Unstable Particles

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Abstract

In these lectures, given in Cargèse during the summer of 1990, we reexamine the problem of unstable particles. In the first part we show the limitations of the exponential decay law, which breaks down at very short and very long times. Then we discuss the problem to know whether an unstable particle is necessarily associated to a pole on some "second sheet" and show that even if one can continue the scattering amplitude through the elastic cut (which is possible only if extra continuity conditions, which do not follow from first principles, are imposed) a pole can be simulated by an entire function. However in a simple case like pion-pion scattering, constraints from analyticity and unitarity make it possible to decide whether a pole exists or not. Hopefully this might be the case for the Z_0 particle also.

Introduction

The subject of unstable particles is a kind of recurrent subject which comes back every time someone is forced to think about it. In fact one of the most important papers on the subject is that of Maurice Lévy (the Director of the "Institut d'Etudes Scientifique de Cargèse"), published in 1959 [1]. This time the pretext to reexamine the problem is the observation of hundreds of thousands of an unstable particle, the Z_0 .

Almost all particles are unstable except the electron, the positron, the neutrinos and also the proton and the antiproton, to the best of our knowledge, in spite of the fact that theoreticians would like them to be unstable. Up to recently the "Particle Data" group made a distinction between "stable particles", decaying only by weak interactions, and "unstable particles" decaying via strong interactions. As a result, the Z_0 was "stable", in spite of its width of 2.42 GeV while the upsilin, with a width of 30 keV was "unstable". I am told that in this year's edition this distinction has been wisely abandoned.

In fact, experiment indicates that one should forget the stable-unstable distinction. It is remarkable that Gell-Mann, at the 1962 Geneva Conference [2] predicted the existence of the "stable" Ω^- particle (which has a very long lifetime of $0.82 \cdot 10^{-10}$ seconds) from the knowledge of the Δ , Σ^* , Ξ^* , which have widths of the order of 100 MeV. In the time interval between the prediction and the discovery, many distinguished theoreticians [3], playing with the Riemann sheet structure associated to the decay channels of these particles argued that the equal spacing rule could not possibly be valid, but Samios and his collaborators showed that it was nevertheless valid [4].

In spite of that, axiomatic field theory makes a big distinction between stable particles, which have an associated asymptotic field, and unstable particles which do not. However, we have to live with unstable particles. We even have to go further and deal with completely confined particles, the quarks and the gluons. In these lectures, I would like to discuss two problems.

- 1) The decay of unstable particles which is known to be non exponential, but, nevertheless approximately exponential if one excludes very short times and very long times after the production of the particle. This is often forgotten, and maybe, some people in the audience are not aware of that. Relatively recently these considerations were resurrected to try to explain the apparent absence of proton decay. We shall see that unfortunately the orders of magnitude are such that this "explanation" fails.
- 2) The association of unstable particles to some pole on some Riemann sheet, and the question to know where is the pole. Naturally this question is essential for Z_0 physics. Essentially everybody believes that a pole is associated to the Z_0 , and one could say that the excellent generalized Breit-Wigner fits of the Z_0 confirm this idea. However, as we shall see, it is not perfectly established in theory. Around 1970-71, L. Fonda and his collaborators wrote a series of papers [5] showing that one could simulate the amplitude produced by the pole by an analytic function without pole, at least in the physical region. I shall show, in a very simple case at least, that of pion-pion resonances, that this procedure may not work if one takes into account all unitarity and analyticity constraints, and that, with high accuracy measurements, it is possible to decide whether there is a pole or not.

1. The Decay of Unstable Particles

Here, we shall follow partly the presentation of Galindo and Pascual in their Quantum Mechanics course [6], which I recommend to you (there is an English translation by

L. Alvarez-Gaumé which will soon appear).

Let us call $|B\rangle$ the unstable state. We assume that B decays into $C+D$. The Hamiltonian is given by

$$H = H_0 + V \quad (1)$$

B is stable under H_0 and V is responsible for the decay. The probability that B remains B after a time t is

$$P(t) = |A(t)|^2 = \left| \langle B | e^{-iHt} B \rangle \right|^2 \quad (2)$$

H is hermitian, and, for $t \rightarrow 0$ we have

$$e^{-iHt} = 1 - iHt$$

and hence

$$P(t) \simeq 1 + O(t^2), \quad (3)$$

instead of $1 - \lambda t$.

