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Vertex Operators in the Fractional Quantum Hall Effect

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Abstract

A second-quantized formalism for electrons confined to a plane in a strong perpendicular magnetic field is constructed using vertex operators. They are seen to arise naturally from a holomorphic representation of Laughlin's first-quantized wave functions, since they have the unique properties of creating coherent states, satisfying anyonic statistics and factorizing matrix elements. While open string vertex operators are sufficient for representing Laughlin's "ground state" wave functions, it is shown that the vertex operators appearing in the theory of closed strings are needed in order to represent both types of anyonic excitations (quasi-holes and quasi-electrons) which appear in the theory of the fractional quantum Hall effect.

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1 Introduction and Summary

The similarity between Laughlin's wave functions appearing in the theory of the fractional quantum Hall effect [1], and the Fubini-Veneziano vertex operators which originated in the theory of open strings [2], has previously been discussed in detail in [3, 4]. It was shown that Laughlin's "ground state" wave functions

$$\lambda^\alpha(z_1, \dots, z_N) = \prod_{i < j}^N (z_i - z_j)^{\alpha^2} \quad \alpha^2 \in 2Z + 1 \quad (2)$$

can be written as a matrix element of vertex operators between the Fock vacuum $|0\rangle$ and a "charged vacuum" $|N\alpha\rangle$ which is labelled by the total charge of the new "ground state":

$$\lambda^\alpha(z_1, \dots, z_N) = \langle N\alpha | U_\alpha(z_1) U_\alpha(z_2) \dots U_\alpha(z_N) | 0 \rangle \quad (3)$$

The vertex operator $U_\alpha(z)$ has the rotation and permutation properties appropriate to represent an anyon with "spin-statistics" label α . In general α may be a vector, in which case $\alpha^2 \equiv \alpha \cdot \alpha = \sum_i \alpha_i^2$.

Interesting new features arise when we try to accommodate quasi-hole (qh) and quasi-electron (qe) excitations within this second-quantized formalism. Because Laughlin's quasi-hole wave function is also *holomorphic*:

$$\lambda_{qh}^\alpha(z_0, z_1, \dots, z_N) = \prod_{i=1}^N (z_0 - z_i) \lambda^\alpha(z_1, z_2, \dots, z_N) \quad (4)$$

it can be written as

$$\lambda_{qh}^\alpha(z_0, z_1, \dots, z_N) = \langle N\alpha + \alpha^{-1} | U_1(z_0) U_\alpha(z_1) U_\alpha(z_2) \dots U_\alpha(z_N) | 0 \rangle. \quad (5)$$

However, it is not possible to represent the quasi-electron wave function

$$\lambda_{qe}^\alpha(\bar{z}_0, z_1, \dots, z_N) = \prod_{i=1}^N (\bar{z}_0 - \frac{\partial}{\partial z_i}) \lambda^\alpha(z_1, \dots, z_N), \quad (6)$$

in terms of the holomorphic vertex operators $U_\alpha(z)$ alone. We find that we must double the set of basic operators in a way which, in the language of dual models (strings), corresponds to the transition between the Veneziano (open string) and the Virasoro (closed string) amplitudes.

These results are a strong indication that the quantum Hall system harbours a hidden scaling symmetry, since the natural home for vertex operators is conformal field theory. We can in fact show that the presence of such a symmetry in the many-body theory is sufficient for explaining why the quantum Hall wave functions are related to vertex operators in an appropriate quantum field theory.

The purpose of this paper is two-fold. In the first part, Sections 2 and 3, we expand on the basic observations remarked on above in an attempt to make it plausible that a field theoretic treatment of the FQHE should involve vertex operators. In order to make a smooth transition from the first- to second-quantized formalism in Section 3, we first briefly re-examine Laughlin's first-quantized many-body theory in a holomorphic representation [5, 6]. This gives a satisfactory description of the ground state wave functions. In the second part, Sections 4 and 5, we turn to the more subtle problem of representing quasi-particle excitations in the vertex operator formalism. In Section 4 the "open string" formalism explained in Section 3 is extended to the "closed string" and the quasi-particle wave functions are given in second-quantized operator form. Finally we sum up and indicate how the operator formalism can be extended to include the hierarchy.

After completing this work we received a paper by Stone [7] which is somewhat similar in spirit but differs greatly in detail from our approach. The relationship between the edge excitations discussed by Stone and the bulk states discussed here requires elucidation.

