delta w(D_L) = \Delta [w(D_1) + w(D_2)] = \pm 1 . \quad (2.4)$$

Therefore, the combination

$$\chi(C_1, C_2) = \frac{1}{2} [w(D_L) - w(D_1) - w(D_2)] \quad (2.5)$$

is an ambient isotopy invariant. The invariant $\chi(C_1, C_2)$ is called the linking number between C_1 and C_2 . Roughly speaking, the value of $\chi(C_1, C_2)$ tells us how many times C_2 winds around C_1 . This quantity can also be expressed in terms of the Gauss integral

$$\chi(C_1, C_2) = \frac{1}{4\pi} \oint_{C_1} \oint_{C_2} dx^\mu \int dy^\nu \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} , \quad (2.6)$$

where the distance $|x-y|$ is computed by means of the ordinary flat metric in R^3 .

This example shows that the same link invariant can be constructed by using very different methods. In eq.(2.5), χ is obtained through computations performed by looking at the link diagrams, whereas eq.(2.6) provides a more direct geometric interpretation of χ . The expression (2.5) has to be computed on a specific link diagram but the result, being ambient isotopic, does not depend on the particular choice of this diagram. Similarly, in the expression (2.6) the integrand depends on the metric $\delta_{\mu\nu}$ of R^3 , but the result of the integral is metric-independent: it exclusively depends on the topology. It is obvious that, for explicit computations, eq.(2.5) is more practical to use than the expression (2.6). Usually, the invariants constructed by operating on the link diagrams are easier to compute than those constructed by means of geometrically more intrinsic methods.

Eqs.(2.5,6) are useful for illustrating the strategy pursued in solving the CS theory. The contributions to $< W(L) >$ obtained, for instance, by computing the Feynman diagrams are the analogue of the expression (2.6) and are in general quite difficult to evaluate. The idea is to find the analogue of eq.(2.5); that is, all the contributions of the Feynman diagrams will be expressed in terms of simple algebraic operations based on the structure of the link diagrams.

As mentioned before, a framed oriented knot C with framing C_f can be interpreted as a knot made of a band; the twist T of this band is simply given by

$$T = \chi(C, C_f) . \quad (2.7)$$

Among the several link invariants that have been discovered, there are the so-called link polynomials. Some of them are strictly connected with the universal link

polynomial described by the CS theory and their construction is simply formulated in terms of the concept of the skein relation. Three oriented links L_+ , L_- and L_0 are skein related if they have diagrams which are identical except in a neighbourhood of a point where they look like as shown in Fig.3.

The Alexander-Conway polynomial $\nabla(L; z)$, associated to the link L , is defined by [26]

- i) ambient isotopy invariance,
- ii) $\nabla(U; z) = 1$,
- iii) $\nabla(L_+; z) - \nabla(L_-; z) = z \nabla(L_0; z)$,

where U is the unknotted knot and condition ii) fixes the normalization. With our notations, $\nabla(L; z)$ is also called the Conway potential function and it is a Laurent polynomial in the variable z with integer coefficients. Two links L_1 and L_2 such that $\nabla(L_1; z) \neq \nabla(L_2; z)$ are not ambient isotopic, of course. However, it is easy to find examples of links which are not ambient isotopic but have the same Alexander-Conway polynomial.

The Jones polynomial $V(L; q) \in Z^\pm[q^{\frac{1}{2}}]$ is defined by [13]

- i) ambient isotopy invariance,
- ii) $V(U; q) = 1$,
- iii) $qV(L_+) - q^{-1}V(L_-) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})V(L_0)$.

The $V(L)$ polynomial is more selective than $\nabla(L)$; however, there still are ambient non-isotopic links with the same Jones polynomial.

The two-variable HOMFLY polynomial $P(L; t, z)$ is defined by [27]

- i) ambient isotopy invariance,
- ii) $P(U; t, z) = 1$,
- iii) $tP(L_+) - t^{-1}P(L_-) = zP(L_0)$,

and it represents essentially the most general polynomial [28] constructed by means of the skein relation involving the configurations shown in Fig.3. In fact, $P(L; t, z)$ reduces to the Alexander-Conway and Jones polynomials with the obvious choices for the values of the variables t and z . The HOMFLY polynomial also does not provide a complete classification of knots or links.

A common feature of all these polynomials is that, by means of the skein relation, they can be easily constructed analysing the link diagrams. By using the conditions i) and iii) recursively, the polynomial of whatever link can be written in terms of

the polynomial of the unknotted knot U , which is conventionally taken to be the identity. At this stage, it is not completely obvious that the construction based on the recursive use of the skein relation is well defined. Several different proofs of the internal consistency of the defining conditions i), ii) and iii) have been produced in the literature, see for instance [13,27,29].

By looking at the defining conditions (2.8-10), one notes that the progress in the construction of the "classical" link polynomials has been made by modifying the skein (or exchange) relation. However, along this line it is hard to imagine how to improve the HOMFLY polynomial significantly. In fact, the strategy leading to the universal link polynomial is to come back to the Jones polynomial and provide it with a Lie algebra interpretation.

We conclude this section by considering a link polynomial of regular isotopy which is of particular interest for the CS theory. The $S(L; \alpha, \beta, z)$ polynomial [8] is defined by

- i) regular isotopy invariance ,
- ii) $S(U_0) = 1$,
- iii) $S(L^{(+)}) = \alpha S(L^{(0)})$, $S(L^{(-)}) = \alpha^{-1} S(L^{(0)})$,
- iv) $\beta S(L_+) - \beta^{-1} S(L_-) = z S(L_0)$,

where $L^{(\pm)}$ and $L^{(0)}$ are shown in Fig.4 and U_0 is the unknotted knot with zero writhe. The use of a simplified notation should cause no problems here. From the definition (2.11) it is clear that $S(L)$ is defined on the equivalence classes of regular isotopic link diagrams or, equivalently, on the set of framed links. The polynomial $S(L)$ was introduced in [8] for describing the observed behaviour of the Wilson line operators in some particular representations. $S(L)$ also can be constructed by using the skein relation recursively and it is related to the HOMFLY polynomial by [8]

$$P(L; t = \alpha\beta, z) = \alpha^{-w(L)} S(L; \alpha, \beta, z) . \quad (2.12)$$

The factor multiplying $S(L)$ in eq.(2.12) just compensates the covariant variation of $S(L)$ under RM of type I. Because of the identity (2.12), the information concerning the link classification problem contained in $S(L)$ and $P(L)$ is essentially the same. However, the $S(L)$ polynomial is particularly meaningful; as we shall see in sect.6, $S(L)$ is in some sense the ancestor of all the polynomials which are obtained in Hecke algebra representations of the braid group. Moreover, the structure of the relations (2.11) naturally extends to the general case described by $E(L)$.

3. FRAMING IN FIELD THEORY

The problem of the computation of the Wilson line expectation values is meaningful provided $\{ \langle W(L) \rangle \}$ are well defined. Usually in field theories, the expectation values of composite operators present in general several problems related, for instance, to the occurrence of divergences or of ambiguities in their definition. We must specify then what is the precise meaning of the expression (1.3).

In the present paper, we consider the case in which the expression (1.3) is supplemented by two further requirements: first, that in the quantum CS theory gauge invariance is preserved and, second, that general covariance is also preserved. In the quantization schemes used to perform the explicit computations reported here, both these conditions are always satisfied.

Many of the properties of the expectation values $\{ \langle W(L) \rangle \}$ follow from general covariance; in particular, the necessity of introducing a framing prescription for the definition of the Wilson line operators is a consequence of general covariance. The precise definition of framing in the CS model is certainly a somewhat technical issue; and yet, this is also one of the crucial aspects of the theory. The importance of framing cannot be overestimated. Indeed, as we shall see, finding the solution of the problem of the framing essentially means finding the solution of the CS theory.

In order to understand what the framing means in the field theory context, it is useful to consider first the simple case of an Abelian CS theory defined by the action

$$S_0 = \frac{k}{8\pi} \int d^3x \, \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \quad (3.1)$$

The action (3.1) is invariant under gauge transformations

$$\Delta A_\mu(x) = \partial_\mu \Lambda(x) \quad ; \quad (3.2)$$

the way to deal with gauge invariance at the quantum level has been widely discussed in the literature.

General covariance follows from the fact that the expression (3.1), which is invariant under general coordinate transformations performed on the vector field $A_\mu(x)$, is metric-independent. This means that in the physical system described by the action

(3.1) there is no notion of the distance between two points because there is no particular metric for defining a distance. This being the case, general covariance would suggest that $\{ \langle W(L) \rangle \}$ is an ambient isotopy invariant for the links $\{L\}$. As it stands, this classical argument fails at the quantum level. Still, it is possible to recover general covariance in a slightly modified version.

In order to define the quantum theory, the well-known BRS formalism [30] can be used. In the Landau gauge, the total action reads

$$S = \frac{k}{4\pi} \int d^3x \left[\frac{1}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - B \partial^\mu A_\mu + \bar{c} \partial^\mu \partial_\mu c \right] \quad , \quad (3.3)$$

where B is a bosonic auxiliary field and c and $\bar{c}$ are the ghost fields. In the gauge-fixing and ghost terms, the ordinary flat metric of R^3 has been used. The introduction of a metric in the Lagrangian has to be confronted with the requirement of general covariance. In our case, since the metric enters in the gauge-fixing procedure only, for gauge-invariant and metric-independent observables general covariance is preserved. The propagators obtained from (3.3) are [7]

$$\langle A_\mu(x) A_\nu(y) \rangle = \frac{i}{k} \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} \quad , \quad (3.4)$$

$$\langle A_\mu(x) B(y) \rangle = -\frac{i}{k} \frac{(x-y)_\mu}{|x-y|^3} \quad , \quad (3.5)$$

$$\langle c(x) \bar{c}(y) \rangle = -\frac{i}{k} \frac{1}{|x-y|} \quad . \quad (3.6)$$

The quadratic action (3.3) defines a free theory and the correlation functions of the fields are well defined; they can obviously be expressed in terms of the expressions (3.4-6).

The integral of the one-form $A_\mu dx^\mu$ on a closed oriented path C in R^3 is gauge-invariant and metric-independent; let us consider the associated expectation values. Given two non-intersecting oriented closed paths C_1 and C_2 , one gets

$$\begin{aligned} \langle \oint_{C_1} A_\mu dx^\mu \oint_{C_2} A_\nu dy^\nu \rangle &= \frac{i}{k} \oint_{C_1} dx^\mu \oint_{C_2} dy^\nu \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} \\ &= i \frac{4\pi}{k} \chi(C_1, C_2) \quad , \end{aligned} \quad (3.7)$$

where $\chi(C_1, C_2)$ is the linking number between C_1 and C_2 . As expected, even if the R^3 metric has been used in the intermediate steps of the computations, the final result (3.7) is metric-independent.

Since the action (3.3) is quadratic in the fields, it is clear that the expectation value of the product of an arbitrary number of $A_\mu dx^\mu$ integrals, performed on different paths, is a topological invariant. For instance, in the case of a four-component link one obtains

$$\begin{aligned} & \langle \oint_{C_1} A_\mu dx^\mu \oint_{C_2} A_\nu dx^\nu \oint_{C_3} A_\rho dx^\rho \oint_{C_4} A_\tau dx^\tau \rangle = \\ & = - \left(\frac{4\pi}{k} \right)^2 [\chi(C_1, C_2)\chi(C_3, C_4) + \chi(C_1, C_3)\chi(C_2, C_4) + \chi(C_1, C_4)\chi(C_2, C_3)] \end{aligned} \quad (3.8)$$

In conclusion, for any oriented link L in R^3 the expectation value of the associated observable, obtained by integrating the one-form $A_\mu dx^\mu$ on the components of L , is well defined and it is an ambient isotopy invariant.

In considering Wilson line operators, however, one has to exponentiate the integral of the one-form $A_\mu dx^\mu$. As a consequence, we must analyse also the case in which the product of two or more $A_\mu dx^\mu$ integrals, performed on the same path, occurs. Let us consider first the simplest case: namely, the expectation value of two $A_\mu dx^\mu$ integrals associated to the same oriented path C parametrized by $\{x^\mu(s); 0 \leq s \leq 1\}$. The "naive" result is

$$\langle \oint_C A_\mu dx^\mu \oint_C A_\nu dy^\nu \rangle = \frac{i}{k} \int_0^1 ds \int_0^1 dt \varepsilon_{\mu\nu\rho} \dot{x}^\mu(s) \dot{x}^\nu(t) \frac{(x(s) - x(t))^\rho}{|x(s) - x(t)|^3} \quad (3.9)$$

The integral appearing in (3.9) is well defined and finite [31], so there is no need to introduce any regularization. The problem is that, differently from the Gauss integral (3.7), the expression (3.9) is not an ambient isotopy invariant [31] and this is in contradiction with general covariance.

How has eq.(3.9) to be interpreted? At first sight, one could believe that in the quantization procedure that we have used there is a breaking of general covariance. This is not the case, however. In reality, the problem connected with the expression (3.9) is related to a typical quantum effect which is well known in field theory and which shows that the classical arguments are quite often misleading in the quantum context. The whole point can be illustrated by an example; consider a renormalizable quantum field theory constructed in terms of a scalar field $\phi(x)$. At the classical level, the knowledge of the field $\phi(x)$ is sufficient for determining the values of $\{\phi^2(x), \phi^3(x), \dots\}$. At the quantum level, the situation is quite different. The field operator $\phi(x)$ is well defined in the sense that the correlation functions of $\phi(x)$ are well defined. However, this is not

enough for uniquely determining what the quantum operators $\{\phi^2(x), \phi^3(x), \dots\}$ mean. This problem is not related to the choice of the quantization scheme which has been used to define the renormalized correlation functions of $\phi(x)$. In other words, even if the field operator $\phi(x)$ is well defined, one still has to define the precise meaning of the composite operators $\{\phi^2(x), \phi^3(x), \dots\}$. In general, different definitions of the same power of $\phi(x)$ are possible and, of course, different definitions can lead to different composite operators.

Let us consider the CS theory now. The vector field $A_\mu(x)$ is the fundamental field of the model. In any consistent and general-covariance-preserving quantization scheme, the integral of the one-form $A_\mu dx^\mu$ on a closed path is a well-defined operator. This means that the expectation values of an arbitrary number of such operators, associated with non-intersecting different paths, are unique and well defined. This is precisely the result that we have obtained by using the covariant BRS formalism. However, the product of two $A_\mu dx^\mu$ integrals on the same path is a composite operator and we must define it. Different definitions of $(\oint_C A_\mu dx^\mu)^2$ are possible and, accordingly, different composite operators can be obtained. The first possibility is to define $(\oint_C A_\mu dx^\mu)^2$ precisely like its classical expression suggests, i.e.

$$\left(\oint_C A_\mu dx^\mu \right)^2 \Big|_{\text{naive}} = \int_0^1 ds \int_0^1 dt \dot{x}^\mu(s) A_\mu(x(s)) \dot{x}^\nu(t) A_\nu(x(t)) \quad (3.10)$$

Eq.(3.10) is called the naive definition; the result (3.9) has been obtained by using eq.(3.10). With the naive definition of $(\oint_C A_\mu dx^\mu)^2$, the vacuum expectation values are finite; unfortunately, general covariance is not maintained.

Another possibility is to define $(\oint_C A_\mu dx^\mu)^2$ by means of a framing procedure [9]. For each knot C parametrized by $\{x^\mu(s); 0 \leq s \leq 1\}$, we introduce a framing contour C_f parametrized by

$$y^\mu(s) = x^\mu(s) + \varepsilon n^\mu(s), \quad (\varepsilon > 0, \quad |\vec{n}(s)| = 1), \quad (3.11)$$

where n^μ is a vector field orthogonal to C . Then we define

$$\left(\oint_C A_\mu dx^\mu \right)^2 \Big|_{\text{framed}} = \lim_{\varepsilon \rightarrow 0} \oint_C A_\mu dx^\mu \oint_{C_f} A_\nu dy^\nu, \quad (3.12)$$

with the convention [9] that the $\varepsilon \rightarrow 0$ limit has to be taken after all the Wick-contractions and integrations have been performed. At the classical level, the definitions

(3.10) and (3.12) coincide. At the quantum level, however, the two definitions lead to very different results. Indeed, according to (3.12) one obtains

$$\left\langle \oint_C A_\mu dx^\mu \right\rangle_{\text{framed}}^2 > = i \left(\frac{4\pi}{k} \right) \lim_{\epsilon \rightarrow 0} \chi(C, C_f) = i \left(\frac{4\pi}{k} \right) \chi(C, C_f) . \quad (3.13)$$

The result (3.13) is now an ambient isotopy invariant for the link defined by $\{C, C_f\}$ but its particular values depend on the choice of the framing C_f . So, by using the framing procedure (3.12), general covariance is maintained in the sense that the expression (3.13) is an ambient isotopy invariant of the framed knot C .

The problem of giving a topological meaning to the "linking number" of a single knot is known as the self-linking problem. The solution (3.13), obtained by means of framing, was already known in mathematics [32]. In the field theory context, the framing procedure is extended from the expectation value (3.13) to the composite operator $(\oint_C A_\mu dx^\mu)^2$ itself, eq.(3.12). The improvement obtained by means of this definition is clear. Not only $< (\oint_C A_\mu dx^\mu)^2 >$, but also any expectation value, involving an arbitrary number of $(\oint A_\mu dx^\mu)$ and $(\oint_C A_\mu dx^\mu)^2$ operators, now preserves general covariance.