Hence we see that for small times we have unavoidably a violation of the exponential decay law.

To find the general structure of the decay law, we insert in $A(t)$ a complete set of states of H :

$$\langle B | e^{-iHt} | B \rangle = \sum_n \langle B | e^{-iE_n t} | B \rangle \langle B | n \rangle^2$$

In fact these states are scattering states (we assume explicitly that the $B+C$ system has no bound states) and it is more correct to write

$$A(t) = \int_0^\infty d\lambda e^{-i\lambda t} \|E_\lambda | B \rangle\|^2, \quad (4)$$

where E_λ represents the projection over the state with eigenvalue. $\lambda = 0$ corresponds to the $C+D$ threshold.

Now, we can define

$$R(t) = \frac{1}{2} [A^*(t^*) + A(t)], \quad (5)$$

i.e. for real t , the real part of $A(t)$. So, from (4), we have

$$R(t) = \int_0^\infty w(\lambda) \cos(\lambda t) d\lambda \quad (6)$$

with

$$w(\lambda) = \|E_\lambda | B \rangle\|^2 \geq 0,$$

and then, conversely, by inverse Fourier cosine

$$w(\lambda) = \frac{1}{2\pi} \int_0^\infty R(t) \cos \lambda t dt \quad (7)$$

Now, if $|R(t)| \leq C \exp -\gamma t$ (7) shows that $w(\lambda)$ is analytically the strip $|\text{Im} \lambda| < \gamma$, because the integral still converges for complex values of λ . This is impossible since $w(\lambda) = 0$ for $\lambda < 0$.

This is more or less the essence of the Paley-Wiener theorem. The essential point is that the spectrum of $B+C$ is lower bounded.

Let me show you, first by hand, that one can get as close as one wishes to exponential decay, for instance $\exp(-|t|^{1-\epsilon})$. Take $w(\lambda) = \exp(-\frac{1}{\lambda^N} - \lambda)$, then

$$A(t) = \int_0^\infty \exp - \left(\lambda t + \frac{1}{\lambda^N} + \lambda \right) d\lambda \quad (8)$$

the line of integration can be deformed, and instead of integrating over $\arg \lambda = 0$ one can integrate over $\arg \lambda = -\phi$

As long as $N\phi < \pi/2$ the contribution from the arc at infinity is zero, and the integral becomes

$$A(t) = \int_0^\infty e^{i\phi} dr \exp[-rt \sin\phi - \frac{1}{r^N} \cos N\phi - r \cos\phi + i(\text{real terms})] \quad (9)$$

Now the quantity

$$X = rt \sin\phi + \frac{1}{r^N} \cos N\phi$$

is maximum at

$$r = \left(\frac{N \cos N\phi}{t \sin\phi} \right)^{\frac{1}{N+1}},$$

and one find that the optimum ϕ is

$$\frac{M}{2(N+1)}$$

then replacing X by its maximum one can carry the integration and get

$$A(t) < \frac{\exp\left(-\frac{N+1}{N} t^{\frac{N}{N+1}} \sin \frac{M}{2(N+1)}\right)}{\cos \frac{M}{2(N+1)}} \quad (10)$$

which is the announced result. Notice, however, that we cannot get the exponential decay by letting N go to infinity because of the factor $\sin(M/2(N+1))$ in the exponential in (10). (10) represents a strict bound, but some minor changes, using the steeped descent method can lead to an estimate very close to the r.h.s. of (10).