2 First-Quantization

Consider first a single electron confined to move in a plane $\mathbf{R}^2(x, y)$ submerged in a perpendicular homogeneous magnetic field $\mathbf{B} = \nabla \times \mathbf{A} = (0, 0, -B)$. In the symmetric gauge $\mathbf{A} = \frac{B}{2}(y, -x, 0)$ the classical Hamiltonian

$$H = \frac{1}{2m}(\mathbf{p} - \frac{e}{c}\mathbf{A})^2 \quad (7)$$

can be diagonalized by rotating to the *Fock basis* (complex conjugation is denoted by a star, while an overbar denotes "helicity"):

$$\begin{aligned}
 a &= \frac{1}{\sqrt{2}}(x + iy + \frac{1}{2}p_x - \frac{1}{2}p_y) \rightarrow a \\
 a^* &= \frac{1}{\sqrt{2}}(x - iy - \frac{1}{2}p_x - \frac{1}{2}p_y) \rightarrow a^\dagger \\
 \bar{a} &= \frac{1}{\sqrt{2}}(x - iy + \frac{1}{2}p_x + \frac{1}{2}p_y) \rightarrow \bar{a} \\
 \bar{a}^* &= \frac{1}{\sqrt{2}}(x + iy - \frac{1}{2}p_x + \frac{1}{2}p_y) \rightarrow \bar{a}^\dagger
 \end{aligned} \tag{7}$$

in units where the *magnetic length*

$$\lambda = \left(\frac{\hbar c}{eB} \right)^{\frac{1}{2}} \tag{8}$$

has been normalized so that $2\lambda = 1$. The column on the right refers to the corresponding quantum mechanical creation and annihilation operators, whose functional (Schrödinger) representation in terms of differential operators is obtained in the usual way by letting $\mathbf{p} \rightarrow -i\hbar\nabla$. They satisfy harmonic oscillator commutation relations for *two independent oscillators*, one for each degree of freedom in the plane:

$$[a, a^\dagger] = [\bar{a}, \bar{a}^\dagger] = 1 \quad [a, \bar{a}] = [a^\dagger, \bar{a}^\dagger] = 0. \tag{9}$$

By definition the vacuum state $|0\rangle$ is annihilated by both a and $\bar{a}$. In this Fock basis the Hamiltonian depends only on one of the oscillators

$$H = \hbar\omega(\bar{N} + \frac{1}{2}), \tag{10}$$

where $\bar{N} = \bar{a}^\dagger \bar{a}$ is the number operator and $\omega = \frac{eB}{mc}$ is the cyclotron frequency.¹

Since $[H, N] = 0$, the states

$$|n, \bar{n}\rangle \equiv (a^\dagger)^n (\bar{a}^\dagger)^{\bar{n}} |0\rangle \quad (n, \bar{n} = 0, 1, 2, \dots) \tag{11}$$

are simultaneous eigenstates of both H and N :

$$\begin{aligned}
 H|n, \bar{n}\rangle &= (\bar{n} + \frac{1}{2})|n, \bar{n}\rangle \\
 N|n, \bar{n}\rangle &= n|n, \bar{n}\rangle.
 \end{aligned} \tag{12}$$

¹We set $\hbar\omega = 1$ in the following. If we reverse the direction of the magnetic field we get instead $H = N + \frac{1}{2}$, where $N = a^\dagger a$.

We now choose a coordinate representation basis in the single-particle Hilbert space which preserves the holomorphic factorization displayed by the operators defined above. This is achieved by a grossly overcomplete set of Gaussian wavepackets called *coherent states*. They are the "closest thing" to classical states that can be constructed in quantum mechanics, in the sense that they are wavepackets whose centre of mass follows the *classical* path in phase space, with the minimum spread allowed by the uncertainty relation².

We can choose the coherent states to be holomorphic in z and $\bar{z}$

$$|z, \bar{z}\rangle = e^{z a^\dagger + \bar{z} \bar{a}^\dagger} |0\rangle \tag{13}$$

provided we choose the Gaussian measure

$$d\mu = d^2 z e^{-|z|^2} \tag{14}$$

on the physical Hilbert space $\mathcal{H}_{z, \bar{z}}$. Notice that we can treat z and $\bar{z}$ as independent parameters in the coherent state formalism. This is the first-quantized version of the fact that "right- and left-movers" in the closed string formalism to be discussed in Section 4 can be treated as independent. This is called "holomorphic factorization" in conformal field theory, and it is gratifying to see it emerging here. Physical results, like scattering amplitudes in string theory, correlation functions in statistical mechanics or probability densities in the FQHE, are obtained by setting $z^* = \bar{z}$. When this is done the parameter z can be interpreted as the classical position of the oscillators in the xy-plane, i.e., as the centre of the Landau orbit executed by the electron ("guiding centre coordinate"). It will be clear from the context whether we are working in the physical Hilbert space $\mathcal{H}_{z, \bar{z}}$ or the larger space $\mathcal{H}_{z, \bar{z}}$.