At this stage, the definition of the composite Wilson line operator by means of the framing procedure is quite natural. For an oriented knot C characterized by a given value e of the "charge", the naive expression of the associated Wilson line operator would be

$$\begin{aligned} W(C; e) &= \exp \left(ie \oint_C A_\mu dx^\mu \right) \\ &= \sum_n \frac{(ie)^n}{n!} \oint_C A_{\mu_1}(x_1) dx_1^{\mu_1} \dots \oint_C A_{\mu_n}(x_n) dx_n^{\mu_n} . \end{aligned} \quad (3.14)$$

In this equation, the products of the $A_\mu dx^\mu$ integrals performed on the same path C would lead to troubles with general covariance. But we already know how to overcome this difficulty; it is sufficient to introduce a framing and take the limit of coincident paths at the end of the computations. More precisely, in a generic product

$$\begin{aligned} \oint_C A_{\mu_1}(x_1) dx_1^{\mu_1} \dots \oint_C A_{\mu_n}(x_n) dx_n^{\mu_n} &= \\ = \int_0^1 ds_1 \dots \int_0^1 ds_n \dot{x}^{\mu_1}(s_1) A_{\mu_1}(x(s_1)) \dots \dot{x}^{\mu_n}(s_n) A_{\mu_n}(x(s_n)) \end{aligned}$$

appearing in eq.(3.14), each term $\dot{x}^{\mu_i}(s_i) A_{\mu_i}(x(s_i))$ is replaced by

$$\dot{x}^{\mu_i}(s_i) A_{\mu_i}(x(s_i)) \rightarrow [\dot{x}^{\mu_i}(s_i) A_{\mu_i}(x(s_i))]_f , \quad (3.15)$$

$$[\dot{x}^{\mu_i}(s_i) A_{\mu_i}(x(s_i))]_f = (\dot{x}^{\mu_i}(s_i) + \epsilon_i \dot{n}^{\mu_i}(s_i)) A_{\mu_i}(x(s_i) + \epsilon_i n(s_i)) , \quad (3.16)$$

where the vector field n^μ characterizes the choice of framing, see eq.(3.11), and the values $\{\epsilon_i\} = \{\epsilon_i, \epsilon_i > 0\}$ can be arbitrarily chosen provided that $\epsilon_i \neq \epsilon_j$ for $i \neq j$. Finally, the composite Wilson line operator associated with a framed knot C is defined as

$$W(C; e)_f = \lim_{\{\epsilon\} \rightarrow 0} \sum_n \frac{(ie)^n}{n!} \int_0^1 ds_1 \dots \int_0^1 ds_n \left\{ [\dot{x}^{\mu_1}(s_1) A_{\mu_1}(x(s_1))]_f \dots \dots [\dot{x}^{\mu_n}(s_n) A_{\mu_n}(x(s_n))]_f \right\} . \quad (3.17)$$

In the case of a non-Abelian gauge group, composite Wilson line operators associated with framed knots are similarly defined; in each term of the field expansion of the path-ordered exponential, the substitution (3.15) is performed and then the $\{\epsilon\} \rightarrow 0$ limit is taken at the end of the computations. Of course, in the case of a generic link, the framing procedure described above must be applied to each component of the link.

From now on, it is understood that Wilson line operators are defined by means of the framing procedure as shown in eq.(3.17).

We can give now the solution of the Abelian CS theory. Consider a generic oriented framed link L in R^3 with m components $\{C_1, \dots, C_m\}$ in which the i -th component C_i has framing $C_{i,f}$ and charge e_i . From eqs.(3.4) and (3.17) it follows that the expectation value of the associated Wilson line operator is simply

$$\langle W(L) \rangle = \exp \left\{ -i \left(\frac{2\pi}{k} \right) \left[\sum_i e_i^2 \chi(C_i, C_{i,f}) + 2 \sum_{i < j} e_i e_j \chi(C_i, C_j) \right] \right\} . \quad (3.18)$$

Note that this result does not depend on the choice of the direction in the $\{\epsilon\}$ space along which the $\{\epsilon\} \rightarrow 0$ limit is performed.

As explained in sect.2, the framed link L can be represented by a particular link diagram in vertical framing that we indicate by the same symbol L . In vertical framing,

the result (3.18) takes the form

$$\langle W(L) \rangle = \exp \left\{ -i \left(\frac{2\pi}{k} \right) \left[\sum_i e_i^2 w(C_i) + 2 \sum_{i < j} e_i e_j \chi(C_i, C_j) \right] \right\}, \quad (3.19)$$

where $w(C_i)$ is the writhe of the component C_i .

The exact solution (3.18) of the Abelian CS theory shows that $\langle W(L) \rangle$ defines an ambient isotopy invariant for framed links in R^3 ; equivalently, in vertical framing $\langle W(L) \rangle$ represents a regular isotopy invariant for link diagrams, eq.(3.19). So, general covariance is manifestly preserved.

Summarizing, the framing is a way of defining the Wilson line operator at the quantum level. The introduction of framing is not motivated by the need to eliminate divergences but by the necessity of preserving general covariance. At the classical level and for each given vector field $A_\mu(x)$, the Wilson line (3.14) is a function on the space on knots in R^3 . At the quantum level, the Wilson line operator, obtained by means of the framing procedure, is defined on the space of knots and of their "first-order" neighbourhoods. In this sense, the framing is similar to the blowing-up procedure. The only ambient isotopic characterization of these knot neighbourhoods is represented precisely by the linking number $\chi(C, C_f)$ between C and its framing C_f . This is why the whole dependence of $\langle W(L) \rangle$ on the framing is expressed by $\chi(C, C_f)$ or, in vertical framing, by $w(C)$. Clearly, this general feature is common to both the Abelian and non-Abelian CS theories.

4. NON-ABELIAN CHERN-SIMONS THEORY

The action of the CS theory is constructed in terms of a vector field $A_\mu(x)$ which takes values on a real Lie algebra $\mathcal{A}$. Like in any gauge theory, each Abelian $U(1)$ subfactor and each simple subalgebra of $\mathcal{A}$ has its own coupling constant. The Abelian CS theory has already been analysed in sect.3. We consider now the case of a real simple Lie algebra $\mathcal{G}$. This algebra can be interpreted as the Lie algebra of a compact simple Lie group G . Following the standard language used in physics, we will often refer to the generic group G , but it is understood that all the statements really concern the associated Lie algebra $\mathcal{G}$.

The vector field is written as $A_\mu(x) = A_\mu^a(x) T^a$, where $\{T^a\}$ are Hermitian matrices representing the generators of G in some defining representation. We use the convention

$$\text{Tr} \left(T^a T^b \right) = \frac{1}{2} \delta^{ab}, \quad (4.1)$$

and the commutators

$$[T^a, T^b] = i f^{abc} T^c \quad (4.2)$$

define the structure constants $\{f^{abc}\}$. The CS action

$$S_{CS} = \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} \text{Tr} \left(A_\mu \partial_\nu A_\rho + i \frac{2}{3} A_\mu A_\nu A_\rho \right) \quad (4.3)$$

can be rewritten as

$$S_{CS} = \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} \left\{ \frac{1}{2} A_\mu^a \partial_\nu A_\rho^a - \frac{1}{6} f^{abc} A_\mu^a A_\nu^b A_\rho^c \right\}; \quad (4.4)$$

the real parameter k is called the coupling constant of the model.

Like in the Abelian case, gauge invariance and general covariance are the basic properties of the model. The action S_{CS} is invariant under gauge transformations

$$\Delta A_\mu(x) = D_\mu \Lambda(x) = \partial_\mu \Lambda(x) + i[A_\mu(x), \Lambda(x)] \quad (4.5)$$

General covariance simply follows from the metric independence of S_{CS} . The physical system defined by the action S_{CS} does not contain any kind of ordinary particles. Indeed, the classical energy-momentum tensor derived from the expression (4.4) vanishes identically. This fact reflects precisely the topological nature of the system.

The quantum CS theory formulated by means of the covariant BRS method is defined by the total action S which contains, in addition to S_{CS} , the gauge-fixing and ghost terms. In the Landau gauge, the total action is

$$S = \frac{k}{4\pi} \int d^3x \left\{ \frac{1}{2} \epsilon^{\mu\nu\rho} A_\mu^a \partial_\nu A_\rho^a - \frac{1}{6} \epsilon^{\mu\nu\rho} f^{abc} A_\mu^a A_\nu^b A_\rho^c - B^a \partial^\mu A_\mu^a + \bar{c}^a \partial^\mu \left(\partial_\mu c^a - f^{abc} A_\mu^b c^c \right) \right\}, \quad (4.6)$$

where, as usual, the $\delta_{\mu\nu}$ metric of R^3 has been used to contract the vector indices in the gauge-fixing and ghost Lagrangian terms. The resulting non-vanishing propagators are [7]

$$\langle A_\mu^a(x) A_\nu^b(y) \rangle = -\frac{i}{k} \delta^{ab} \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3}, \quad (4.7)$$

$$\langle A_\mu^a(x) B^b(y) \rangle = -\frac{i}{k} \delta^{ab} \frac{(x-y)_\mu}{|x-y|^3}, \quad (4.8)$$

$$\langle c(x)^a \bar{c}^b(y) \rangle = -\frac{i}{k} \delta^{ab} \frac{1}{|x-y|}. \quad (4.9)$$

The expression (4.6) is the action of a renormalizable quantum field theory; so the correlation functions of the fields are well defined and can be constructed, for example, by means of ordinary perturbation theory.

The main properties of the quantum field theory defined by the action (4.6) have been explored at second order of perturbation theory in [7] and have been analysed in [33,34]; the results obtained in [7] have been confirmed and extended to all orders of perturbation theory.

It has been proved [33,34] that the theory is not only renormalizable but actually finite; both the β -function and the anomalous dimensions vanish. This means that there is an exact scale invariance in the correlation functions with A_μ transforming (at the full quantum level) exactly as a vector field. Vanishing of the β -function is in agreement with the general arguments presented in [6]. Vanishing of the anomalous dimension of A_μ is also in perfect agreement with the constraints imposed by the topological character of the theory. Indeed, $A_\mu dx^\mu$ should be a one-form in the quantum theory and the vanishing of the A_μ anomalous dimension guarantees that this is precisely the case.

The Landau gauge is particularly convenient because no external spurious scales (i.e. dimensioned parameters) are introduced in the action. Another interesting feature of the Landau gauge is that the total action (4.6) is invariant also under symmetry transformations whose generators carry a vector index. These generators connect commuting with anticommuting fields and, together with the BRS and anti-BRS operators, form an algebra which closes on-shell on the translations [33,35]. This allows for a supersymmetric interpretation of the model. At first sight, the existence of this supersymmetry may appear to be a coincidence without any particular significance. On the contrary, this SUSY has really a deep physical meaning. Remember that the anticommuting variables appearing in (4.6) have been introduced to compensate for the unphysical pure-gauge degrees of freedom contained in the bosonic fields. The existence of a SUSY connecting bosonic and fermionic variables tells us that there are in fact no physical degrees of freedom. Of course, the conclusion that there are no particles propagating in the CS model is not a novelty; this is just a consequence of general covariance. The advantage of having a supersymmetric realization of general covariance at the quantum level is technically of great help in studying the properties of the theory [33]; it also opens up new interesting possibilities. For example, one could try to compute the partition function of the CS theory in a generic three-manifold by means of an appropriately modified index theorem [36]. Anyway, in this paper we are interested in finding $\{ < W(L) > \}$ for the CS theory in R^3 and, in this case, the vacuum-to-vacuum diagrams are completely irrelevant because they cancel in the computation of the expression (1.3).

Summarising, the non-Abelian CS quantum field theory in R^3 , defined by the action (4.6), exists, is finite and its correlation functions admit a well-defined expansion in powers of $(\frac{2\pi}{k})$.

Exercise: prove that the statement

“If a gauge-invariant regularization is used, then (for $G = SU(N)$) the CS one-loop coupling constant k_1 depends on the bare coupling constant k_0 as $k_1 = k_0 + N$ ”

is false. Construct also an explicit counterexample in which, by using a gauge-invariant regularization, one has $k_1 = k_0 + 4.738/(1+\pi^2)$ and prove that there is in fact an infinite number of counterexamples.

Exercise: verify explicitly at the two-loop level that, with the regularization introduced in [7], gauge-invariance is preserved in the limit in which the cut-off is removed.

Exercise: verify explicitly at the two-loop level that, for whatever choice of the regularization, in the renormalized CS theory gauge-invariance can be preserved.

$\tilde{\alpha}$ appearing in eqs.(4.11,12) is related to the renormalized coupling constant k by

$$\tilde{\alpha} = \frac{k}{8\pi} \quad (4.13)$$

Eqs.(4.11-13) uniquely fix the normalization condition for k . (Eq.(4.11) shows also that there are no radiative corrections to the two-point function!) In particular, k enters the canonical commutation relations between the fields as shown in eq.(3.2) of [6] or eq.(2.6) of [11]. Also, k appears in the Gauss Law constraint as shown in eq.(3.4) of [6] or in eq.(2.26) of [11]. Finally, the topological argument concerning the quantization of the CS coupling constant, presented in [6], precisely refers to the renormalized coupling constant k .

In the covariant BRS formalism, the quantization condition on k does not play any relevant role in the definition of the quantum CS model in R^3 by means of perturbation theory. Indeed, in perturbation theory $\lambda = (\frac{2\pi}{k})$ can be considered a free parameter. Each term of the perturbative expansion in powers of λ respects all the symmetries of the action and, clearly, the quantization condition on k has no consequences on it. Of course, one can always require that in the renormalized theory k is an integer; what has to be noted is that the construction of the quantum theory itself is not modified by this requirement. This does not mean that the quantization condition on k is not important; for several reasons, the fact that k is an integer is crucial. For example, for the internal consistency of the CS theory defined on a generic three-manifold, it is important that k is an integer.

It is well known that any result obtained in quantum field theory has a definite and intrinsic meaning only when it is expressed in terms of the renormalized parameters. Accordingly, in what follows all the formulae are expressed in terms of the renormalized coupling constant k , precisely as in [7-11].

Exercise: prove that in the CS theory (and, in general, in any quantum field theory) the way in which the renormalized coupling constant k depends on the bare coupling constant k_0 is completely irrelevant and physically meaningless (have you ever heard that the bare value of the electric charge has been measured in electrodynamics?).

Exercise: verify explicitly at the two-loop level and for whatever choice of the regularization that, provided gauge invariance and general covariance are maintained in the renormalized CS theory, $\langle W(L) \rangle$ depends on the renormalized coupling constant k in a universal and unique way; this dependence has been produced precisely in [9].

Let us now define the composite Wilson line operator in the non-Abelian case. The classical expression of the Wilson line $W_\rho(C)$, associated with an oriented knot C in R^3

Of course, it is well known that the particular choice of the regularization is totally irrelevant. In quantum field theory, physics never depends on the particular choice of the regularization because physics exclusively depends on (and is defined by) the renormalized theory. This is actually the reason why quantum field theories are predictive and can be used to describe physical phenomena.

The paper [33] contains also the proof that all the symmetries of the action (4.6) are not anomalous and, therefore, can be maintained at the full quantum level. As expected, the only free parameter in the renormalized action turns out to be precisely the renormalized coupling constant k of the theory. The normalization condition on k can be fixed in several equivalent ways. Like in any Lagrangian field theory, the proper vertices and the dressed propagators satisfy the renormalized Schwinger-Dyson equations following from an action principle. The action principle is formulated in terms of the renormalized action which can be written in the form shown in eq.(4.6) and this fixes the renormalized coupling constant k . Equivalently, consider the effective action Γ of the CS theory. As is well known, Γ is BRS invariant and admits an expansion in powers of the fields. The terms of this expansion which are quadratic in the fields determine the inverse of the dressed propagators, whereas the terms of higher order in powers of the fields define the proper vertices. Let Γ have an expansion of the form

$$\Gamma = \int \frac{d^3p}{(2\pi)^3} A_\mu^a(p) M_{ab}^{\mu\nu}(p) A_\nu^b(-p) + \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} N_{abc}^{\mu\nu\rho}(p,q) A_\mu^a(p) A_\nu^b(q) A_\rho^c(-p-q) + \dots \quad (4.10)$$

As explained in [33], the exact form of $M_{ab}^{\mu\nu}(p)$, resummed to all orders of perturbation theory, is given by

$$M_{ab}^{\mu\nu}(p) = i \tilde{\alpha} \delta^{ab} \epsilon^{\mu\nu\rho} p_\rho \quad (4.11)$$

where $\tilde{\alpha}$ is a real coefficient. Consider now the three-point proper vertex. Because of gauge invariance (the BRS symmetry is not anomalous), the normalization of the A_μ^a fields can always be chosen in such a way that the equality

$$p_\mu N_{abc}^{\mu\nu\rho} = -\frac{\tilde{\alpha}}{3} f^{abc} \epsilon^{\mu\nu\rho} p_\mu \quad (4.12)$$

holds for the exact three-point proper vertex. With this convention on the vector field normalization, gauge transformations are given precisely by eq.(4.5) and the coefficient

and an irreducible representation ρ of G , is shown in eq.(1.2). Let C be parametrized by $\{x^\mu(s); 0 \leq s \leq 1\}$; the classical expression (1.2) admits an expansion in powers of the vector field

$$W_\rho(C) = \text{Tr}_\rho \left\{ 1 + i \int_0^1 ds \dot{x}^\mu(s) A_\mu(x(s)) - \int_0^1 ds \int_0^s dt \dot{x}^\nu(t) A_\nu(x(t)) \dot{x}^\mu(s) A_\mu(x(s)) \right. \\ \left. - i \int_0^1 ds \int_0^s dt \int_0^t du \dot{x}^\tau(u) A_\tau(x(u)) \dot{x}^\nu(t) A_\nu(x(t)) \dot{x}^\mu(s) A_\mu(x(s)) + \dots \right\} \quad (4.14)$$

In this equation, it is assumed that A_μ takes values in the Lie algebra of G and the trace operation is performed in the ρ representation.