In fact, one can approach still slightly closer the exponential decay, for the exact necessary and sufficient condition on $A(t)$ due to the fact that $w(\lambda)$ vanishes for $\lambda < 0$ is [7]

$$\int_{-\infty}^{\infty} \frac{|\ln |A(t)|| dt}{1+t^2} < \infty \quad (11)$$

This allows for instance

$$A(t) \sim \exp - \frac{t}{(\log t)^{1+\epsilon}}, \text{ etc...} \quad (12)$$

However, in practice, reasonable models exhibit much more marked deviations from the exponential law. The weight function $w(\lambda)$ cannot be chosen at will. It is imposed from outside and it is constrained to have a normal threshold behaviour [1] [6].

For

$$w(\lambda) \sim \lambda^\nu f(o), \quad f(o) \neq o, \quad (13)$$

one gets

$$\int w(\lambda) e^{-i\lambda t} dt \simeq \frac{\Gamma(\nu+1) f(o)}{(it)^{\nu+1}} \quad (14)$$

i.e. a negative power behaviour.

Now we describe the Pascual Galindo model, in which one has a separable interaction between an unstable state $|B\rangle$ and the continuum of scattering states $|\delta(x-x_0)\rangle$ with energy x_0 . The main interest of the model is that it allows complete explicit calculations showing deviations from exponential decay as well as approximate exponential decay for intermediate times.

A state is defined as $\{\alpha, f(x)\}$ [13], where α represents the amplitude of the unstable particle and $f(x)$ that of the continuum. We can introduce a norm

$$|\alpha|^2 + \int_0^\infty dx |f(x)|^2 \quad (15)$$

the action of the free hamiltonian on a state is given by

$$H_0 \{\alpha, f(x)\} \rightarrow \{\alpha, x f(x)\} \quad (16)$$

Which means that the energy of $|B\rangle$ is taken to be unity. The action of the separable interaction is given by

$$v \{\alpha, f(x)\} \rightarrow \{<v|f>, \alpha v(x)\} \quad (17)$$

Notice that this interaction is Hermitian.

Let us assume that

$$\int \frac{dy |v(y)|^2}{y} < 1 \quad (18)$$

Then the system has no bound state (in fact this is necessary and sufficient). The decay amplitude can be calculated to be

$$P(t) = |A(t)|^2 \quad (19)$$

$$A(t) = \int_0^\infty dy e^{-iyt} \frac{|v(y)|^2}{\left[y - 1 + p \int d\tau \frac{|v(\tau)|^2}{\tau - y}\right]^2 + \pi^2 |v(y)|^4} < 1$$

$$= \int_0^\infty dy e^{-iyt} F(y) \quad (20)$$

We see that $F(y)$ looks like a Breit-Wigner form, if $v(y)$ is small, and sufficiently slowly varying near $y=1$, so that the principal value integral can be neglected or replaced by a constant.

If we make these drastic simplifications we get

$$A(t) \simeq \int_0^\infty dy e^{-iyt} \frac{\frac{\Gamma}{2\pi}}{(y-y_0)^2 + \left(\frac{\Gamma}{2}\right)^2} \quad (21)$$

with the identification

$$\pi^2 |v(1)|^4 = \frac{\Gamma^2}{4}$$

$$y_0 = 1 - p \int \frac{d\tau |v(\tau)|^2}{\tau - 1} \quad (22)$$

Now if we make the further step to replace (19) by an integral running from $-\infty$ to $+\infty$, we get

$$A(t) = \int_{-\infty}^{+\infty} dy \frac{e^{-iyt} \frac{\Gamma}{2\pi}}{(y - y_0)^2 + \left(\frac{\Gamma}{2}\right)^2} \quad (23)$$

The integral has 2 poles in the complex plane at $y = y_0 \pm i\frac{\Gamma}{2}$, and to calculate (23) it is sufficient to replace the integration contour from $-\infty$ for $+\infty$ by a horizontal line in the lower half plane plus a contour around the pole in the lower half plane.