Eq.(4.14) is the analogue of eq.(3.14) obtained in the Abelian case. The only difference is that the path-ordering is non-trivial now, and therefore the order of the different $(\dot{x}^\mu A_\mu)$ factors in (4.14) cannot be modified (modulo cyclic permutations, of course). Precisely like in the Abelian case, at the quantum level the product of $A_\mu dx^\mu$ integrals performed on the same path are composite operators and must be defined. In order to preserve general covariance, a framing procedure is adopted [9]. We introduce a framing contour C_f parametrized by

$$y^\mu(s) = x^\mu(s) + \varepsilon n^\mu(s) \quad , \quad (\varepsilon > 0 \quad , \quad |\vec{n}(s)| = 1) \quad , \quad (4.15)$$

and each $\dot{x}^\mu A_\mu$ term appearing in eq.(4.14) is replaced by $[\dot{x}^\mu A_\mu]_f$ defined in eq.(3.16). Then, the composite Wilson line operator $W_\rho(C)$ associated with the framed knot C is defined as

$$W_\rho(C) \Big|_f = \lim_{\{\varepsilon\} \rightarrow 0} \text{Tr}_\rho \left\{ 1 + i \int_0^1 ds [\dot{x}^\mu(s) A_\mu(x(s))]_f \right. \\ \left. - \int_0^1 ds \int_0^s dt [\dot{x}^\nu(t) A_\nu(x(t))]_f [\dot{x}^\mu(s) A_\mu(x(s))]_f + \dots \right\} \quad , \quad (4.16)$$

and the $\{\varepsilon\} \rightarrow 0$ limit has to be taken at the end of the computations. At this point, we can give the precise definition of the vacuum expectation values $\{ \langle W(L) \rangle \}$ for the non-Abelian CS theory. We consider framed links in $\mathbb{R}^3$; each framed component C_i ($i = 1, 2, \dots, m$) of L is associated with an irreducible representation ρ_i of G . The

vacuum expectation value

$$\langle W(L) \rangle = \frac{\langle 0 | W_{\rho_1}(C_1) \dots W_{\rho_m}(C_m) | 0 \rangle}{\langle 0 | 0 \rangle} \quad (4.17)$$

is computed by means of the action (4.6) and $W_\rho(C_i)$ are defined according to the framing prescription shown in eq.(4.16). $\{ \langle W(L) \rangle \}$ are now well defined and describe an invariant of ambient isotopy for framed links [9]. In vertical framing, we will denote the expectation value by

$$E(L) = \langle W(L) \rangle \quad ; \quad (4.18)$$

by definition, $E(L)$ is an invariant of regular isotopy for link diagrams.

At second non-trivial order of perturbation theory [9], it turns out that $\langle W(L) \rangle$ does not depend on the choice of the particular direction in the $\{\varepsilon\}$ space along which the limit (4.16) is performed. I will assume that this is indeed the case to all orders of perturbation theory.

Differently from the Abelian case considered in sect.3, the action (4.6) does not define a free theory and, by making use of the Feynman rules, the computation of $E(L)$ in closed form is hard to perform. Let us consider the first three terms of the expansion of $E(C)$ in powers of $\lambda = (\frac{2\pi}{k})$. Let ρ be the representation of G associated with the knot C . We denote by $Q(\rho)$ the value of the quadratic Casimir operator of G in the ρ representation

$$T_{(\rho)}^a T_{(\rho)}^a = Q(\rho) \cdot 1 \quad , \quad (4.19)$$

and by c_ρ the value of Q in the adjoint representation. One obtains [9]

$$E(C) = \dim \rho \left[1 - i Q(\rho) w(C) - \frac{1}{2} \lambda^2 Q^2(\rho) w^2(C) \right] + O(\lambda^3) \quad , \quad (4.20)$$

where $w(C)$ is the writhe of C and

$$b(C) = b_1(C) + b_2(C) \quad , \quad (4.21)$$

with

$$b_1(C) = -\frac{1}{32} \oint_C dx^\mu \int^z dy^\nu \int^y dz^\rho \varepsilon^{\alpha\beta\gamma} \varepsilon_{\mu\alpha\sigma} \varepsilon_{\nu\beta\lambda} \varepsilon_{\rho\gamma\tau} I^{\sigma\lambda\tau}(x, y, z) \quad , \quad (4.22)$$

$$I^{\sigma\lambda\tau}(x, y, z) = \int d^3w \frac{(w-x)^\sigma (w-y)^\lambda (w-z)^\tau}{|w-x|^3 |w-y|^3 |w-z|^3}, \quad (4.23)$$

and

$$b_2 = \frac{1}{8\pi^2} \oint_C dx^\mu \int dy^\nu \int dz^\rho \int dw^\sigma \epsilon_{\sigma\nu\alpha} \epsilon_{\rho\mu\beta} \frac{(w-y)^\alpha (z-x)^\beta}{|w-y|^3 |z-x|^3}. \quad (4.24)$$

It can be shown that $b(C)$, in spite of its complicated expression, is well defined and represents an ambient isotopy invariant. The value of $b(C)$ for the unknotted knot U has been computed explicitly from expressions (4.21-24); the result is [9]

$$b(U) = -\frac{1}{12}. \quad (4.25)$$

For a generic knot, the value of $b(C)$ is simply related to the second coefficient of the Alexander-Conway polynomial [9].

Now, let $G = SU(N)$ and let all the components of the links be associated with the fundamental (N -dimensional) representation of $SU(N)$. In this case, $E(L)$ satisfies [9] (at order λ^2 included) the skein relation of the $S(L; \alpha, \beta, z)$ polynomial defined in sect.2 with particular values of the parameters α, β and z which are consistent (at this order in λ) with the expressions

$$\alpha = q^{\frac{N^2-1}{2N}}, \quad (4.26)$$

$$\beta = q^{\frac{1}{2N}}, \quad (4.27)$$

$$z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}, \quad (4.28)$$

where

$$q = \exp\left(-\frac{2\pi i}{k}\right). \quad (4.29)$$

As we shall see, this is precisely the exact solution to all orders in λ . Namely, for $G = SU(N)$ and for link components in the N representation of $SU(N)$, $E(L)$ satisfies the skein relation of the $S(L)$ polynomial with parameters given in eq.(4.26-29). In the definition (2.11) of $S(L)$, the normalization for the unknot was conventionally fixed to be the identity. The normalization of $E(L)$, however, cannot be arbitrarily chosen

because it is uniquely fixed by eqs.(4.17,18). As explained in sect.12, for the unknot U_0 with zero writhe (in the N representation of $SU(N)$) one has

$$E(U_0) = \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}}. \quad (4.30)$$

Note that the expression (4.30) admits a well-defined Taylor expansion in powers of $\lambda = (\frac{2\pi}{k})$. Each term of this expansion represents precisely the result that one would obtain in perturbation theory. Differently from the case of ordinary gauge theories (and of ordinary field theories in general), the series defined by the perturbative expansion in the CS theory is summable. This also is a consequence of general covariance. In fact, there are no Dyson instabilities because, with a vanishing Hamiltonian, the CS theory is well defined and exists for both signs of k . But there are also no non-perturbative effects and there are no renormalons either. Quite remarkably, the perturbative approach to the CS model in R^3 exactly defines the theory. I have no complete explanation for all that except the following plausibility argument. In computing the expectation values $\langle W(L) \rangle$, one considers a set of operators defined within a limited domain of R^3 . This means that, in the functional integral, only the values of the fields inside some finite region of R^3 are important. On the other side, there are no physical degrees of freedom in the CS theory; in fact, the equations of motion derived from the action S_{CS} are $F_{\mu\nu} = 0$. Essentially, the vector field A_μ is a pure gauge. Now, all the values that a pure-gauge configuration can take inside some finite region of R^3 can be represented by gauge transformations connected to the identity in the whole R^3 . As a consequence, only the group structure of the infinitesimal gauge transformations (4.5) is relevant. But this is precisely what is taken carefully into account by the covariant BRS formalism and this is why (in R^3) the perturbative approach gives the right and complete answer.

All the contributions to $E(L)$ found in perturbation theory have the structure shown in eq.(4.20); e.g., the gauge group dependence appears in the form of Casimir operators of the group G multiplying rather complicated expressions which represent link invariants. In general, it is difficult to evaluate these expressions for generic links. On the other hand, suppose that one knows the values of all these invariants as functions of the links. Then the entire dependence of $E(L)$ on the group G and its representations is contained in the coefficients appearing in the sum of these invariants.

In principle, the expression (4.30) could be obtained by means of a direct resummation of its perturbative expansion. In practice, this possibility is excluded because of obvious technical reasons. The problem then is how to produce the explicit expressions of $\{E(L)\}$. The idea is to combine the general properties of the CS field theory

with the results obtained by means of some explicit computations. These computations are performed in a certain simple situation in which the complete answer (to all orders in λ) can easily be achieved. The whole point is that gauge invariance and general covariance impose very stringent constraints on the properties of the expectation values. These constraints, supplemented by some definite numerical results, uniquely fix $\{E(L)\}$.

5. WILSON LINE PROPERTIES

In sect.4, we have seen that $\{< W(L) >\}$ have a precise meaning and define an invariant $E(L)$ of regular isotopy for link diagrams. In this section, we want to explore the main properties of $E(L)$ which are consequences of its particular field theory description. In doing this, we will not use perturbation theory. We concentrate instead on the general aspects which follow from the existence of the quantized CS theory. The results obtained in this section are consequences of the discrete symmetries of the field theory, of general covariance, of the properties of the ambient space, of the gauge symmetry and of the particular structure of the Wilson line operators.

For each classical value of the vector field $A_\mu^a(x)$ and for each irreducible unitary representation ρ of G , the holonomy matrix associated with an oriented open path γ in R^3 , connecting the points x and y as shown in Fig.5(a), is given by

$$U_\rho(\gamma; x, y) = P \exp \left(i \int_\gamma A_\mu(x) dx^\mu \right), \quad (5.1)$$

where $A_\mu(x) = A_\mu^a(x) T_{(\rho)}^a$ and $\{T_{(\rho)}^a\}$ are the Hermitian generators of G in the ρ representation. By definition, the holonomy satisfies

$$U_\rho(\gamma; x, y) = U_\rho(\gamma_1; x, z) U_\rho(\gamma_2; z, y), \quad (5.2)$$

where γ_1, γ_2 and z are shown in Fig.5(b). Under gauge transformations

$$A_\mu \mapsto A_\mu^\Omega = \Omega^{-1} A_\mu \Omega - i \Omega^{-1} \partial_\mu \Omega, \quad (5.3)$$

$U_\rho(\gamma; x, y)$ transforms as

$$U_\rho^\Omega(\gamma; x, y) = \Omega^{-1}(x) U_\rho(\gamma; x, y) \Omega(y). \quad (5.4)$$

Let γ be parametrized by $\{x^\mu(s); 0 \leq s \leq 1\}$ with $x^\mu(0) = x^\mu$ and $x^\mu(1) = y^\mu$, then

the field expansion of $U_\rho(\gamma; x, y)$ takes the form

$$U_\rho(\gamma; x, y) = 1 + i \int_0^1 ds A(s) - \int_0^1 ds \int_0^1 dt A(t) A(s) - i \int_0^1 ds \int_0^1 dt \int_0^1 du A(u) A(t) A(s) + \dots, \quad (5.5)$$

where

$$A(s) = \dot{x}^\mu(s) A_\mu(x(s)). \quad (5.6)$$

Let us denote by γ^{-1} the path obtained by reversing the orientation of γ ; from eq.(5.1) it follows that

$$U_\rho(\gamma^{-1}; y, x) = U_\rho^{-1}(\gamma; x, y), \quad (5.7)$$

and since ρ is unitary, one obtains

$$U_\rho^\dagger(\gamma; x, y) = U_\rho^{-1}(\gamma; x, y). \quad (5.8)$$

When ρ is equivalent to its complex conjugate representation ρ^* ,

$$T_{(\rho)}^a = -V T_{(\rho)}^{a*} V^{-1}, \quad (5.9)$$

one immediately finds

$$U_{\rho^*}(\gamma; x, y) = U_\rho^*(\gamma; x, y) = V^{-1} U_\rho(\gamma; x, y) V. \quad (5.10)$$

The classical Wilson line associated with an oriented closed path C is

$$W_\rho(C) = \text{Tr} [U_\rho(\gamma; x, x)]. \quad (5.11)$$

From eq.(5.4), it follows that $W_\rho(C)$ is gauge invariant and, from eqs.(5.7,8), one gets

$$W_\rho^*(C) = W_\rho(C^{-1}). \quad (5.12)$$

In addition to eq.(5.12), when ρ is equivalent to ρ^* one has

$$W_\rho(C) = W_\rho^*(C) = W_\rho(C). \quad (5.13)$$

Let us consider now the Wilson line operators, defined by the framing prescription (4.16), at the quantum level. The first issue which has to be considered is gauge

invariance. Differently from the Abelian case, the expression (4.16) is not gauge invariant before the $\{\varepsilon\} \rightarrow 0$ limit is performed. So the question is if gauge invariance is restored (in the $\{\varepsilon\} \rightarrow 0$ limit) in the expectation values. It is not difficult to see that this is indeed the case. The simplified argument is the following. Under gauge transformations, the expression (4.16) is not invariant (for finite $\{\varepsilon\}$) and its variation gives a non-vanishing breaking term Δ . The operator Δ contains products of line integrals performed on framed paths; the integrands are made of products of the one-form $A_\mu dx^\mu$ multiplied by numerical functions (expressed in terms of the gauge parameters) which vanish in the $\{\varepsilon\} \rightarrow 0$ limit like powers of $\{\varepsilon; -\varepsilon_j\}$. On the other hand, in the $\{\varepsilon\} \rightarrow 0$ limit the expectation values of the product of $A_\mu dx^\mu$ integrals are in any case finite. Therefore, the insertion of the breaking term Δ inside the expectation values gives a contribution which necessarily vanishes in the $\{\varepsilon\} \rightarrow 0$ limit. Gauge invariance is in fact preserved by the framing procedure.

The consequences of eqs.(5.12,13) for the expectation values $\langle W(L) \rangle$ can easily be derived. First of all it is clear that the algebraic properties of the generators of G are left unchanged by the introduction of the framing. Second, the inversion operation $C \rightarrow C^{-1}$ is well defined for framed knots, see eqs.(3.11,16) and (4.16); moreover, $\chi(C, C_f) = \chi(C^{-1}, C_f^{-1})$, or $w(C) = w(C^{-1})$. Finally, the effect of complex conjugation in the functional integral is well known. Since the measure is real, the complex conjugation modifies the sign of the total action (4.6). In our case, this is equivalent to changing the sign of the renormalized coupling constant k . As a consequence, from eq.(5.12) it follows that

$$E^*(L; -k) = E(L^{-1}; k). \quad (5.14)$$

Consider now the case in which a particular representation ρ_i , associated to the component C_i of L , is equivalent to its complex conjugate. Then, eqs.(5.12,13) imply that $E(L)$ is invariant under $C_i \rightarrow C_i^{-1}$. Therefore, when all the representations ρ_i associated to the components C_i of L are real, $E(L)$ is invariant under the inversion operation $C_i \rightarrow C_i^{-1}$ of whatever component of L . In this case, we will say that $E(L)$ represents a regular isotopy invariant for unoriented link diagrams.

The particular structure of the action (4.6) gives a further constraint on the dependence of $E(L)$ on k . As shown in [9], it is always possible to assign definite parity properties to the fields in such a way that a modification of the orientation of the three-manifold is equivalent to a change in the sign of the renormalized coupling constant k . Because of that, if $\tilde{L}$ denotes the mirror image of the framed link L , one obtains [9]

$$E(\tilde{L}; -k) = E(L; k). \quad (5.15)$$

From the definitions (4.17,18) it also follows that when $L = L_1 \cup L_2$, e.g. when the link L is the distant (disjoint) union of L_1 and L_2 , one has

$$E(L_1 \cup L_2) = E(L_1) E(L_2) . \quad (5.16)$$

Eq.(5.16) is just a consequence of general covariance and of the uniqueness of the vacuum.

Finally, the particular structure of the Wilson line operator combined with the properties of the ambient space permit the derivation of very useful satellite formulae. The remaining part of this section is devoted to this subject. Let us start by considering an oriented framed link which is the distant union of two unknotted knots U_1 and U_2 . Let ρ_1 and ρ_2 be the irreducible representations of G associated with U_1 and U_2 and let us consider the case in which the framings U_{1f} and U_{2f} of U_1 and U_2 are such that

$$\chi(U_1, U_{1f}) = 0 = \chi(U_2, U_{2f}) . \quad (5.17)$$

From eq.(5.16), we know that

$$< W_{\rho_1}(U_1) W_{\rho_2}(U_2) > = < W_{\rho_1}(U_1) > < W_{\rho_2}(U_2) > . \quad (5.18)$$

One can imagine that eq.(5.18) has been obtained in the following way. Because of general covariance, $< W_{\rho_1}(U_1) W_{\rho_2}(U_2) >$ does not depend on how "far" U_2 is from U_1 . In particular, one can send U_2 to infinity and then, by the unicity of the vacuum, eq.(5.18) follows. Now one can ask what happens in the opposite limit, namely, when U_2 gets closer and closer to U_1 and, at the end, it coincides with U_1 . Of course, this limit has to be carefully defined; in particular, one must take into account the two framings and the two path-orderings.