It is easy to see that the integral reduces to the contribution of this pole, which gives:

$$A(t) \simeq -2\pi i \frac{\Gamma/2\pi}{\pi} \frac{e^{-iy_0 t - \frac{\Gamma}{2}t}}{-i\Gamma} \simeq e^{-iy_0 t - \frac{\Gamma}{2}t}$$

and hence

$$B(t) = |A(t)|^2 = \exp -\Gamma t \quad (24)$$

So that we get, with these approximations, the exponential decay law.

However this is certainly not always valid because $v(y) = \text{constant}$ would lead to the fact that the integral $\int \frac{|v(y)|^2 dy}{y}$ would diverge, and furthermore the integral from $-\infty$ to 0 should not be included. Then it is not anymore possible to deform the contour.

An explicit model has been worked out by Galindo and Pascual. It is given by

$$|v(y)|^2 = \frac{\sqrt{2}}{\pi} \frac{g^2}{1 + y^2} \quad (25)$$

then

$$y - 1 + P \int \alpha \tau \frac{|v(\tau)|^2}{\tau - y} = (y - 1) \left[1 - \frac{g^2}{1 + y^2} \right] \quad (26)$$

then Γ previously defined is $\sqrt{2}g^2$.

The expression

$$y - 1 + \int_0^\infty \frac{d\tau |v(\tau)|^2}{\tau - y},$$

has a pole on the second sheet of the y cut plane if $g^2 < 1$.

We show on Figure 1, Figure 2 and Figure 3 the results of explicit calculations for several values of g^2 .

Figure 1 and Figure 2 correspond to $g^2 = 0.01$. The lifetime of the particle is $T = 70$ in dimensionless units, and we see on Figure 1 that the breakdown of the exponential law occurs for $t \geq 30$ T and on Figure 2 that we have also a violation of the exponential law for short times $t < \frac{T}{20}$, with a characteristic quadratic behaviour near $t=0$.

Figure 3 shows that there are situations where there is no time interval where the exponential decay is even approximately valid. This occurs for $g^2 \geq 0.3$, corresponding to a "lifetime" of $T=2$.

It is amusing that in the case $g^2 = 0.01$, already for $t=3000$, i.e. $t=40$ T, the asymptotic formula (14) is valid, giving

$$P(t) \simeq \frac{2}{\pi^2} (g^2)^2 \frac{(\Gamma(3/2))^2}{t^3} \sim 10^{-15}. \quad (27)$$

To see when the exponential holds and when it does not hold, let us take a very crude model:

$$A(t) = \int_0^\infty \left(\frac{E}{m}\right)^k \frac{2\pi\Gamma e^{-iEt}}{(E-m)^2 + \frac{\Gamma^2}{4}} dE, \quad (28)$$

k integer.

From the contour argument, already used previously, we have

$$e^{-\Gamma t/2} = \int_{-\infty}^{+\infty} \left(\frac{E}{m}\right)^k \frac{2\pi\Gamma e^{-iEt}}{(E-m)^2 + \frac{\Gamma^2}{4}} dE, \quad (29)$$

Now we try to estimate, the magnitude of what we have added to make (28) calculable, namely

$$\Delta = \int_{-\infty}^0 \frac{e^{-iE_k} \left(\frac{E}{m}\right)^k dE}{(m-E)^2 + \left(\frac{\Gamma}{2}\right)^2}, \quad (30)$$

we can rotate the line of integration along the negative imaginary axis and get

$$\begin{aligned} |\Delta| &< \int_0^{\infty} \frac{e^{-\gamma t} \left(\frac{t}{m}\right)^k 2\pi\Gamma}{m^2 + \Gamma^2} dy \\ &\leq 2\pi k! \left(\frac{\Gamma}{m}\right)^{k+2} \left(\frac{1}{\Gamma t}\right)^{k+1} \end{aligned} \quad (31)$$

So, we will get the exponential regime if

$$(\Gamma t)^{2k+2} e^{-\Gamma t} \gg 4\pi^2 (k!)^2 \left(\frac{\Gamma}{m}\right)^{2k+4} \quad (32)$$

k, is of course model dependent, we can take

- k=0 then, (30) is satisfied if $t_1 < t < t_2$

$$\begin{cases} \Gamma t_1 = 2\pi \left(\frac{\Gamma}{m}\right)^2 \\ \Gamma t_2 = \ln \frac{1}{4\pi^2} + 4 \ln \left(\frac{m}{\Gamma}\right) \\ \quad + 4 \ln 2 + 2 \ln \ln \left(\frac{m}{\Gamma}\right) \end{cases}$$

- k=1 $t_1 < t < t_2$

$$\begin{aligned} \Gamma t_1 &= \sqrt{2\pi} \left(\frac{\Gamma}{m}\right)^{3/2} \\ \Gamma t_2 &= \ln \frac{1}{4\pi^2} + 6 \ln \left(\frac{m}{\Gamma}\right) \\ &\quad + 4 \ln \ln \left(\frac{m}{\Gamma}\right) + 4 \ln 6 \end{aligned} \quad (33)$$

Let us give now some examples in "real" life. Consider the decay of the lambda particle, here m is of the order of 1 GeV and t is 10^{-10} sec equivalent to 1cm, equivalent to 10^{-14} GeV⁻¹.

With k=1, we get

$$\begin{cases} \Gamma t_1 = 2 \times 10^{-21} \\ \Gamma t_2 = 200 \end{cases}$$

So that deviations from exponential decay occur at either infinitely too early times or at later times where essentially no particles are left so that the observation is impossible.

Another example which created some excitement some years ago is that of proton decay. Estimates of grand unified theories indicated a lifetime which could be as low as 10^{30} years [8]. On the other hand, no decay was observed and the apparent lifetime was larger than 10^{30} years [9]. On the other hand, the age of the universe is less than 10^{10} years, so that protons were created less than 10^{10} years ago. It would be tempting to say that we are still in the quadratic regime, $P(t) \sim 1 - ct^2$ so that the apparent lifetime is much bigger than the real lifetime.

However, for any lifetime larger than 10^{30} years, we have, with $m = \text{GeV } \Gamma/m < 10^{-51}$, and with $k=1$

$$t_1/T < 10^{-75}$$

i.e. $t_1 < 10^{-27}$ sec, so that we are well inside the first three minutes!

Another case about which one might worry is that of K_0 decay if we take K_{short} decay, with a lifetime of $0.9 \cdot 10^{-10}$ seconds if we take $m=0.3$ GeV we expect deviations to exponential decay around 100 to 200 lifetimes, i.e. 10^{-8} seconds and this is precisely the order of magnitude of the K_{long} decay lifetime. But it must be realized that after 100 lifetimes there will be nothing left of the K_0 short and one should not be impressed by the coincidence of the time after which K_0 short ceases to decay exponentially with the K_0 long lifetime.

To conclude this section, I would like to say that all these considerations can be extended to the "relativistic" Breit-Wigner form which is used for instance to describe the Z_0

$$A(t) = \int_0^{\infty} s \frac{\Gamma e^{-iEt} dE}{(s-M)^2 + (M\Gamma)^2} \quad (34)$$

To make the exponential decay law appear, we have to add an integration along the negative imaginary axis:

$$\hat{A}(t) = \int_{-\infty}^0 + \int_0^{\infty} s \frac{\Gamma e^{-iEt} dE}{(s-M)^2 + (M\Gamma)^2}, \quad (35)$$

So that a single pole is present in the quadrant. $\hat{A}(t)$ is a pure exponential and the corrective term is bounded by

$$\frac{\Gamma}{M^2} \int y^2 e^{-\gamma t} dy.$$

2. Is an Unstable Particle a Pole

In the previous model, we have seen that if $v(y)$ is analytic, for instance if

$$(v(y))^2 = \frac{\sqrt{2}g^2}{M} \frac{\sqrt{y}}{1+y^2} \quad (36)$$

a pole on a second sheet is associated to the unstable particle, given by

$$0 = \lambda - 1 + \frac{g^2}{1+\lambda^2} (1 - \lambda - \sqrt{-2\lambda}) \quad (37)$$

However, if v is not analytic we cannot make a continuation to the second sheet, and yet, we have an unstable particle.