The limit in which U_2 coincides with U_1 is defined in the following way. First of all, U_1 and U_2 can always be represented by two planar circles which are parallel to each other, as shown in Fig.6. Moreover, the circles U_1 and U_2 are placed in R^3 in such a way that they define the two bases of a cylinder with eight δ ($\delta > 0$). We are interested in the $\delta \rightarrow 0$ limit, of course, and we already know that this limit exists because $< W_{\rho_1}(U_1) W_{\rho_2}(U_2) >$ does not depend on δ .

Let U_1 be parametrized by $\{x^\mu(s), 0 \leq s \leq 1\}$. Then one can parametrize U_2 by

$$y^\mu(s) = x^\mu(s) + \delta v^\mu , \quad |\vec{v}| = 1 , \quad (5.19)$$

where $\vec{v} = \delta \vec{v}$ is the constant vector which connects each point x of U_1 with its corresponding point y in U_2 , as shown in Fig.6. With the parametrization (5.19) of U_2 ,

the product of the two path-orderings on U_1 and U_2 can obviously be reduced to the single ordering defined by the s parameter. In fact, the path-ordering between the two one-forms

$$A_\mu^a(x) T_{(\rho_1)}^a dx^\mu \quad \text{and} \quad A_\nu^b(y) T_{(\rho_2)}^b dy^\nu ,$$

which have to be integrated along U_1 and U_2 respectively, is totally ineffective because $\{T_{(\rho_1)}^a\}$ and $\{T_{(\rho_2)}^b\}$ act on different spaces and therefore commute.

Let us consider the framings U_{1f} and U_{2f} . Since we are in the conditions specified by eq.(5.17), we can take the unit vector $\vec{v}$, appearing in eq.(5.19), as the vector defining both the framings U_{1f} and U_{2f} . For instance, one can choose U_{1f} specified by

$$z^\mu(s) = x^\mu(s) + \delta_1 v^\mu , \quad \delta_1 > 0 , \quad (5.20)$$

and U_{2f} specified by

$$w^\mu(s) = y^\mu(s) + \delta_2 v^\mu , \quad \delta_2 > 0 . \quad (5.21)$$

According to the framing prescription (3.15,16) and (4.16), the Wilson operators $W_{\rho_1}(U_1)$ and $W_{\rho_2}(U_2)$ are characterized by the sets $\{\delta_1\}$ and $\{\delta_2\}$ and we are interested in the limit in which $\{\delta_1\} \rightarrow 0$, $\{\delta_2\} \rightarrow 0$ and then $\delta \rightarrow 0$. Here is the crucial point: this limit is nothing but the limit (performed along a particular direction in the $\{\varepsilon\}$ space) entering the framing procedure which defines a single Wilson line operator W' in U_1 with framing U_{1f} .

The G representation associated to W' can immediately be obtained; since the generators are

$$T_{(\rho_1)}^a + T_{(\rho_2)}^a = T_{(\rho_1)}^a \otimes 1 + 1 \otimes T_{(\rho_2)}^a , \quad (5.22)$$

W' is defined in the $\rho = \rho_1 \otimes \rho_2$ representation of G . This is in agreement with the fact that

$$\text{Tr}_{\rho_1} \text{Tr}_{\rho_2} = \text{Tr}_{\rho_1 \otimes \rho_2} , \quad (5.23)$$

In conclusion

$$W' = W_{\rho \otimes \rho_2}(U_1) , \quad (5.24)$$

with framing U_{1f} . Let the tensor product $\rho_1 \otimes \rho_2$ have the decomposition

$$\rho_1 \otimes \rho_2 = \oplus_i \rho(i) \quad (5.25)$$

in terms of the irreducible representations $\{\rho(t)\}$ of the group G . It follows then that

$$\langle W_{\rho_1}(U_1) W_{\rho_2}(U_2) \rangle = \langle W_{\rho_1 \otimes \rho_2}(U_1) \rangle = \sum_t \langle W_{\rho(t)}(U_1) \rangle \quad (5.26)$$

Eq.(5.26) combined with eq.(5.18) finally give

$$\langle W_{\rho_1}(U_1) \rangle \langle W_{\rho_2}(U_2) \rangle = \sum_t \langle W_{\rho(t)}(U_1) \rangle \quad (5.27)$$

Eq.(5.27) gives origin to highly non-trivial constraints for the expectation values of the unknots. Of course, for the moment the only exact expression that we know for the unknots concerns the trivial ($\rho = 0$) representation of the group G : in this case $\langle W_{(\rho=0)}(U) \rangle = 1$. Clearly, for any knot C also it is true that $\langle W_{(\rho=0)}(C) \rangle = 1$.

Now eq.(5.26) can easily be generalized. For example, suppose that U_1 and U_2 are two components of the link shown in Fig.7(a) with framings specified by eq.(5.17). The whole argument used to derive eq.(5.26) can be repeated without any modification and the result obtained in this case is

$$\langle W_{\rho_1}(U_1) W_{\rho_2}(U_2) W_{\rho_3}(U_3) \rangle = \sum_t \langle W_{\rho(t)}(U_1) W_{\rho_2}(U_2) \rangle \quad (5.28)$$

where the two-component links appearing in the sum of the RHS of eq.(5.28) are shown in Fig.7(b) and the irreducible representations $\{\rho(t)\}$ are those defined in eq.(5.25).

Eqs.(5.26,28) are particular examples of satellite formulae. Let us briefly recall what satellites (in the knot theory context) are. Consider a link L in R^3 with m components $\{C_1, \dots, C_m\}$. Starting from L one can construct another link L' called a satellite of L ; L' is obtained by substituting one of the components, say C_i , of L by another link P called the pattern link. In order to specify how this substitution has to be performed, the pattern link P is usually shown with respect to a reference unknotted circle in R^3 whose complement defines the tube in which the substitution $C_i \rightarrow P$ is performed; an example is shown in Fig.8. The original link L is called a companion of L' . Of course, in looking for satellite formulae for $\langle W(L) \rangle$, one has to take care of the different representations of G associated with the Wilson line operators and of their framings also.

The same construction, which has been used before for deriving eqs.(5.26,28), allows one to obtain a very important satellite formula.

Consider a generic oriented framed link in R^3 and let C be one of its components. We indicate by $\{W(L; C, C_f, \rho)\}$ the Wilson line operators associated with L when C_f is the framing of C and the irreducible representation ρ of G is associated with C ; the framings and the representations of all the remaining components of L are kept fixed. Now we concentrate on a particular satellite of L obtained by substituting for C the two-component pattern link P shown in Fig.9. The link P is fixed by imposing two conditions: first, that the linking number between its two components C_1 and C_2 is

$$\chi(C_1, C_2) = \chi(C, C_f) \quad (5.29)$$

and, second, that the framings C_{1f} and C_{2f} are such that

$$\chi(C_1, C_{1f}) = \chi(C_2, C_{2f}) = \chi(C, C_f) \quad (5.30)$$

Let ρ_1 and ρ_2 be the irreducible representations associated with C_1 and C_2 . We indicate by $W(L; P[\rho_1, \rho_2])$ the Wilson line operator associated to the satellite of L obtained by substituting C by the pattern link P shown in Fig.9 and specified by the conditions (5.29,30). A straightforward extension of the argument used to derive eq.(5.26) shows that

$$\langle W(L; P[\rho_1, \rho_2]) \rangle = \sum_t \langle W(L; C, C_f, \rho(t)) \rangle \quad (5.31)$$

where the sum is performed on the irreducible representations of G appearing in the decomposition (5.25).

Eq.(5.31) represents one of the most important results of this paper. The framing structure of $W(L)$ is crucial for deriving this equation which, in turn, essentially permits us to solve the CS theory.

Since the exact solution (3.18,19) of the Abelian CS theory is known, one can check in this particular case all the results obtained in this section. For an Abelian gauge group, the decomposition (5.25) takes the simple form

$$e_1 \otimes e_2 = (e_1 + e_2) \quad (5.32)$$

where e_1 and e_2 are the charges of the Wilson lines. It is easy to see that eqs.(5.14-16) and eqs.(5.26,28,31) are satisfied. In the non-Abelian case, the explicit expression of $\langle W(L) \rangle$ has been computed in [9] at second non-trivial order of perturbation theory. Again, all the general properties of $\langle W(L) \rangle$ discussed in this section are indeed satisfied (at order λ^2 included).

Note that in the CS theory the naive use of a reflection positivity argument may lead to wrong conclusions (in particular for the unknot). Finding the correct version of reflection positivity in the quantum CS theory is an interesting open problem.

Summarizing, the link invariants obtained in the CS theory are of a very particular kind. First of all, the dependence of $E(L)$ on the renormalized coupling constant k is constrained by eqs.(5.14,15). Second, because of general covariance, the factorization property (5.16) must be satisfied. Finally, satellite formulae like eqs.(5.26,28) and, more generally, of the form shown in eq.(5.31) must hold. As we shall see, all these conditions impose such strong constraints on $E(L)$ that, by adding some extra numerical inputs to our data, the universal link polynomial $E(L)$ turns out to be uniquely determined.

The problem now is how to incorporate all this information in some definite scheme permitting the reconstruction of $E(L)$. A possibility is to operate directly on the link diagrams by using a set of simple rules derived from the three-dimensional quantum CS theory. For this purpose, it is convenient to introduce a more detailed and, in a certain sense, systematic description of link diagrams. As is well known, this can be done by making use of the Artin braid group.

6. BRAID GROUP

As mentioned in sect.2, the study of the topology of links in R^3 can be reduced to the analysis of the link diagrams; more precisely, one has to consider the properties of the classes of link diagrams modulo the Reidemeister moves. A further step consists of formulating the problem in an algebraic context; the braid group enters precisely at this stage.

The Artin braid group B_n is generated by $(n-1)$ elements $\{g_1, g_2, \dots, g_{n-1}\}$ which are called the generators and satisfy the relations [37]:

$$g_i g_j = g_j g_i, \quad |i - j| > 1, \quad (6.1)$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad i < n-1. \quad (6.2)$$

Each generator or product of generators admits a suggestive n -string graphical presentation. One can associate to g_i the picture shown in Fig.10(a) and the product of two generators, say $g_2^{-1} g_1$, can be represented as in Fig.10(b). The graphs associated to the product of generators of B_n can be understood as diagrams corresponding to the projections on a plane of n strings in R^3 oriented, for instance, in the upward direction.

Of course, the elements of B_n are in correspondence with the sets of the products of generators modulo the equivalence relation defined by eqs.(6.1,2). Because of the above mentioned presentation in terms of diagrams, eqs.(6.1,2) define also an equivalence relation between diagrams. In particular, two diagrams are equivalent iff the associated strings in R^3 are connected by smooth deformations in R^3 performed by keeping the two planes, to which the endpoints of the strings belong, fixed. So, conditions (6.1,2) give an algebraic realization of a somewhat restricted form of ambient isotopy; its associated equivalence classes are described precisely by the elements of B_n .

Note that to the overcrossing configuration L_+ , occurring between the i -th and the $(i+1)$ -th strings, is associated the generator g_i , whereas to the undercrossing configuration L_- is associated g_i^{-1} . Let us consider the RM. Invariance under RM of type *II* is automatically satisfied because, by definition, $g_i g_i^{-1} = 1 = g_i^{-1} g_i$. Invariance under RM of type *III* also is satisfied because of the defining relation (6.2). RM of type *I* only are excluded from this scheme and the reason is very simple: RM of type *I* are described in terms of diagram configurations in which the string orientations

necessarily go upwards and downwards simultaneously. Only by taking into account the closure of braids can RM of type I be formulated.

The closure $\widehat{\sigma}$ of an element $\sigma \in B_n$ is a diagram obtained by connecting (in an ordering-preserving way) the endpoints of the strings of a graph associated to σ , as shown in Fig.11(a), and it will be indicated like in Fig.11(b). Closures of braids describe diagrams of oriented links in R^3 .

The fact that any link diagram can be represented as the closure of a braid may seem a rather trivial result; yet, it has important consequences. For example, this means that any link is a satellite of the unknot with a pattern link made of oriented strings. So, if one finds a general formula for satellites, then the whole problem of computing the link invariants $\{E(L)\}$ reduces to the problem of computing the unknots.

Given two generic elements σ_1 and σ_2 of B_n , the closure of σ_1 and the closure of $\sigma_2^{-1}\sigma_1\sigma_2$ are related by RM of type II , whereas in general $\sigma_1 \neq \sigma_2^{-1}\sigma_1\sigma_2$. Therefore, invariance under RM of type II requires that we must identify the closure of conjugate elements of B_n , i.e. elements related by the adjoint action of the group. For the conjugacy classes of B_n , invariance under RM of type III is satisfied.

Invariance under RM of type II and III only defines precisely regular isotopy. So, the first conclusion is that the equivalence classes of regular isotopic link diagrams are described by the conjugacy classes of the braid groups B_n for arbitrary n . Let us denote by B_∞ the union of the braid groups B_n for arbitrary n equipped with the natural inclusion $B_n \subset B_{n+1}$. The functions defined on the conjugacy classes of B_∞ describe link invariants of regular isotopy.

Let us make an example. The set Z of the integer numbers is a group with respect to the usual sum operation. Consider now a homomorphism φ of B_n into Z . Since Z is an Abelian group, conjugate elements of B_n have the same image under φ . Therefore, φ defines a function of the conjugacy classes of B_n into Z . Note that the generators of B_n are conjugate elements; so, it is sufficient to specify φ on a single generator only. When

$$\varphi(g_i) = 1, \quad (6.3)$$

the regular isotopy invariant described by φ is the writhe number mentioned in sect.2. The conjugacy classes of $\sigma \in B_n$ will be denoted by $<\sigma>$.

In order to recover ambient isotopy invariance, we must include RM of type I . It is easy to see that each RM of type I connects $<\sigma>$ with $<\sigma g_n^{\pm 1}>$. Therefore, the equivalence classes of ambient isotopic links are described by the set $\mathcal{M}$ of the

conjugacy classes of B_∞ modulo the identification

$$\sigma \in B_n \leftrightarrow \sigma g_n^{\pm 1} \in B_{n+1}. \quad (6.4)$$

Of course, any function defined in $\mathcal{M}$ describes a link invariant of ambient isotopy.

Summarizing, any oriented link diagram can be represented as the closure $\widehat{\sigma}$ of an element $\sigma \in \bigcup_n B_n$. The diagrams $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ describe ambient isotopic oriented links if σ_1 and σ_2 differ by a sequence of elementary moves (Markov moves) of two types [38]:

M1) σ_1 and σ_2 are conjugate elements of B_n ;

M2) if $\sigma_1 \in B_n$, then $\sigma_2 = \sigma_1 g_n^{\pm 1} \in B_{n+1}$.

Any function invariant under Markov moves is called a Markov trace and defines a link invariant of ambient isotopy.

Usually, the method employed for constructing such functions is to make use of some specific representation of the braid group. Between them, Hecke algebra representations play a prominent role in connection with the classical link polynomials. Consider the algebra H_n generated by a set of $(n-1)$ elements $\{T_1, T_2, \dots, T_{n-1}\}$ with defining relations

$$T_i T_j = T_j T_i, \quad |i-j| > 1, \quad (6.5)$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad i < n-1, \quad (6.6)$$

$$T_i^2 + a T_i + b = 0, \quad i = 1, 2, \dots, n-1, \quad (6.7)$$

where a and b are (in general complex) parameters. By comparing eqs.(6.1,2) with eqs.(6.5,6) it is clear that the homomorphism

$$g_i \mapsto T_i \quad (6.8)$$

defines a representation of B_n in H_n . The quadratic relation (6.7) satisfied by the $\{T_i\}$ generators greatly simplifies the construction of link invariants. In fact, because of eqs.(6.5-7), each element of H_{n+1} can always be written as a sum of products of generators in which the generator T_n appears at most once. Technically, it is stated that H_{n+1} admits a presentation as a (H_n, H_n) -bimodule; the map from the set of (H_n, H_n) -bimodules into H_{n+1} is an isomorphism [39]. In practice, this means that one can define in a natural (and unique) way a correspondence between the conjugacy classes of H_{n+1} and those of H_n . By means of the Ocneanu trace [13,27,39], a recursive

procedure can then be defined for constructing link invariants for a generic link in terms of the value of the unknot. This construction clearly shows that the $S(L)$ polynomial defined in sect.2 is in fact the fundamental invariant from which all the other classical polynomials can be derived (for Hecke algebra representations of B_n).

We conclude this section by mentioning the colored (or pure) braid group C_n which is a particular subgroup of B_n . The relation between B_n , C_n and the permutation group P_n can be understood as follows. Each element $\sigma \in B_n$ defines a permutation $\bar{\sigma} \in P_n$ of the n end-points of the strings of the associated diagrams. The map π from B_n to P_n

$$\pi : \sigma \mapsto \pi(\sigma) = \bar{\sigma} \quad (6.9)$$

defines a homomorphism of B_n into P_n for which

$$\text{Ker } \pi = C_n \quad , \quad (6.10)$$

where C_n is the colored braid group. Clearly, the closures of elements of C_n describe links with n components. In general, the number of components of the link associated with the closure of $\sigma \in B_n$ is equal to the number of cycles contained in $\bar{\sigma} \in P_n$.

Different examples of B_n representations are known to be related to the classical polynomials. On the other hand, we know that in the CS model the gauge group structure is one of the fundamental aspects of the theory. So, we are interested in the particular B_n representations whose construction is based on some Lie algebra structure. Representations of this kind have indeed been discovered: the so-called R -matrix representations.

7. R-MATRIX AND BRAIDS

In this section, we briefly review some results concerning braid group representations associated with ordinary simple Lie algebras. The material presented here will not be used for solving the CS model. The purpose of this section is just to recall some general features of the quasi-tensor category of quasi-triangular quasi-Hopf algebras which are useful for identifying the $E(L)$ polynomial obtained in the CS theory.

Suppose that a finite dimensional linear space V_i is associated with the i -th string of the diagrams representing the B_n elements. For the moment, we assume that all the spaces $\{V_i\}$ have the same dimension D . Let us try to find a matrix representation of B_n in the resulting space $V(n) = V_1 \otimes \dots \otimes V_n$ of the form

$$g_i \mapsto G_i = 1 \otimes \dots \otimes R \otimes \dots \otimes 1 \quad , \quad (7.1)$$

where R acts on $V_i \otimes V_{i+1}$. The defining relations (6.1) are obviously satisfied, whereas eq.(6.2) implies

$$R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23} \quad , \quad (7.2)$$

where the matrices

$$\begin{aligned} R_{12} &= R \otimes 1 \quad , \\ R_{23} &= 1 \otimes R \quad , \end{aligned} \quad (7.3)$$

act on $V_1 \otimes V_2 \otimes V_3$. Eq.(7.2) is called the quantum Yang-Baxter equation (QYBE). Each non-singular solution of eq.(7.2) provides an explicit representation of B_n , eq.(7.1).

Assuming that R admits an enhancement μ , Turaev has shown [22] how to construct link invariants. The matrix μ , acting on the space associated to a single string, is called an enhancement for R if, in $V_1 \otimes V_2$, the following conditions are satisfied :

$$R \mu_1 \otimes \mu_2 = \mu_1 \otimes \mu_2 R \quad , \quad (7.4)$$

$$\text{Tr}_{V_2} (R \mu_2) = \alpha \cdot 1_{V_1} \quad , \quad (7.5)$$

$$\text{Tr}_{V_2} (R^{-1} \mu_2) = \alpha^{-1} \cdot 1_{V_1} \quad . \quad (7.6)$$

In eqs.(7.5,6), the matrices $R\mu_2$ and $R^{-1}\mu_2$ act on $V_1 \otimes V_2$ but the trace is performed in V_2 only. On the RHS of eqs.(7.5,6), the identity operator is defined in V_1 . In general,

μ_i denotes the enhancement matrix acting on V_i and the product

$$\mu^{\otimes n} = \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_n \quad (7.7)$$

is called the measure in $V(n)$.

Consider now a link diagram L described by the closure $\widehat{\sigma}$ of an element $\sigma \in B_n$ and let B be the matrix representing σ in $V(n)$. The function

$$Y(L) = \text{Tr}(B\mu^{\otimes n}) \quad (7.8)$$

is a regular isotopy invariant for link diagrams. Indeed, eq.(7.4) implies that $Y(L)$ is invariant under Markov moves $M1$ of the first type. Moreover, from eqs.(7.5,6) it follows that

$$\begin{aligned} Y(L^{(+1)}) &= \alpha Y(L^{(0)}) , \\ Y(L^{(-1)}) &= \alpha^{-1} Y(L^{(0)}) , \end{aligned} \quad (7.9)$$

where $L^{(\pm)}$ and $L^{(0)}$ are shown in Fig.4. Eqs.(7.9) mean that $Y(L)$ is a generalized $S(L)$ polynomial, in the sense that the covariance properties of $Y(L)$ under RM of type I (or Markov moves $M2$) coincide with those of the $S(L)$ polynomial. Finally, assume that the R matrix satisfies the polynomial equation

$$\sum_{p=a}^b c_p R^p = 0 , \quad (7.10)$$

for some given coefficients $\{c_p\}$. Then, $Y(L)$ obviously satisfies

$$\sum_{p=a}^b c_p Y(L_p) = 0 , \quad (7.11)$$

where the link diagrams $\{L_p\}$ coincide except in a neighbourhood of a point where L_p has a sequence of p consecutive crossings. Eq.(7.11) is the generalization of the skein relation iv) appearing in (2.11).

Given some explicit representation of the R matrix and of the enhancement μ , the direct computation of the trace entering the definition (7.8) of $Y(L)$ may become rather laborious. There is a particular case in which the computation of $Y(L)$ greatly simplifies. Suppose that the R matrix has only two different eigenvalues and then the polynomial equation (7.10) becomes quadratic. In this case, eq.(7.1) defines a Hecke algebra representation of B_n and then, by means of the Ocneanu trace, $Y(L)$ can easily be reconstructed. Given the normalization of the unknot, $Y(L)$ is completely fixed by the knowledge of the two eigenvalues of the R matrix and of the parameter α .

Unfortunately, when the polynomial equation (7.10) is of order higher than two, the normalization of the unknot together with the knowledge of the eigenvalues of the R matrix and of α are no longer sufficient [40] for uniquely determining $Y(L)$. In these conditions, the direct computation of the trace (7.8) has to be performed (in general).

Of course, once $Y(L)$ has been constructed, one can easily define a new invariant of ambient isotopy just by multiplying $Y(L)$ by the factor $\alpha^{-w(L)}$.

It has been shown [18] that a quasi-triangular Hopf (QTH) algebra structure, defined in terms of the so-called q -deformations of a simple Lie algebra $\mathcal{G}$, is naturally associated to the QYBE. As a result, for any D -dimensional irreducible representation ρ of $\mathcal{G}$, a particular solution of the QYBE can be constructed [17,18,20,21,41]. The associated B_n representation, defined by eq.(7.1), is called an R -matrix representation. For generic ρ , the expression of R is rather complicated. However, a simple physical interpretation of this construction has been discovered by Kohno [19]. Consider a set of n point-like particles moving on a plane. The positions of these particles can be parametrized by complex coordinates $\{z_i\}$ with $i = 1, \dots, n$. Assuming that any two particles never occupy the same position, the configuration space of the system is $X_n = \{(z_1, \dots, z_n) \in C^n : z_i \neq z_j \text{ for } i \neq j\}$. The world-lines of the particles, corresponding to a closed path in X_n , provide a geometric description of an element of the pure braid group C_n . If these particles are identified under permutations, the fundamental group of the resulting configuration space $Y_n = X_n/P_n$ is precisely the braid group B_n . Suppose now that to each particle is associated a linear space of dimension D in which an irreducible representation ρ of $\mathcal{G}$ is defined. Let $\{T_i^a\}$ be the generators of $\mathcal{G}$ in the ρ representation acting on the space H_i associated to the i -th particle. The total space associated to the n particles is $H(n) = H_1 \otimes \cdots \otimes H_n$. The one-form

$$\omega = \frac{\lambda}{\pi} \sum_{i \neq j} \Omega_{ij} d \ln(z_i - z_j) , \quad (7.12)$$

where

$$\Omega_{ij} = T_i^a \otimes T_j^a \quad (7.13)$$

and λ is a free parameter, represents a flat connection in X_n and, since ω is symmetric under permutations of the particles, ω extends to a flat connection $\tilde{\omega}$ in Y_n . For each element $\gamma \in \pi_1(Y_n)$, the associated holonomy

$$h(\gamma) = P \exp \int_{\gamma} \tilde{\omega} \quad (7.14)$$

gives then a matrix representation h of B_n on $H(n)$. Quite remarkably, h is equivalent [19] to the representation (7.1) obtained by means of the standard algebraic quantum

group construction when the deformation parameter q is given by

$$q = \exp(-i\lambda) \quad (7.15)$$

In the case of two particles only, the generator of B_2 is represented by

$$h(g_1) = \Pi_{12} q^{\alpha_{12}} \quad (7.16)$$

where Π_{12} is the permutation operator acting on $H_1 \otimes H_2$.

The particular structure of the Kohno connection (7.12) implies that, in the generic case of n ($n \geq 2$) particles, each generator of B_n is represented in $\hbar$ by a matrix which is equivalent to the expression (7.16) but acts on a certain appropriate subspace of $H(n)$. The intertwining operator depends on the particular generator under consideration. Since $\hbar$ is equivalent to the R -matrix representation (7.1), this means that the matrix R is equivalent to the expression (7.16).

It should be noted that Kohno's construction is entirely based on the structure of the ordinary Lie algebra $\mathcal{G}$. Unfortunately, in this approach the inclusion $B_n \subset B_{n+1}$ is not realized in a natural way because the intertwining operators connecting the generators have to be modified in going from B_n to B_{n+1} . On the other hand, in the R -matrix algebraic approach the naive inclusion $B_n \subset B_{n+1}$ is imposed by construction. However, the price to be paid is that the fundamental role of $\mathcal{G}$ is somehow obscured. If one insists on requiring that the B_n generators have the form shown in eq.(7.1), then the q -deformations of $\mathcal{G}$ seems to play a natural role.

For generic values of the deformation parameter q , which of the two methods one decides to adopt is a matter of taste. Both these methods can naturally be extended to the case in which different representations $\{\rho_i\}$ of $\mathcal{G}$ are associated to the different particles, or, to the different strings of the graphical presentation of B_n .

Drinfeld has shown [12] that the QTH algebra associated to the QYBE actually provides a particular realization of a more general structure called the quasi-triangular quasi-Hopf (QTQH) algebra. In this new context, the QYBE is in general modified; still, the braid group representations described by the quasi-tensor category defined by the QTQH algebra remain the same. The advantage of using a QTQH algebra is that the "generalized" R -matrix [12] can be taken to be precisely

$$R = \Pi_{12} q^{T_{(\rho_1)}^* T_{(\rho_2)}} \quad (7.17)$$

where $\{T_{(\rho_1)}^*\}$ and $\{T_{(\rho_2)}^*\}$ are the $\mathcal{G}$ generators in the representations ρ_1 and ρ_2 . In these conditions, a non-trivial associativity isomorphism must be introduced for satisfying

the pentagon- and hexagon-relations defining a QTQH algebra [12]. The link invariants obtained from the braid representations associated with this quasi-tensor category are called by Drinfeld the universal link invariants [12]. The QTQH algebra is the algebraic context which naturally fits the more geometrical approach of Kohno. The generalized Kohno construction, defined for arbitrary representations of $\mathcal{G}$, provides the general structure of the B_n representations constructed by means of ordinary Lie algebras. Up to equivalence, the form of the B_n generators is shown precisely in eq.(7.17).

As mentioned before, for q generic, the standard algebraic quantum group method and Kohno's method are equivalent. When q is a root of unity, Kohno's construction is still well defined, whereas the definition and the meaning of the quantum group itself present some problems. In this sense, and as far as braid group representations are concerned, Kohno's approach seems to be more simple and in fact more fundamental than the standard quantum group approach.

A few possible methods for computing the link invariants (7.8) can in principle be used. As already mentioned, in the case of Hecke algebra B_n representations, the construction of the $S(L)$ polynomial can easily be performed by using, for instance, the skein relation. For B_n representations which are not of the Hecke type, one should find the explicit expression of the R matrix and of the enhancement μ . The direct computation of the trace (7.8) is then straightforward but, in practice, quite tedious. Kirillov and Reshetikhin [20,21] have developed a nice method for computing the link invariants in terms of the q -deformed $6j$ -symbols of the quantum groups. Depending on your ability in computing these $6j$ -symbols, this method looks more or less practical. Particular examples of link invariants which cannot be obtained in Hecke algebra representations have been considered for instance in [42]. New examples of this kind have been produced also in [43]. What is still missing in all the different approaches is a systematic and simple method of producing these new invariants in such a way that the properties of the ordinary Lie algebras have a transparent and fundamental role. The quantum CS field theory naturally suggests how to solve this problem.

8. CHERN-SIMONS MONODROMIES

Like in any field theory, $\langle W(L) \rangle$ can be interpreted as a scalar product between two states. Given a surface which separates R^3 into two parts P_1 and P_2 , the computation of the functional integral in R^3 can be divided into two steps. First, the integral is performed separately in P_1 and P_2 with given boundary conditions on the surface for the fields. In this way, one obtains the wave functions of one "bra" and one "ket" state vectors. Second, the scalar product between these two states gives the result of the whole functional integral in R^3 . In this sense, one can consider the space of the states of the CS theory in the presence of Wilson line operators.

Let us consider the case of a surface intersecting a link L on which a Wilson line operator $W(L)$ is defined. The states of the system are associated with a punctured surface; the positions of the punctures are defined by the intersection points of the surface with the oriented link L . Each puncture is characterized by the quantum numbers of a representation ρ of G inherited from $W(L)$. As observed in [6], the non-trivial information carried by the states of the theory is represented by the monodromy matrices acting on the state wave functions when the punctures are exchanged according to the action of the braid group. This method of computing the monodromy matrices looks very similar to Kohn's construction described in sect.7. In fact, finding the form of the monodromy matrices obtained in the CS theory is equivalent to determining the structure of the R matrix entering the braid representations realized in the state space of the CS model.

The direct computation of the state wave functions associated to a generic link and a generic punctured surface by means of the functional integral is rather complicated. On the other hand, our real interest does not concern the wave functions themselves but the monodromy matrices; our wish is to compute them exactly, to all orders in $\lambda = (\frac{2\pi i}{k})$. The crucial observation [6] which permits us to simplify and actually solve this problem is that the structure of the monodromy matrices is independent of the particular link under consideration and also of the measure giving the pairing between bra and ket vectors. This is just a consequence of the properties of the transition amplitudes in quantum field theory. This being the case, the computation of the monodromy matrices can consistently be performed in a particularly simple situation in which all the complications due to the pairing and the details of the links are absent.

Consider the Hamiltonian approach to the CS theory based on a $(2+1)$ decomposition of R^3 [11]. As is well known, one of the components of the vector field, say $A_3^a(x)$, appears in the action without "time" derivatives acting on it and can be interpreted as a Lagrange multiplier enforcing the Gauss Law constraint. Of course, in this formalism the state wave functions are not normalizable: the group volume of time-independent gauge transformations should be factorized. Since we are interested in the monodromy matrices only, however, we can safely ignore the pairing. More generally, it is well known that the expectation values $\langle W(L) \rangle$, being vacuum-to-vacuum amplitudes, cannot be obtained by means of a naive Hamiltonian formulation. But in the Hamiltonian approach we do not want to compute $\langle W(L) \rangle$ at all; we want just to find the form of the monodromy matrices. So, we consider the state wave functions associated with a fixed-time plane $\mathcal{P}$ cutting two (or more) oriented parallel straight lines running along the time direction. One can imagine that two Wilson operators, in the irreducible representations ρ_1 and ρ_2 of G , are associated with these lines so that the two punctures on $\mathcal{P}$ are characterized by the quantum numbers of the ρ_1 and ρ_2 representations. The monodromy matrix M associated with an exchange of the two punctures on the plane $\mathcal{P}$ (corresponding to an overcrossing with writhe $w = 1$) has been computed in [11]. The result is

$$M = \Pi_{12} q^{T_{(\rho_1)}^a T_{(\rho_2)}^a} \quad , \quad (8.1)$$

where Π_{12} is the permutation operator and

$$q = \exp\left(-\frac{2\pi i}{k}\right) \quad . \quad (8.2)$$

In the Hamiltonian approach, the exact results (8.1,2) can be obtained because the Gauss Law constraint can easily be solved and, moreover, without the Lagrange multipliers $\{A_3^a(x)\}$, the CS action becomes quadratic in the remaining fields and actually defines a completely free theory [11]. In particular, the dependence (8.2) of the deformation parameter q on the renormalized coupling constant k is a simple consequence of the canonical commutation relations between the fields. More details on the derivation of eqs.(8.1,2) can be found in [11].

Eqs.(8.1,2) can also be obtained by noting that a particular solution of the Gauss Law constraint in the CS theory in the presence of punctures on a plane, eq.(3.4) of [6], is given precisely by the Kohn connection (7.12) in which the parameter λ coincides with $\frac{2\pi i}{k}$. Needless to say, the results (8.1,2) are in perfect agreement with the perturbative computations of $\langle W(L) \rangle$ performed in [9]. There is agreement also with the general constraints (5.14,15).

For several reasons, the "shift"-conjecture formulated in [6], concerning the dependence of q on the renormalized coupling constant k , turns out to be false. As for the η -invariant argument presented in [6], it is not an argument in favour of the shift at all*.

In conclusion, the monodromy matrix (8.1) of the CS theory has precisely the structure of the generalized R matrix (7.17) entering the definition of the quasi-tensor category associated to the QTDH algebras in which the deformation parameter q is given in eq.(8.2). The link invariant $E(L)$ obtained in the CS theory is nothing but the universal link invariant considered by Drinfeld.

It is well known [44-48] that braid group representations associated to the quasi-tensor category of QTDH algebras can be found also in several two-dimensional systems. For instance, monodromy representations of B_n are realized in the critical Ising model, WZNW models, generalized conformal Toda systems, rational conformal field theories in general, etc. In this sense, the three-dimensional CS theory describes the universal structure of the braid group representations underlying all these different models. This property is also called the universality of the CS theory.