One might object that this model is rather artificial and that, in real life, unitarity and analyticity force you to associate a pole on the second sheet to an unstable particle.

Long ago it has been "shown" by Zimmerman [10] that one can continue the elastic scattering amplitude $F(s, \cos\phi)$ across the elastic cut. This looks very simple. One projects the amplitude on a certain angular momentum to get the partial wave amplitude:

$$f_l(s) = \frac{\sqrt{s}}{2k} e^{i\delta_l} \sin \delta_l \quad (38)$$

then

$$S_L(s) = 1 + \frac{2k}{\sqrt{s}} f(s) = e^{2i\phi_L(s)}, \quad (39)$$

and if $S_L(s)$ is analytic in some neighbourhood of the real axis, which is guaranteed by field theory [11] [12] we can define $S_L(s)$ for complex s , $\text{Im}s > 0$ and we can continue the unitarity relation on the real axis

$$S_L(s) \bar{S}_L(s) = 1, \quad (40)$$

Where the bar denotes complex conjugation, by

$$S_L(s) \bar{S}_L(\bar{s}) = 1, \quad (41)$$

(41) defines a continuation of $S_L(s)$ in a domain symmetric with respect to the real axis of the initial analyticity domain of $S_L(s)$. In this way, a zero of S on the first sheet, with $\text{Im}s > 0$, gives a pole on the second sheet, with $\text{Im}s < 0$

Then, once this is done for partial waves, one can, if one wants to, resum the partial wave series and obtain a continuation of the full amplitude.

All this seems fine. However some years ago I discovered a difficulty which is considered to be very serious by the experts as shown for instance by the talk of A.S. Wightman at the Vienna Conference in 1968 [13]. I shall try to describe this because, though this result is officially published in a festschrift in honor of N. N. Bogoliubov [14] it has never been distributed to libraries in the West, except to contributors. It is a pity, for this book contains many interesting articles like the one of V. Glaser on positivity.

The point is that without violating any sacred principle you can arrange to have $S_L(s)$ to have an accumulation of zeros along the real axis, the "physical region". You have to do this in a careful way, so that the infinite product over zeros converges, but this can be done, for instance by taking:

$$S_L(s) = \prod_{N=1}^{\infty} \prod_{k=0}^N \left(\frac{s - s_0 - \frac{k}{N} - \frac{1}{N^3}}{s - s_0 - \frac{k}{N} + \frac{1}{N^3}} \right) \quad (42)$$

on the segment $0 < \text{Re}(s - s_0) < 1$ $\text{Im } s = 1/N^3$, we have $N+1$ zeros. It is easy to see that, for complex s , the product converges. It is also possible to show that $|S_L(s + i\epsilon)| \rightarrow 1$ for $\epsilon \rightarrow 0$ for almost every s . However the real axis is clearly a natural boundary for the function.

Though, as I said, field theory experts consider this to be a catastrophe, one can add an extra postulate, which does not follow from the axioms of axiomatic field theory, which is that $f_L(s)$ or $S_L(s)$ is continuous on the real axis. Then the catastrophe is avoided

and the continuation to the second sheet can be carried through. Most physicists will find this acceptable.

Naturally generalisation to N channels can be made without problem. This is essential for the Z_0 problem, because in lowest order in QCD and if hadronization is disregarded, the Z_0 case can be regarded as an N channel problem, the N channels being the lepton-antilepton and quark-antiquark channels. As soon as gluon production is allowed, this is no longer true, and this is not true if one looks at the real final states, made of multitudes of pions, protons etc.

A tough problem is the true 3 body problem. Then, after postulating the continuity of 2 body amplitudes to avoid the catastrophe previously mentioned, J. Bros [15] has been able to prove that this continuation to "a" second sheet through the physical region where 3 body final states are energetically allowed is possible. However, on this second sheet, other singularities than poles can be met, like cuts for instance.

However, in the best of all cases, the scattering amplitude can be continued through the elastic cut or the inelastic cut. This does not say that necessarily a pole on some hidden sheet is associated to an unstable particle. If you look back at the particle hunts of the 60's with bubble chambers, you see that people were content to call "particle" a bump seen in some projection of a Dalitz plot.

Without going to very high mathematics you can see by yourself that you can relatively well reproduce a Breit-Wigner shape by a sum of gaussians. A Breit-Wigner shape

$$|A| = \frac{\Gamma/2}{|E - E_0 + i\Gamma/2|}$$

correspond to a pole on the second sheet while a gaussian shape can be analytically continued but does not exhibit any pole. Let me give first a very simple example: take a half width $\Gamma/2 = 1$, and let $x = E - E_0$. Compare

$$f = \frac{1}{1 + x^2} \quad (\text{Breit - Wigner}) \quad \text{and} \quad (43)$$

$$g = 0.8 \exp - 0.5x^2 + 0.2 \exp - 3x^2 \quad (\text{sum of gaussians}) \quad (44)$$

we get

x	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4
$\frac{g}{f}$	1.00	1.00	1.00	1.00	1.00	0.99	0.96	0.89

In fact this is first a preview of a theorem which says that any function defined on the real axis can be approximated with an arbitrary accuracy by an entire function.

These considerations have been made in 1970-71 by L. Fonda and his collaborators in a series of articles [5]. They also point out that one can make a phase shift rise through $\pi/2$ without having an associated pole on the second sheet even through a rising phase shift crossing $\pi/2$ was believed by phase shift analyzers to be the characteristic signal of a resonance.

They also showed that these "particles" exhibit the approximate exponential decay for times which are neither very small, nor very large.

Somewhat, little attention was paid by most physicists to the work of these "iconoclasts" and everybody continued to associated unstable particles and poles. However, I

was personally very much disturbed and tried to understand what was going on. If you think a little you realize that the price you pay when you replace a Breit-Wigner form by a sum of gaussians is that your amplitude grows very fast in some directions, when you go away from the real axis (physical region) on the physical sheet (i.e. $\text{Im} E > 0$ if you start from $E + i\epsilon$, $\epsilon > 0$). This is unpleasant but there is one simple case where one has bounded the magnitude of the scattering amplitude in the complex plane, inside its domain of analyticity, it is the case of pion-pion scattering.

Let me try to summarize what we know of the pion-pion scattering amplitude.

- 1) It has a large analyticity domain [16]. If we call $F(s, t, u)$ the amplitude, with one redundant variable, since $s+t+u=4m_\pi^2$, m_π pion mass, s square of the c.m. energy of the s channel. t square of the 4 momentum transfer in the s channel, etc., the amplitude (case of $\pi_0\pi_0 \rightarrow \pi_0\pi_0$) is completely symmetric, with 3 physical regions defined by $s > 4m_\pi^2$, $t < 0$, $u < 0$ and permutations, and is analytic in a domain containing the union of the domain s

$$\left\{ |t| < 4m_\pi^2 \text{ and } s - 4m_\pi^2 \notin \mathbb{R}^+, \text{ and } u - 4m_\pi^2 \notin \mathbb{R}^+ \right\} \quad (45)$$

and permutations.

- 2) Inside this domain of analyticity it has been possible, by using a rather tricky combination of analyticity and unitarity to find absolute bounds on the amplitude. Initially the bounds that I obtained for an amplitude normalized in such a way that $F(s=4, t=0, u=0)$ represents the scattering length at threshold, were very poor [17] as indicated on Fig.4, where we show upper and lower bounds at the symmetry point, i.e., at $s=t=u=4m_\pi^2/3$, but with time and efforts, they were considerably improved [18]

Perhaps more physically significant is the lower bound on the $\pi_0\pi_0 \rightarrow \pi_0\pi_0$ scattering length which was obtained first by Bonnier and Vinh Mau [19] later improved [20] [21] to reach the best possible value in the paper of Caprini and Dittà [22] of - 1.65 Compton wave length. This is about a factor 10 times what one observes, but, considering that essentially no dynamical ingredient has been included, except the absence of bound states of the pion-pion system, it is rather remarkable.