* I would like to recall that how the deformation parameter q depends on the renormalized coupling constant k is a well-defined problem, whereas how q depends on the bare coupling constant k_0 is a meaningless question. A simple second order computation [9] is sufficient for showing that the "famous shift" conjecture is not in agreement with the facts. Of course, this does not mean that this conjecture has not been useful; on the contrary, it has been of primary importance for the development of the theory. New ideas and conjectures are always useful. But, in order to achieve some concrete progress, it is also important to recognize when some of these conjectures do not fit the reality of facts.

9. DEFINING RELATIONS

In this section, the information obtained from the three-dimensional CS field theory is organized in a set of rules permitting the reconstruction of $E(L)$. These rules are formulated for link diagrams corresponding to oriented framed links in vertical framing. The link diagrams are represented by the closure of elements of the braid group. Each string of the diagrams is associated with an irreducible representation of G .

Let us consider first the information in which the closure of braids is not involved.

I. PROJECTION DECOMPOSITION

We have seen in sect.5 that one of the main properties of the Wilson line operators is represented by some kind of "fusion rule" as shown in eq.(5.26). The elementary blocks entering the decomposition (5.25) are the irreducible representations of G . By the way, this is precisely what is expected to follow from gauge invariance. So, given two irreducible representations ρ_1 and ρ_2 , the tensor product $\rho_1 \otimes \rho_2$ is decomposed

$$\rho_1 \otimes \rho_2 = \oplus_i \rho(t) \quad (9.1)$$

in terms of the irreducible components $\{\rho(t)\}$ of G and the relation shown in Fig.12 holds. The boxes introduced in Fig.12 are projectors in the sense shown in Fig.13.

II. CROSSING

We know that to each crossing between strings is associated a monodromy matrix which is equivalent to the expression (8.1). On the other hand,

$$\begin{aligned} T_{(\rho_1)}^a \otimes T_{(\rho_2)}^a &= \frac{1}{2} \left[\left(T_{(\rho_1)}^a + T_{(\rho_2)}^a \right)^2 - T_{(\rho_1)}^a T_{(\rho_1)}^a - T_{(\rho_2)}^a T_{(\rho_2)}^a \right] \\ &= \frac{1}{2} [Q(\rho_1 \otimes \rho_2) - Q(\rho_1) - Q(\rho_2)] \end{aligned} \quad (9.2)$$

Therefore, by making use of the projection decomposition (I), the crossing relations shown in Figs.14,15 must hold. Note that, when $\rho_1 = \rho_2$, the order of ρ_1 and of ρ_2 in the projectors appearing in Figs.14,15 can be exchanged; in doing this, one gets a factor $(+1)$ if $\rho(t)$ is symmetric and a factor (-1) if $\rho(t)$ is antisymmetric. Of course, the braid relations (6.1,2) are assumed to be satisfied.

The introduction of projectors in the crossing relations looks very similar to the Kirillov-Reshetikhin construction [20,21]. In the method presented here, the explicit form of these projectors will never be used. So, we never pose the problem of finding the explicit matrix form of the crossing relations (or of the associated representations of the braid group), the idea being that only the tensor structure defined by the ordinary Lie algebra is relevant for the construction of the link invariants.

Let us consider now the case in which the partial or total closure of braids is involved.

III. TWISTING

A change of framing in $\langle W(L) \rangle$ is represented by a change in the writhe of the single components of the link diagram. Each elementary operation $\Delta w = \pm 1$ on each component can be performed by means of a Markov move of the second type (M2). The perturbative computations performed in [9] show that a $\Delta w = \pm 1$ modification of the framing of a string in a representation ρ of G results in the presence of a multiplicative factor $\alpha^{\pm 1}$ which (at order λ^2 included) is consistent with the expression

$$\alpha = q^{Q(\rho)} \quad (9.3)$$

As we shall see in sect.11, this is precisely the only possible value of α (with $\alpha \rightarrow 1$ when $k \rightarrow \infty$) which is consistent with the form (8.1) of the monodromy matrix and with the satellite formula (5.31). So, we assume that the relations shown in Fig.16 hold.

IV. PROJECTION COMPATIBILITY

Eqs.(5.16,28) and (5.31) require that the irreducible representations of G appearing in the decomposition (9.1) of the tensor product associated with two strings (and represented by the projection decomposition rule) can be used to construct satellites. Therefore, we require that (in the framework of regular isotopic diagrams) the satellite formula shown in Fig.17 holds.

V. FACTORIZATION

We have seen in sect.5 that, because of the peculiar properties of the CS field theory, for the distant union $L_1 \cup L_2$ the following relation

$$E(L_1 \cup L_2) = E(L_1) E(L_2) \quad (9.4)$$

must hold.

According to the results of the previous sections, in particular eq.(8.2), the deformation parameter q entering these defining relations is given by

$$q = \exp\left(-\frac{2\pi i}{k}\right) \quad (9.5)$$

This concludes the list of the five defining relations of the universal link polynomial; in the remaining sections, I will illustrate how to use them for reconstructing $E(L)$.

10. THE EXTENDED JONES POLYNOMIAL

In this section we consider the case in which $G = SU(2)$ and the representations associated with the link components are arbitrary. The universal polynomial $E(L)$ is called in this case the extended Jones polynomial. The irreducible $SU(2)$ representations are labelled by J with $2J = 0, 1, 2, \dots$. In our notations

$$Q(J) = J(J+1) \quad (10.1)$$

and the dimension $D(J)$ of the J -representation is $D(J) = 2J+1$.

The fundamental blocks for reconstructing $E(L)$ are the values $\{E_0[J]\}$ of the unknot U_0 (with zero writhe) in representation J ; let us compute them. Consider the closure of the braid shown in Fig.12 when $\rho_1 = \rho_2 = (J = 1/2)$. From the projection decomposition (I), projection compatibility (IV) and factorization (V), it follows that

$$E_0[1/2] E_0[1/2] = E_0[1] + E_0[0] \quad (10.2)$$

This relation is just a particular case of eq.(5.26). We have three variables $E_0[0]$, $E_0[1/2]$ and $E_0[1]$ and only one relation; so, we look for two new equations involving these variables.

Consider the closure of the braid shown in Fig.14 when $\rho_1 = \rho_2 = (J = 1/2)$. From twisting (III) it follows that the LHS is equal to $q^{3/4} E_0[1/2]$. On the other hand, from crossing (II) and projection compatibility (IV), it follows that the RHS is equal to

$$q^{\frac{1}{4}} E_0[1] - q^{-\frac{1}{4}} E_0[0] \quad (10.3)$$

The minus sign appearing in eq.(10.3) is due to the fact that, in taking the closure of the RHS of the equation shown in Fig.14, we must exchange ρ_1 with ρ_2 . Under this exchange, the $J = 1$ representation is symmetric, whereas the $J = 0$ representation is antisymmetric and therefore keeps a factor (-1) . In conclusion, from (II), (III) and

(IV), it follows that

$$E_0[1/2] = q^{-\frac{1}{4}} E_0[1] - q^{-\frac{3}{4}} E_0[0] \quad (10.4)$$

Similarly, by taking the closure of the braid shown in Fig.15, one obtains

$$E_0[1/2] = q^{\frac{1}{4}} E_0[1] - q^{\frac{3}{4}} E_0[0] \quad (10.5)$$

Combining eqs.(10.4,5), one gets

$$E_0[1/2] = (q^{\frac{1}{4}} + q^{-\frac{1}{4}}) E_0[0] \quad (10.6)$$

$$E_0[1] = (q + 1 + q^{-1}) E_0[0] \quad (10.7)$$

If one inserts eqs.(10.6,7) into eq.(10.2), one discovers that two solutions for $E_0[0]$ are possible. The first possibility is that $E_0[0]$ vanishes; however, $E_0[0] = 0$ is excluded by definition, see eq.(4.17,18). So, only the second solution

$$E_0[0] = 1 \quad (10.8)$$

is acceptable. Eq.(10.8) gives precisely the right answer, for we already know that $E_0[0]$ must be equal to unity. Summarizing, by using all the five defining relations (I)-(V) of sect.9, we have found eq.(10.8) and

$$E_0[1/2] = q^{\frac{1}{4}} + q^{-\frac{1}{4}} \quad (10.9)$$

$$E_0[1] = q + 1 + q^{-1} \quad (10.10)$$

How to determine $E_0[3/2]$ should be clear now; it is sufficient to take the closure of the braid shown in Fig.12 when $\rho_1 = 1/2$ and $\rho_2 = 1$ and use eqs.(10.9,10). One finds

$$E_0[3/2] = q^{\frac{3}{4}} + q^{\frac{1}{4}} + q^{-\frac{1}{4}} + q^{-\frac{3}{4}} = \frac{q^2 - q^{-2}}{q^{\frac{1}{4}} - q^{-\frac{1}{4}}} \quad (10.11)$$

By induction, it is easy to see that

$$E_0[J] = \frac{q^{J+\frac{1}{2}} - q^{-(J+\frac{1}{2})}}{q^{\frac{1}{4}} - q^{-\frac{1}{4}}} \quad (10.12)$$

Two more examples illustrate how to use the defining relations (I)-(V). For computing the Hopf 2^2_1 link (shown in Fig.18) in the case in which the representations are

J_1 and J_2 , it is sufficient to consider twice the crossing relation of Fig.14 and make use of the property of the projectors shown in Fig.13. One gets then the relation shown in Fig.19. Consequently,

$$E(2_1^2; J_1, J_2) = q^{-(Q(J_1)+Q(J_2))} \sum_{J=|J_1-J_2|}^{J_1+J_2} q^{Q(J)} E_0[J] , \quad (10.13)$$

and therefore, for $J_1 \geq J_2$ for instance,

$$E(2_1^2; J_1, J_2) = \frac{q^{2J_1J_2}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \sum_{m=0}^{2J_2} q^{m^2-m(2J_1+2J_2+1)} \cdot \left[q^{J_1+J_2-m+\frac{1}{2}} - q^{-(J_1+J_2-m+\frac{1}{2})} \right] . \quad (10.14)$$

The closure of three consecutive crossings, shown in Fig.14, for $\rho_1 = \rho_2 = J$ gives the value $E(3_1; J)$ for the trefoil 3_1 ($< g_1^3 > 2$) knot in representation J (shown in Fig.20)

$$E(3_1; J) = \frac{q^{3J^2}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \sum_{n=0}^{2J} (-1)^n q^{\frac{3}{2}n(n-4J-1)} \left(q^{2J-n+\frac{1}{2}} - q^{-(2J-n+\frac{1}{2})} \right) . \quad (10.15)$$

Before considering more complicated examples, it is useful to explain the precise relation between the universal link polynomial $E(L)$ when $G = SU(2)$ and the Jones polynomial.

Suppose that all the components of the links are associated with the $J = 1/2$ representation of $SU(2)$. In this particular case, the projection decomposition takes the form shown in Fig.21, whereas the crossing relations are shown in Fig.22. So we have three configurations (overcrossing, undercrossing and no-crossing) expressed in terms of two projectors only. Therefore, there is a linear relation between them, shown in Fig.23, which is precisely the ordinary skein relation with particular values of the parameters. Taking into account the properties of $E(L)$ under twisting, we conclude that for $G = SU(2)$ and for $J = 1/2$ only, $E(L)$ satisfies the skein relation (2.11) of the $S(L)$ polynomial with

$$\alpha = q^{\frac{3}{4}} , \quad (10.16)$$

$$\beta = q^{\frac{1}{4}} , \quad (10.17)$$

$$z = q^{\frac{1}{2}} - q^{-\frac{1}{2}} , \quad (10.18)$$

and with the normalization of the unknot given in eq.(10.9). Therefore, the combination

$$V(L) = \frac{q^{-\frac{1}{2}w(L)}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} \cdot E(L) \quad (10.19)$$

is an ambient isotopy invariant and coincides precisely with the Jones polynomial, see eqs.(2.9,12). Note that, even if there are certain similarities, Kauffman's bracket polynomial [49] does not coincide with $E(L)$ for $G = SU(2)$ and $J = 1/2$.

Since the Jones polynomial can entirely be reconstructed by means of the skein relation, and since the skein relation has been derived from $(1)-(V)$, it is clear that the defining relations $(1)-(V)$ completely determine $\{E(L)\}$ when $G = SU(2)$ and $J = 1/2$.

Now we want to prove that, even when arbitrary representations $\{J_i\}$ are associated to the link components $\{C_i\}$, the rules $(1)-(V)$ still permit us to reconstruct $E(L)$. The proof is very simple: it is sufficient to use satellites. By means of the relations shown in Figs.21,22, one can express the projector on $J = 1$, for instance, in terms of $J = 1/2$ configurations, as shown in Fig.24. Then, the satellite formula shown in Fig.17 takes the form shown in Fig.25. This means that a link component C_i , in the $J = 1$ representation, can be substituted by the linear combination of the pattern links with two components (both with $J = 1/2$) shown in Fig.25. By induction, it is clear that for any representation J it is possible to find a satellite formula in terms of an appropriate number of $J = 1/2$ representations. This method is also called cabling. In conclusion, any link with arbitrary representations of $G = SU(2)$ can be expressed in terms of new links containing only $J = 1/2$ and this concludes the proof of the reconstruction of the extended Jones polynomial.

The existence of the cabling procedure shows that the conditions $(1)-(V)$ completely determine $\{E(L)\}$ for $G = SU(2)$. Of course, the naive use of cabling is not very practical; quite often, a clever use of satellites greatly simplifies the construction of $E(L)$. Eqs.(10.14,15) for instance have been obtained without cabling.

Some more examples are in order. Let us denote by $< \sigma >_n$ the conjugacy class of $\sigma \in B_n$. Consider the link diagram associated with the conjugacy class $< g_1 g_2 >_3$ of $g_1 g_2 \in B_3$ in the $J = 1$ representation of $SU(2)$. Here also there is no need to use cabling. Indeed, from (III) it follows that

$$E(< g_1 g_2 >_3; 1) = q^2 E(< g_1 >_2; 1) = q^4 E_0[1] . \quad (10.20)$$

Similarly, for the link diagram associated with $< g_1^2 g_2^{-1} >_3$ in the $J = 1$ representation

tation of $SU(2)$, one has

$$\begin{aligned} E(< g_1^2 g_2^{-1} >_3; 1) &= q^{-2} E(< g_1^2 >_2; 1) \\ &= q^{-2} (q^2 E_0[2] + q^{-2} E_0[1] + q^{-4} E_0[0]) \\ &= E_0[2] + q^{-4} E_0[1] + q^{-6} E_0[0] . \end{aligned} \quad (10.21)$$

The link diagram associated with $< g_2^{-1} g_1 g_2^2 g_1 >_3$ is shown in Fig.26. Let the strings be in the $J = 1$ representation of $SU(2)$. In order to compute $E(< g_2^{-1} g_1 g_2^2 g_1 >_3; 1)$, we first derive the satellite formula for $g_1^{-1} \in B_2$ shown in Fig.27. Then

$$E(< g_2^{-1} g_1 g_2^2 g_1 >_3; 1) = q^{-1} E(2_1; 1, 2) - q E(2_1; 1, 1) + q^2 E(2_1; 1, 0) . \quad (10.22)$$

On the other hand, from eq.(10.13) it follows that

$$E(2_1; 1, 2) = q^4 E_0[3] + q^{-2} E_0[2] + q^{-6} E_0[1] , \quad (10.23)$$

$$E(2_1; 1, 1) = q^2 E_0[2] + q^{-2} E_0[1] + q^{-4} E_0[0] , \quad (10.24)$$

$$E(2_1; 1, 0) = E_0[1] . \quad (10.25)$$

Combining eqs.(10.22-25), one finds

$$\begin{aligned} E(< g_2^{-1} g_1 g_2^2 g_1 >_3; 1) &= q^3 E_0[3] - (q^3 - q^{-3}) E_0[2] \\ &\quad + (q^2 - q^{-1} + q^{-7}) E_0[1] - q^{-3} E_0[0] . \end{aligned} \quad (10.26)$$

Let us compute now $E(4_1; 1)$ for the 4_1 knot associated with the diagram described by $< g_1 g_2^{-1} g_1 g_2^{-1} >_3$ for $J = 1$. By using crossing (II), we have, for $J = 1$,

$$g_2^{-1} = -q^2 g_2^2 + (q^3 - q + 1) g_2 + (q^2 - q + q^{-1}) . \quad (10.27)$$

(Eq.(10.27) represents the generalized skein relation for $J = 1$.) Therefore

$$\begin{aligned} E(4_1; 1) &= -q^2 E(< g_1 g_2^{-1} g_1 g_2^2 >_3; 1) + (q^3 - q + 1) E(< g_1 g_2^{-1} g_1 g_2 >_3; 1) \\ &\quad + (q^2 - q + q^{-1}) E(< g_1 g_2^{-1} g_1 >_3; 1) . \end{aligned} \quad (10.28)$$

For the conjugacy classes entering eq.(10.28) one has

$$< g_1 g_2^{-1} g_1 g_2^2 >_3 = < g_2^{-1} g_1 g_2^2 g_1 >_3 , \quad (10.29)$$

$$< g_1 g_2^{-1} g_1 g_2 >_3 = < g_1 g_2 >_3 , \quad (10.30)$$

$$< g_1 g_2^{-1} g_1 >_3 = < g_2^{-1} g_1^2 >_3 . \quad (10.31)$$

Therefore, by means of eqs.(10.20,21,26) and eq.(10.12), one finds

$$E(4_1; 1) = q^7 - q^5 + q + 1 + q^{-1} - q^{-5} + q^{-7} . \quad (10.32)$$

The result (10.32) was already reported in [16].