Now the initial strategy of Bonnier and Vinh Mau was to use bounds on the full amplitude and to calculate a bound on the S wave by the formula

$$f_0(s) = \frac{2}{s-4} \int_{t=4}^0 F(s, t, u) dt \quad (46)$$

and since bounds on F exist for any $|t| < 4m_\pi^2$ if one avoids the s and u cuts one sees that one gets a bound on $f_0(s)$ for

$$|s-4| < 8m_\pi^2,$$

so that the domain in which we have an absolute bound on $f_0(s)$ contains a piece of the physical region

$$4m_\pi^2 < s < 12m_\pi^2,$$

This bound can be translated into a bound on $S=1+2k/\sqrt{s} f_0(s)$, with the remark that $|S|$ is bounded by 1 on the segment $4m_\pi^2 < s < 8m_\pi^2$. Naturally the bound becomes infinite on the border of the initial domain but it is finite and calculable on, for instance, the semi continuous contour, with a global upper bound B .

Now we want to use this information to know if the S wave amplitude has, or has not a pole on the second sheet and we want to show that if it is known with sufficient accuracy in the physical region this question is decidable, at least in principle.

Now suppose we have two candidates S_1 and S_2 for the amplitude, (here we take a given isospin state so that we have $|S|=1$ on the physical region), both known with an uncertainty ϵ on the physical region $4m_\pi^2 < s < 12m_\pi^2 - \eta$, but that S_1 has a pole on the second sheet, i.e., a zero on the first sheet, while the second has not. Starting from the information that $|S_1 - S_2| < 2\epsilon$ above and below the physical cut and that $|S_1 - S_2| < B$ along the semi continuous contour, it is possible, using the theory of subharmonic functions, to find "equipotentials" along which $|S_1 - S_2| < |\epsilon|^{\alpha} |2B|^{1-\alpha}$, $0 < \alpha < 1$ (see technical note). Such an equipotential E_α is qualitatively drawn by a dotted line on the figure.

Suppose that along this equipotential one has $|S_1| > |\epsilon|^{\alpha} |2B|^{\alpha}$, which will happen if ϵ is sufficiently small. Then the variation of the phase of S_1 and S_2 along the closed contour made of E_α and the cut $4m_\pi^2 < s < 12m_\pi^2 - \eta$ is the same and therefore according to Rouche's theorem S_1 and S_2 have the same number of zeros. In particular, if S_1 has one zero in this region for $\text{Im } s > 0$, S_2 has this zero too and it is possible to know in principle if there is or there is not a pole on the second sheet. In practice, because B is very large, ϵ must be very small, but not infinitesimally small. Naturally one would like to make this argument work for higher energies and other processes, but it is comforting that at least in a very limited case one can find an answer to this question.

In the case of the Z_0 , one might try to construct an analogue of this argument by using the Jost-Lehmann-Dyson representation and the positivity of the e^+e^- cross section. However, as I said before, one has to remain in the approximate framework where final states are t^+t^- and $q\bar{q}$ pairs so as to deal with a problem with a finite number of channels.

Technical Note.

Let f be an analytic function limited by a domain with a border made of two arcs, Γ_1 and Γ_2 , assume that $|f| < B_1$ on Γ_1 , $|f| < B_2$ on Γ_2 . Then construct a condenser with voltage $\log B_1$ on Γ_1 and $\log B_2$ on Γ_2 . Then the voltage at any point inside is the logarithm of an upper bound on f at this particular point in the complex plane.

In mathematical terms $\log f$ is less than the largest subharmonic function taking the value $\log B_1$ on Γ_1 and $\log B_2$ on Γ_2 .

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