The last example of this section concerns the three-component link described by $< g_1^2 g_2^2 g_2 >_3$ with the $SU(2)$ representations $J = 1$, $J = 2$ and $J = 3$, as shown in Fig.28. By means of the satellite formula for the $J = 1$ and $J = 2$ components, it is easy to find

$$\begin{aligned} E(< g_1^2 g_2^2 g_2 >_3; 1, 2, 3) &= q^{22} E_0[6] + 2q^{10} E_0[5] + 3E_0[4] + 3q^{-8} E_0[3] \\ &\quad + 3q^{-14} E_0[2] + 2q^{-18} E_0[1] + q^{-20} E_0[0] . \end{aligned} \quad (10.33)$$

11. GENERAL PROPERTIES

As we have seen, the defining relations (I)-(V) have been "very strongly" suggested by the three-dimensional CS theory. Now we want to prove explicitly that in the $E(L)$ polynomial, defined by (I)-(V), all the general properties of $\{< W(L) >\}$ obtained in sect.5 are indeed satisfied. In this section, a generic compact simple Lie group G is considered. Some general properties of the universal link polynomial are also discussed.

SATELLITES

Let us derive first the satellite formula (5.31). When $\chi(C, C_f) = 0$, eq.(5.31) follows immediately from (I) and (IV,V). Consider now the case in which $\chi(C, C_f) = 1$. From (II) and (III), the relation shown in Fig.29 can easily be obtained and then, from (IV), one finds the satellite formula shown in Fig.30, which coincides precisely with eq.(5.31). Note that eq.(9.3) gives the only possible value of the twisting variable α which is consistent with crossing (II), eq.(5.31) and has the correct asymptotic behaviour for $k \rightarrow \infty$. Two or more consecutive applications of the substitution rule shown in Fig.29 reproduce eq.(5.31) in its full generality. So, the property (5.31) is satisfied.

MIRROR IMAGE

Let $\tilde{L}$ be the mirror image of the link diagram L ; i.e. $\tilde{L}$ is related to L by exchanging overcrossing with undercrossing (and vice-versa). Then, from (II) and (III), it follows that $E(\tilde{L})$ is simply obtained from $E(L)$ by substituting q by q^{-1} . Since $q = \exp(-2\pi i/k)$, this means that eq.(5.15) is satisfied.

INVERSION

Let L^{-1} be the link diagram obtained by reversing the orientations of all the components of L . Under the inversion operation $L \rightarrow L^{-1}$, the writhe number of each crossing is not modified. Therefore, any link invariant, constructed by means of operations performed on link diagrams and in which only the difference between over/under-crossing is relevant, is left unchanged under inversion. The universal link polynomial defined by (I)-(V) is precisely of this type and so

$$E(L) = E(L^{-1}) \quad (11.1)$$

This property was already known to hold for the Jones, HOMFLY and $S(L)$ poly-

nomials and reflects the isotropy properties of the ambient space. On the other hand, the complex conjugation of $E(L)$ is equivalent to exchanging q for q^* . Since $q = \exp(-2\pi i/k)$, eq.(5.14) is easily seen to be satisfied.

CHARGE CONJUGATION

Consider an oriented framed link L in R^3 whose m components $\{C_i\}$ are characterized by the representations $\{\rho_i\}$ of G . Let L^* be the link obtained from L by substituting $\rho_i \rightarrow \rho_i^*$ for $i = 1, 2, \dots, m$. The projection decompositions of the couples (ρ_i, ρ_j) and (ρ_i^*, ρ_j^*) have the same structure and the corresponding crossing relations coincide because $Q(\rho) = Q(\rho^*)$. Therefore

$$E(L^*) = E(L) \quad (11.2)$$

Since the link L^{-1} (obtained from L by inversion) coincides with L^* , eq.(11.2) is in agreement with eq.(11.1). Suppose now that C , which is one of the components of L , is associated with a real representation ρ . Reversing the orientation of C only ($C \rightarrow C^{-1}$) and all the other components of L are kept fixed) is equivalent to performing the substitution $\rho \rightarrow \rho^*$ in L . Since in our case ρ is real, the projection decomposition (I) and all the crossing relations (II) are invariant under $\rho \rightarrow \rho^*$. Therefore, the universal link polynomial $E(L)$ is invariant under $C \rightarrow C^{-1}$. When all the representations $\{\rho_i\}$ are real, $E(L)$ represents an invariant of regular isotopy for unoriented link diagrams, in perfect agreement with the conclusions of sect.5.

GENERALIZED SKEIN RELATIONS

Consider the crossing relation, shown in Fig.14, when ρ_1 is equivalent to ρ_2 . On each irreducible component $\rho(t)$, appearing in the decomposition of the tensor product

$$\rho_1 \otimes \rho_2 = \oplus_i \rho_i(t) \quad (11.3)$$

the crossing operation is realized by multiplication with

$$\lambda(t) = (-1)^{s(t)} q^{\frac{1}{2}[Q(\rho(t)) - Q(\rho_1) - Q(\rho_2)]} \quad (11.4)$$

where $s(t) = 0$ (1) if $\rho(t)$ is symmetric (antisymmetric) under a permutation of ρ_1 and ρ_2 . Let us denote by K the crossing operation between two strings and by K^p the

presence of p consecutive crossings. Then K satisfies

$$\prod_i [K - \lambda(t)] = \sum_p c_p K^p = 0, \quad (11.5)$$

where the product is performed on the different values of $\{\lambda(t)\}$. As a consequence

$$\sum_p c_p E(L_p) = 0, \quad (11.6)$$

which represents a generalized skein relation of the type considered in eq.(7.11). An explicit example of generalized exchange relation is shown in eq.(10.27). As mentioned before, eq.(11.6) is not sufficient for reconstructing $E(L)$. In fact, the defining relations (I)-(V) contain much more information than just eq.(11.6). In particular, conditions (I)-(V) describe the essential features of the quasi-tensor structure associated with closed braids which characterize $E(L)$.

A remarkable aspect of the whole construction is that, in order to find $E(L)$, the explicit forms of the particular braid group representations are not required. In other words, for finding the universal link invariants, knowledge of the associativity isomorphism entering the definition of the QTH algebra [12] is not necessary. This is in agreement with the fact that only the structure of the quasi-tensor category determined by the commutativity isomorphism considered in [12] is relevant. The explicit realization of these ideas by means of the defining relations (I)-(V) represents the most important result of the present paper.

PARTIAL CLOSURE

Consider the insertion of the partial closure $\mathcal{P}(\rho_1; \rho_2, \rho_3)$ of a projector, shown in Fig.31, into some closed braid. Since only the representation ρ_1 effectively contributes to this braid, $\mathcal{P}(\rho_1; \rho_2, \rho_3)$ is equivalent to a single string in the ρ_1 representation multiplied by an eventually non-trivial c-number γ , as shown in Fig.32. The value of γ can simply be obtained by taking the closure of both sides of the equation shown in Fig.32. From (IV), it follows that

$$\gamma = \frac{E_0[\rho_3]}{E_0[\rho_1]}. \quad (11.7)$$

For certain representations ρ_1 and for particular values of the deformation parameter q , it may happen that $E_0[\rho_1]$ is vanishing. In spite of that, the use of the partial-closure rule shown in Fig.32 for computing $E(L)$ is always consistent. In other words, even if in

the intermediate steps of the computation of $E(L)$ the relation of Fig.32 with γ given in eq.(11.7) is used, the final answer for $E(L)$ still has no pathologies for whatever value of q and coincides with the result that one would obtain by using a different method. The point is that, if $E_0[\rho_1]$ is vanishing, then the computation of $E(L)$ in the presence of a string in the ρ_1 -representation always produces a vanishing term in the numerator which compensates the presence of $E_0[\rho_1]$ in the denominator. This fact is just a consequence of the tensor structure underlying the defining relations (I)-(V). Actually, this phenomenon is not completely unknown; in fact, it is strictly related to the consistency of the fusion rules in conformal field theories. I will elaborate on this point elsewhere; here, I simply give an example. When $k = 3$, one has $q^3 = 1$ and therefore, for $G = SU(2)$, $E_0[1] = 0$. Consider now the three-component link associated with $< g_1^2 g_2^2 >_3$ for the representations $J = 1/2$, $J = 1$ and $J = 1/2$ shown in Fig.33. By using crossing (II) and the partial-closure rule, one obtains

$$\begin{aligned} E(< g_1^2 g_2^2 >_3; 1/2, 1, 1/2) &= \frac{q E_0[3/2] + q^{-2} E_0[1/2]}{E_0[1]} E(2_1; 1/2, 1) \\ &= \frac{(q E_0[3/2] + q^{-2} E_0[1/2])^2}{E_0[1]}. \end{aligned} \quad (11.8)$$

Even for $q^3 = 1$, the expression (11.8) is well defined. Indeed, eq.(11.8) can be rewritten as

$$E(< g_1^2 g_2^2 >_3; 1/2, 1, 1/2) = q^{-3} (1 + q^3)^2 E_0[1] \quad (11.9)$$

and for $q^3 = 1$ this quantity is not divergent; on the contrary, it vanishes.

CONNECTED SUMS

Given two link diagrams L_1 and L_2 , the link diagram $L_1 \# L_2[\rho]$, corresponding to the connected sum of L_1 and L_2 obtained by acting on two strings in the ρ representation, is shown in Fig.34. The universal link polynomial satisfies

$$E(L_1 \# L_2[\rho]) = \frac{E(L_1) E(L_2)}{E_0[\rho]}. \quad (11.10)$$

Proof. Let L_1 and L_2 be described by the closure of the braids σ_1 and σ_2 respectively, then $L_1 \# L_2[\rho]$ is described by the closure of the braid shown in Fig.35. A straightforward extension of the partial-closure rule (shown in Fig.32) gives the relation illustrated in Fig.36 and the result (11.10) immediately follows.

The identity shown in Fig.36 is very useful also for the construction of invariants for generic three-manifolds obtained by surgery. I will elaborate on this subject elsewhere.

Eq.(11.10) is the generalization of the usual rule for connected sums satisfied by the HOMFLY polynomial. An equivalent version of the relation (11.10) appears in [6].

Eq.(11.10) can be further generalized to the case of multi-connected sums. For example, consider the link diagrams L_1 and L_2 , shown in Fig.37, in which the $J = 1/2$ representation of $SU(2)$ is associated with the strings explicitly depicted in the picture. We want to express now the universal link polynomial $E(L_1 \oplus_2 L_2)$, for the double-connected sum $L_1 \oplus_2 L_2$ shown in Fig.37, in terms of $E(L_1)$, $E(L_2)$, $E(L'_1)$ and $E(L'_2)$, where L'_1 and L'_2 are shown in Fig.38. For open strings, one has the relation shown in Fig.39, where the complex coefficients $x_{1,2}$ and $y_{1,2}$ are given by

$$x_{1,2} = \frac{1}{E_0[1/2]E_0[1]} \left(q^{-1/2} E(L_{1,2}) + q^{1/2} E(L'_{1,2}) \right), \quad (11.11)$$

$$y_{1,2} = \frac{1}{E_0[1/2]E_0[1]} \left(q^{1/2} E(L_{1,2}) - q^{-1/2} E(L'_{1,2}) \right). \quad (11.12)$$

Eqs.(11.11,12) are obtained by combining the relation shown in Fig.39 with the crossing relations (III) and by taking the closure. On the other hand, by using the decomposition into the irreducible components shown in Fig.39, one has

$$E(L_1 \oplus_2 L_2) = x_{1,2} E_0[1] + y_{1,2} E_0[0]. \quad (11.13)$$

By substituting the values of $\{x_{1,2}, y_{1,2}\}$, shown in eqs.(11.11,12), into eq.(11.13) one obtains the desired result.

AMBIENT ISOTOPY

Because of the covariance properties of $E(L)$ under Reidemeister moves of type I, it is very easy to obtain ambient isotopy invariants from $E(L)$. Let $\{C_1, \dots, C_m\}$ be the components of L and $\{\rho_1, \dots, \rho_m\}$ be the associated representations of G . Then

$$X(L) = E(L) q^{-\sum_{i=1}^m w(C_i) Q(\rho_i)} \quad (11.14)$$

is clearly an ambient isotopy invariant. For equivalent representations, $\rho_i \sim \rho \quad \forall i$, one can define

$$X'(L) = E(L) q^{-w(L) Q(\rho)}, \quad (11.15)$$

which also represents an ambient isotopy invariant for links.

12. GROUP $SU(N)$

The building blocks of the universal link polynomial are the values $\{E_0[\rho]\}$ of the unknot U_0 (with zero writhe) for the different representations $\{\rho\}$ of G . In the classical polynomials mentioned in sect.2, the unknot is usually normalized to unity and, in any case, its particular value seems to play no relevant role. On the contrary, one of the main improvements introduced by the $E(L)$ polynomial is to recognize that the values of the unknots for different representations have well-defined relations between them following from the tensor structure associated with the gauge group.

Once the normalization for the trivial $\rho = 0$ representation is fixed (by definition, in the CS theory in R^3 this normalization is the unity), the values $\{E_0[\rho]\}$ for the non-trivial representations are uniquely determined. In the universal link polynomial, the unity is associated to the presence of no knots at all in R^3 . The fact that, in knot theory, the absence of knots has to be considered on the same footing as the presence of any other knot should not be a surprise. It is well known that the properties of knots are defined precisely with respect to the ambient space. This also means that, once the normalization of the ambient space R^3 is fixed, the universal link polynomial associated to any other ambient space (with or without knots) should also be fixed. In principle, this picture emerges naturally from the field theory point of view. In practice, however, several technical problems arise. Presumably, these problems can unambiguously be solved and the universal link polynomial defined on a generic three-manifold will be explicitly produced. Of course, several conjectures can be formulated and different invariants of three-manifolds can be introduced from an abstract point of view. What I mean here by the universal link polynomial on a generic manifold is the particular invariant defined by (and derived from) the quantum CS theory.

In this section, we want to solve first a much more simple problem: how to compute $E_0[\rho]$ for the unknot U_0 (in R^3) associated with an irreducible representation ρ of the gauge group $G = SU(N)$. Let us start with the fundamental (N -dimensional) representation F of $SU(N)$. In the decomposition

$$F \otimes F = A \oplus S, \quad (12.1)$$

A (S) denotes the antisymmetric (symmetric) representation of dimension $N(N-1)/2$

$(N(N+1)/2)$. Since the values of the quadratic Casimir operator are

$$Q(F) = \frac{N^2 - 1}{2N}, \quad (12.2)$$

$$Q(A) = \frac{N^2 - N - 2}{N}, \quad (12.3)$$

$$Q(S) = \frac{N^2 + N - 2}{N}, \quad (12.4)$$

the same argument which has been used to derive eqs.(10.8-10) now gives

$$E_0[F] = \frac{q^{\frac{N}{2}} - q^{-\frac{N}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \quad (12.5)$$

$$E_0[A] = \frac{q^{\frac{N-1}{2}} - q^{-\frac{N-1}{2}}}{q - q^{-1}} E_0[F], \quad (12.6)$$

$$E_0[S] = \frac{q^{\frac{N+1}{2}} - q^{-\frac{N+1}{2}}}{q - q^{-1}} E_0[F]. \quad (12.7)$$

Moreover, the same argument which has been used for deriving eqs.(10.16-18) shows that, in the case in which all the components of the links are associated to the fundamental representation of $G = SU(N)$, $E(L)$ satisfies the generalized skein relations of the $S(L)$ polynomial with parameters

$$\alpha = q^{\frac{N^2-1}{2N}}, \quad (12.8)$$

$$\beta = q^{\frac{1}{2N}}, \quad (12.9)$$

$$z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}. \quad (12.10)$$

Eqs.(12.5,8-10) were quoted in sect.4. The general form of the expression (12.5) was conjectured in [6], but the correct dependence of the deformation parameter q on the renormalized coupling constant k is shown in eq.(8.2).

The value of the unknot for a generic irreducible representation of $SU(N)$ can be obtained, for instance, by means of cabling. Indeed, any irreducible representation of $SU(N)$ can be obtained by taking the tensor product of an appropriate number of fundamental representations. On the other hand, when all the link components are associated with the F representation, the $S(L)$ skein relations together with eq.(12.5) uniquely determine $E(L)$. So, not only $\{E_0[\rho]\}$ are fixed, but the defining relations (1)-(V) uniquely determine $E(L)$ for a generic link diagram L whose components are associated with arbitrary representations of $G = SU(N)$. This concludes the proof of the reconstruction of the universal link polynomial in the case of $\{A_{N-1}\}$ algebras. This also means that, as announced in the Introduction, the non-Abelian $SU(N)$ Chern-Simons theory in R^3 is completely solved.

A few examples are in order. Consider $G = SU(3)$ and let the irreducible representations be labelled by their dimension. The 15 and 15' representations refer to the (2, 1) and (4, 0) Dinkin labels. Then, for the unknot U_0 (with zero writhe) one finds

$$E_0[3] = q^{-1} \frac{1 - q^3}{1 - q}, \quad (12.11)$$

$$E_0[6] = q^{-2} (1 + q^2) \frac{1 - q^3}{1 - q}, \quad (12.12)$$

in agreement with eqs.(12.5,7); moreover

$$E_0[8] = q^{-2} (1 + q)^2 (1 + q^2), \quad (12.13)$$

$$E_0[10] = q^{-3} (1 + q^2) \frac{1 - q^5}{1 - q}, \quad (12.14)$$

$$E_0[15] = q^{-3} \frac{(1 - q^3)(1 - q^5)}{(1 - q)^2}, \quad (12.15)$$

$$E_0[15'] = q^{-4} \frac{1 + q^3 (1 - q^3)(1 - q^5)}{1 + q (1 - q)^2}, \quad (12.16)$$

$$E_0[21] = q^{-5} \frac{1 + q^3 (1 - q^3)(1 - q^7)}{1 + q (1 - q)^2}, \quad (12.17)$$

$$E_0[24] = q^{-4} (1 + q)(1 + q^2)(1 + q^3) \frac{1 - q^3}{1 - q}, \quad (12.18)$$

$$E_0[27] = q^{-4} \frac{(1-q^3)^2(1-q^6)}{(1-q)^3(1+q)} \quad (12.19)$$

Consider now $E(2_1^2; \rho_1, \rho_2)$ associated with the Hopf 2_1^2 link with writhe $w = 2$ and with components in the ρ_1 and ρ_2 representations of $SU(3)$. One has

$$E(2_1^2; 6, 3) = q^{4/3} E_0[10] + q^{-5/3} E_0[8] \quad , \quad (12.20)$$

$$E(2_1^2; 8, 3) = q E_0[15] + q^{-1} E_0[6] + q^{-3} E_0[3] \quad , \quad (12.21)$$

$$E(2_1^2; 10, 3) = q^{-2} E_0[15] + q^2 E_0[15'] \quad , \quad (12.22)$$

$$E(2_1^2; 6, 6) = q^{5/3} E_0[6] + q^{1/3} E_0[15] + q^{23/3} E_0[15'] \quad , \quad (12.23)$$

$$E(2_1^2; 10, 6) = q^{-1/3} E_0[24] + q^4 E_0[21] + q^{-4} E_0[15] \quad . \quad (12.24)$$

Finally, for the trefoil 3_1 knot with writhe $w = 3$ and representation 6 , the universal link polynomial $E(3_1; 6)$ takes the form

$$E(3_1; 6) = q^4 E_0[15'] - q^{-2} E_0[15] + q^{-5} E_0[6] \quad . \quad (12.25)$$

For the remaining simple Lie algebras, which are not of the $\{A_n\}$ type, the proof of the complete reconstruction of $E(L)$ cannot be done in the same simple way that we used for $G = SU(N)$. Of course, in some particular cases of simple links, the defining relations (I)-(V) immediately give the answer. For example, when $G = SO(7)$ one obtains

$$E_0[7] = 1 + q^{-5/2} \frac{1-q^6}{1-q} \quad , \quad (12.26)$$

$$E_0[21] = q^{-5/2} \frac{1-q^6}{1-q} \left[1 + q^{-3/2} \frac{1-q^5}{1-q^2} \right] \quad , \quad (12.27)$$

$$E_0[27] = q^{-5/2} \frac{1-q^6}{1-q} \left[1 + q^{-5/2} \frac{1-q^7}{1-q^2} \right] \quad . \quad (12.28)$$

13. CONCLUSIONS

In the present paper, the explicit solution of the non-Abelian $SU(N)$ Chern-Simons theory in three dimensions has been produced. The general properties of the universal link polynomial, defined by the expectation values of the Wilson line operators, have been derived for a generic real simple Lie algebra. The $E(L)$ polynomial obtained in the CS theory describes the link invariants associated with the braid group representations defined by the quasi-tensor category of quasi-triangular quasi-Hopf algebras.

One of the most remarkable results which has been obtained is the discovery of the very simple way in which the tensor structure of ordinary Lie algebras enters the construction of these link polynomials. Finding the expressions of $E(L)$ for different links in R^3 is now reduced to a simple problem of very elementary algebra.

Of course, this is only the starting point for new further developments in knot theory. For example, it is reasonable to expect that, since the link polynomials in R^3 for different representations of the gauge group are known, the rules for general surgery should now easily be found. Much work remains to be done and several problems are still open, for example:

- 1) finding a proof of the complete reconstruction of $E(L)$ in the case of a generic simple Lie algebra;
- 2) defining an extension for non-compact groups;
- 3) considering the inclusion of new Lagrangian terms ("model building");
- 4) performing a systematic study of the $E(L)$ properties and its relevance for the knot classification problem;
- 5) finding an alternative (and eventually simplified) construction of $E(L)$.

I would like to conclude by formulating a conjecture. Prove (or disprove) the following

Conjecture: the universal link polynomial $E(L)$ provides a complete classification of the (finite dimensional) matrix representations of the braid group.

ACKNOWLEDGEMENTS. I have benefited from several useful discussions with L. Alvarez-Gaumé, D. Birmingham, S. Ferrara, A. Ganchev, H. Grosse, C. Imbimbo, N. Maggiore, J.M. Maillet, V. Petkova, N. Reshetikhin, B. Schroer, S.P. Sorella, R. Stora and G. Veneziano. I wish to thank in particular M. Martellini and M. Mintchev for their valuable and helpful comments. I also wish to thank J. Rostant for her patient collaboration in improving the final version of this paper. The kind hospitality of CERN Theory Division is gratefully acknowledged.

REFERENCES

- [1] W. Siegel, Nucl. Phys. B156 (1979) 135;
J. Schonfeld, Nucl. Phys. B185 (1981) 157.
R. Jackiw and S. Templeton, Phys. Rev. D24 (1981) 2291;
S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. 48 (1982) 975; and Ann. Phys. (NY) 140 (1982) 372;
F. Wilczek and A. Zee, Phys. Rev. Lett. 51 (1983) 2250;
C.R. Hagen, Ann. Phys. (NY) 157 (1984) 342;
A.M. Polyakov, Mod. Phys. Lett. A3 (1988) 325.
- [2] R.B. Laughlin, Phys. Rev. Lett. 60 (1988) 2677;
Y.H. Chen, F. Wilczek, E. Witten and B.I. Halperin, Int. J. Mod. Phys. B3 (1989) 1001;
G.W. Semenoff and N. Weiss, Phys. Lett. B250 (1990) 117;
- [3] D. Arovas, R. Schrieffer, F. Wilczek and A. Zee, Nucl. Phys. B251 (1985) 117;
A. Zee, *Semionics: a theory of high temperature superconductivity*, Santa Barbara preprint NSF-ITP-90-08, 1990;
F. Wilczek, *States of Anyonic Matter*, Princeton preprint IASSNS-HEP-90/22, 1990.
- [4] A.S. Schwarz, Lett. Math. Phys. 2 (1978) 247; Comm. Math. Phys. 67 (1979) 1; Abstracts of the Baku Topological Conference 1987.
- [5] M.F. Atiyah, *New Invariants of Three- and Four-Dimensional Models*, in *The Mathematical Heritage of Hermann Weyl*, Proc. Symp. Pure Math. 48, ed. R. Wells (Amer. Math. Soc., 1988).
- [6] E. Witten, Commun. Math. Phys. 121 (1989) 351.
- [7] E. Guadagnini, M. Martellini and M. Mintchev, Phys. Lett. B227 (1989) 111.
- [8] P. Cotta-Ramusino, E. Guadagnini, M. Martellini and M. Mintchev, Nucl. Phys. B330 (1990) 557.
- [9] E. Guadagnini, M. Martellini and M. Mintchev, Nucl. Phys. B330 (1990) 575.

- [10] E. Guadagnini, M. Martellini and M. Mintchev, Nucl. Phys. B (Proc. Suppl.) 16 (1990) 588.
- [11] E. Guadagnini, M. Martellini and M. Mintchev, Nucl. Phys. B336 (1990) 581.
- [12] V.G. Drinfeld, *Quasi-Hopf Algebras and Knizhnik-Zamolodchikov Equations*, in Problems of Modern Quantum Field Theory, ed. A.A. Belavin, A.V. Klyuk and A.B. Zamolodchikov (Springer-Verlag, 1990).
- [13] V.F.R. Jones, Bull. Amer. Math. Soc. 12 (1985) 103; Ann. Math. 126 (1987) 335.
- [14] E. Witten, *Quantization of Chern-Simons Gauge Theory with Complex Gauge group*, Princeton preprint IASSNS-HEP-89/65.
- [15] E. Guadagnini, N. Maggiore and S.P. Sorella, Phys. Lett. B247 (1990) 543.
- [16] E. Guadagnini, Phys. Lett. B251 (1990) 115.
- [17] P.P. Kulish, N. Yu. Reshetikhin and E.K. Sklyanin, Lett. Math. Phys. 5 (1981) 393;
- A.A. Belavin and V.G. Drinfeld, Funct. Anal. Appl. 16 (1982) 159;
- V.G. Drinfeld, Sov. Math. Dokl. 27(1983) 68;
- N. Yu. Reshetikhin and L.D. Faddeev, Theor. Math. Phys. 56 (1984) 847;
- M. Jimbo, Commun. Math. Phys. 102 (1986) 537.
- [18] V.G. Drinfeld, Proc. Int. Congr. Math., Berkeley, 1 (1986) 798;
- M. Jimbo, Lett. Math. Phys. 10 (1985) 63;
- A.N. Kirillov and N.Y. Reshetikhin, *q-Weyl Group and a Multiplicative Formula for Universal R-Matrices*, preprint 1990.
- [19] T. Kohno, Ann. Inst. Fourier, Grenoble, 37 (1987) 139; Adv. Studies in Pure Math. 16 (1988) 255; Nagoya University preprints No 3 and No 6, 1989.
- [20] N.Y. Reshetikhin, *Quantized Universal Enveloping Algebras, the Yang-Baxter Equations and Invariants of Links, I and II*, LOMI preprints E-4-87 and E-17-87, Steklov Institute, Leningrad 1987.
- [21] A.N. Kirillov and N.Y. Reshetikhin, *Representations of the Algebra $U_q(sl(2))$, q-orthogonal polynomials and invariants of links*, LOMI preprint E-9-88, Steklov Institute, Leningrad 1988.
- [22] V.G. Turaev, Invent. Math. 92 (1988) 527; *The Yang-Baxter Equation and Invariants of Links*, LOMI preprint E-3-87, Steklov Institute, Leningrad 1988.
- [23] N.Y. Reshetikhin and V.G. Turaev, Commun. Math. Phys. 127 (1990) 1; and *Invariants of 3-manifolds via link polynomials and quantum groups*, preprint 1990.
- [24] D. Rolfsen, *Knots and Links*, (Publish or Perish, 1976).
- [25] L.H. Kauffman, *On Knots*, (Princeton Univ. Press, 1987).
- [26] J.H. Conway, in *Computational Problems in Abstract Algebra*, ed. J. Leech, Pergamon Press, 1969.
- [27] P. Freyd, D. Yetter, J. Hoste, W.B.R. Lickorish, K. Millet and A. Ocneanu, Bull. Amer. Math. Soc. 12 (1985) 239;
- J.H. Przytycki and P. Traczyk, Kobe J. Math. 4 (1987) 115.
- [28] A.S. Lipson, Bull. London Math. Soc. 20 (1988) 532.
- [29] W.B.R. Lickorish and K. Millet, Topology 26 (1987) 107.
- [30] C. Becchi, A. Rouet and R. Stora, Commun. Math. Phys. 42 (1975) 127; and Ann. Phys. (NY) 98 (1976) 287.
- [31] G. Calugareanu, Czech. Math. J. 11 (1961) 588.
- [32] G. Calugareanu, Rev. Math. Pures Appl. (Bucarest) 4 (1959) 5;
- W.F. Pohl, J. Math. Mech. 17 (1968) 975.
- [33] F. Delduc, O. Piguet, C. Luechesi and S.P. Sorella, Nucl. Phys. B346 (1990) 313.
- [34] A. Blasi and S. Collina, Nucl. Phys. B345 (1990) 472.
- [35] F. Delduc, F. Gieres and S.P. Sorella, Phys. Lett. B225 (1989) 367;
- D. Birmingham, M. Rakowski and G. Thompson, Nucl. Phys. B329 (1990) 83.
- [36] S. Cecotti and L. Girardello, Phys. Lett. 110B (1982) 39;
- E. Witten, Nucl. Phys. B202 (1982) 253.
- [37] E. Artin, Ann. of Math. 48 (1947) 101.
- [38] J. Birman, Ann. Math. Stud. 82 (1974) 321; and Can. J. Math. 28 (1976) 264.
- [39] P. De la Harpe, M. Kerwaire and C. Weber, L'Enseignement Mathématique 32 (1986) 271.
- [40] W.B.R. Lickorish, Math. Proc. Camb. Phil. Soc. 105 (1989) 103.

- [41] M. Rosso, Commun. Math. Phys. 117 (1988) 581, and 124 (1989) 307;
N. Burroughs, Commun. Math. Phys. 127 (1990) 109;
H. Wenzel, Inv. Math. 92 (1988) 349, and Commun. Math. Phys. 133 (1990) 383.
- [42] Y. Akutsu and M. Wadati, J. Phys. Soc. Jpn. 56 (1987) 839 and 3039;
Y. Akutsu, T. Deguchi and M. Wadati, J. Phys. Soc. Jpn. 56 (1987) 3464.
- [43] A. Liguori and M. Mintchev, *Yang-Baxter Equation and Representations of Braid Group*, Pisa University preprint IFUP-TH 31/90.
- [44] L.A. Takhtadzhian and L.D. Faddeev, Vsp. Mat. Nauk. 34 (1979) 13;
P.P. Kulish and E.K. Sklyanin, J. Sov. Math. 19 (1982) 1596.
- [45] M. Wadati, T. Deguchi and Y. Akutsu, Phys. Rep. 180 (1989) 247 and references therein.
- [46] B. Schroer, *New Methods and Results in Conformal QFT_2 and the "String idea"*, Cargèse Lectures 1987;
- [47] J. Fröhlich, *Statistics of Fields, the Yang-Baxter Equation and the Theory of Knots and Links*, Cargèse Lectures 1987.
- [48] A. Tsuchiya and Y. Kanie, Adv. Stud. Pure Math. 16 (1988) 297;
G. Moore and N. Seiberg, Phys. Lett. B212 (1988) 451;
G. Moore and N. Reshetikhin, Nucl. Phys. B328 (1989) 557;
L. Alvarez-Gaumé, C. Gomez and G. Sierra, Nucl. Phys. B319 (1989) 155;
M. Wadati, Y. Yamada and T. Deguchi, J. Phys. Soc. Jpn. 58 (1989) 1153;
A.C. Ganchev and V.B. Petkova, Phys. Lett. B 233 (1989) 374.
- [49] L.H. Kauffman, Topology 26 (1987) 395.

Fig.2
Possible crossing configurations

Fig.1
Reidemeister moves

Fig.3

Skein related configurations

Fig.4

Configurations related by RM of type I

Fig.5

An oriented path γ in R^3

Fig.6

The unknots U_1 and U_2 in R^3

Fig.7

Links related by eq.(5.28)

Fig.8

Example of pattern link P which has to be substituted for C_i

Fig.9

Pattern link P used in eq.(5.31); the framings C_{1f} and C_{2f} are not shown here

Fig.10

Graphical presentation of B_n generators

Fig.11

Closure $\hat{\sigma}$ of the element σ

Fig.12

Projection Decomposition

Fig.13

Property of projectors

Fig.14

Overcrossing

Fig.16

Twisting

Fig.15

Undercrossing

Fig.17

Projection Compatibility

Fig.18

Hopf link

Fig.19

Computation of the 2_1^2 link

Fig.20

Trefoil in representation J

Fig.21

Projection decomposition for $\rho_1 = \rho_2 = (J = 1/2)$

Fig.22

Crossing relations for $\rho_1 = \rho_2 = (J = 1/2)$

$$q^{1/4} \begin{array}{c} \nearrow \\ \searrow \end{array} = -q^{1/4} \begin{array}{c} \nwarrow \\ \nearrow \end{array} = (q^{1/2} - q^{1/2}) \begin{array}{c} \uparrow \\ \uparrow \end{array}$$

Fig.23

Skein relation for $SU(2)$ and $J = 1/2$

$$(q^{1/2} + q^{-1/2}) \begin{array}{c} \uparrow \quad \uparrow \\ \text{---} \quad \text{---} \\ \downarrow \quad \downarrow \end{array} = q^{1/4} \begin{array}{c} \nearrow \\ \searrow \end{array} + q^{-1/2} \begin{array}{c} \uparrow \\ \uparrow \end{array}$$

Fig.24

Substitution rule of $J = 1$ in terms of $J = 1/2$

Fig.25

Satellite formula for $J = 1$

Fig.26

Two-component link diagram associated with the conjugacy class $< g_2^{-1} g_1 g_2^2 g_1 >_3$

Fig.27

Satellite formula for $\langle g_1^{-1} \rangle_2$ in representation $J = 1$

Fig.28

Link diagram described by $\langle g_1^2 g_2^2 g_3 \rangle_3$

Fig.29

General substitution rule

Fig.30

Satellite formula (5.31) for $\chi(C, C_f) = 1$

Fig.31

Partial closure of a projector

Fig.32

Effective correspondence inside closed braids called partial-closure rule

Fig.33

$\text{Link} < g_1^2 g_2^2 >_3$

Fig.34

Connctcted sum $L_1 \# L_2[\rho]$

Fig.35

Braid representation of $L_1 \# L_2[\rho]$

Fig.36

Example of generalized partial-closure rule

Fig.37

Double-connected sum

Fig.38

Link diagrams L'_1 and L'_2

Fig.39

Decomposition into irreducible components of two open string configurations