The states (13) have the desirable properties

$$\begin{aligned}
 a|z, \bar{z}\rangle &= z|z, \bar{z}\rangle & a^\dagger|z, \bar{z}\rangle &= \partial|z, \bar{z}\rangle \\
 \bar{a}|z, \bar{z}\rangle &= \bar{z}|z, \bar{z}\rangle & \bar{a}^\dagger|z, \bar{z}\rangle &= \bar{\partial}|z, \bar{z}\rangle,
 \end{aligned} \tag{15}$$

where $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$.

²For a nice introduction to the coherent state or "holomorphic functional" representation, see for example [8].

In order to see explicitly the semi-classical nature of these coherent states, compute the expectation value of a point in phase space to find that the coherent state $|z, z^*\rangle$ corresponds classically to an electron at rest at position $\sqrt{2}(\Re z, \Im z)$.

Since the vacuum wave function is trivial with our choice of normalization of the coherent states,

$$\langle z, \bar{z} | 0 \rangle = 1 \quad (16)$$

(i.e., the Gaussian has been absorbed in the measure), the eigenstates $|n, \bar{n}\rangle$ defined above are entire functions (analytic in the entire complex plane) in the coherent representation. In fact, they are simply homogeneous polynomials in z and $\bar{z}$ of total degree $n + \bar{n}$:

$$\psi_{n, \bar{n}}(z, \bar{z}) \equiv \langle z, \bar{z} | n, \bar{n} \rangle = z^n \bar{z}^{\bar{n}}. \quad (17)$$

Representations of Hilbert space on spaces $\mathcal{F}_k$ of entire k -variate functions have been studied by Bargmann [9] for finite k , and by Segal [10] for $k \rightarrow \infty$ which is relevant for field theory. The case encountered here, $\mathcal{F}_2$, is a concrete realization of Schwinger's [11] pair of bosonic oscillators which both he and Bargmann used in ingenious investigations of the rotation group.

In a very strong magnetic field the electrons will all be in the lowest Landau level, $\bar{n} = 0$, and Laughlin [1] showed that it is a good approximation to treat them as independent, i.e., if B is sufficiently large the Coulomb interaction is negligible. The ground state of this system can therefore be regarded as an ensemble of non-interacting 2-dimensional harmonic oscillators, whose eigenstates can be built entirely from one-particle states $|n, 0\rangle = (a^\dagger)^n |0\rangle$. In other words, the Hilbert space is just N copies of the 1-electron Hilbert space discussed above, $\mathcal{H}_N = \mathcal{H}_{z_1, \bar{z}_1} \otimes \dots \otimes \mathcal{H}_{z_N, \bar{z}_N}$. Since electrons obey Fermi statistics, the state must be completely antisymmetric in all oscillators, i.e., it must be an *odd* power of the Vandermonde or Slater determinant:

$$|\alpha\rangle = \prod_{0 \leq i < j}^N (a_i^\dagger - a_j^\dagger)^{\alpha^2} |0\rangle. \quad (18)$$

The parameter $\alpha^2 = 2k + 1$ ($k = 0, 1, \dots$) is the inverse *filling factor* which is the

fraction of the total number of available (degenerate) states of a given Landau level which are actually occupied.

Laughlin's wave-functions are obtained by writing $|\alpha\rangle$ in the coherent representation:

$$|z, \bar{z}\rangle \equiv |z_1, \bar{z}_1; z_2, \bar{z}_2; \dots; z_N, \bar{z}_N\rangle = e^{z_1 \bar{a}_1^\dagger + \bar{z}_1 a_1^\dagger + \dots + z_N \bar{a}_N^\dagger + \bar{z}_N a_N^\dagger} |0\rangle \quad (19)$$

$$\lambda^\alpha(z_1, \dots, z_N) = \langle z, \bar{z} | \alpha \rangle = A^{\alpha^2}, \quad (20)$$

where A is the fundamental anti-symmetric polynomial in $z_1, z_2, \dots, z_N$:

$$A = \prod_{0 \leq i < j}^N (z_i - z_j). \quad (21)$$

Any homogeneous anti-symmetric polynomial P can be written as $P = AS$, where S is a symmetric polynomial which is always a polynomial in the elementary symmetric functions or equivalently, the powersums. In short, the non-interacting case reduces to simple polynomial algebra³. Laughlin's wave-function λ^α is a holomorphic function of the z_i only because we have forced all the electrons into the lowest Landau level, and absorbed the Gaussian vacuum wave-function into the measure on $\mathcal{H}_N$:

$$d\mu = \prod_{i=1}^N d^2 z_i e^{-|z_i|^2}. \quad (22)$$

Laughlin's quasi-particle wave functions also fit nicely into this holomorphic representation. A *quasi-hole* at position z_0 (c -number) in the xy -plane is created by the quasi-hole operator

$$Q_{z_0} = \prod_{i=1}^N (z_0 - a_i^\dagger) \quad (23)$$

acting on the ground state $|\alpha\rangle$, while the *quasi-electron* is created by the adjoint operator. In the coherent basis the wave functions are therefore:

$$\begin{aligned} \lambda_{qh}^\alpha(z_0, z_1, \dots, z_N) &= \langle z, \bar{z} | Q_{z_0} |\alpha\rangle = \prod_{i=1}^N (z_0 - z_i) \lambda^\alpha \\ \lambda_{qe}^\alpha(z_0, z_1, \dots, z_N) &= \langle z, \bar{z} | Q_{z_0}^\dagger |\alpha\rangle = \prod_{i=1}^N (\bar{z}_0 - \bar{z}_i) \lambda^\alpha. \end{aligned} \quad (24)$$

³In a very recent paper, Stone [7] elaborates greatly on this point.

3 Second-Quantization

We now wish to construct many-body states using the powerful machinery of second-quantization, with the ultimate aim of uncovering an underlying quantum field theory which, presumably, will provide a complete theoretical understanding of the quantum Hall system. As usual the fundamental starting assumption is that the dynamical variables labelling the single-particle states suffice for also describing a collection of identical and indistinguishable particles, even when they are interacting.

The basic object in our second-quantized formalism is a *bosonic quantum field operator* $\hat{Q}$ which by definition is a functional of the single-particle modes z^n in the lowest Landau level. They are created from the many-particle (Fock) vacuum by the operators $\hat{b}_n^\dagger$ ($n > 0$) and annihilated by the adjoint operators $\hat{b}_n = \hat{b}_{-n}^\dagger$ ($n > 0$), satisfying the bosonic oscillator commutation relations $[\hat{b}_m, \hat{b}_n^\dagger] = m\delta_{m,n}$. We define

$$\hat{Q}(z) = \hat{Q}_0(z) + i \sum_{n \neq 0} \frac{\hat{b}_n^\dagger}{n} z^n, \quad (25)$$

where the appearance of the zero-mode,

$$\hat{Q}_0(z) \equiv \hat{q} - i\hat{p} \log z \quad [\hat{q}, \hat{p}] = i, \quad (26)$$

is dictated by translation invariance (or the form of the Green function in two dimensions: $\langle Q(z)Q(w) \rangle \simeq \log(z-w)$).

A generic *N-body wave function* $\Psi(x_1, x_2, \dots, x_N)$ is usually constructed from the N-body state $|\Psi\rangle$ by computing the overlap with the eigenstates of the x -representation. These can be constructed from the Fock vacuum $|0\rangle$ using the field operator $\hat{\psi}(x)$ evaluated at the N points $x = x_1, x_2, \dots, x_N$ labelling the state, i.e.,

$$|x_1, x_2, \dots, x_N\rangle = \hat{\psi}(x_1)\hat{\psi}(x_2)\dots\hat{\psi}(x_N)|0\rangle. \quad (27)$$

In short

$$\Psi(x_1, \dots, x_N) \equiv \langle \Psi | x_1, \dots, x_N \rangle = \langle \Psi | \hat{\psi}(x_1)\hat{\psi}(x_2)\dots\hat{\psi}(x_N) | 0 \rangle. \quad (28)$$

We wish to recover Laughlin's wave functions in a similar way from the second-quantized theory using the quantum field operator $\hat{Q}(z)$ defined above. However,

while we in the first-quantized formalism had to put relative phase-information (statistics) into the many-body wave function by hand, by forcing it to be antisymmetric in the interchange of any two particle labels, an important property of the second-quantized formalism is that this information should be built into the field operators themselves. Hence, apart from the obvious problem that a product of Q -operators can hardly be expected to give rise to the wonderfully factorized wave functions constructed by Laughlin, they are bosonic and are therefore inappropriate for representing electrons. The resolution of this impasse is well known to string theorists, but perhaps less familiar to the condensed matter community.

In $2+1$ dimensions the rigorous distinction between fermions and bosons disappears to some extent [12], since bosons can easily be built from fermions and *vice versa*. In fact, any statistics is possible and we can “anyonize” the boson $Q(z)$ (we drop the caret in the following) by simply exponentiating it into a so-called *vertex operator*:

$$U_\alpha(z) \equiv : e^{i\alpha Q(z)} : \quad (29)$$

The normal ordering denoted by double dots is the usual one where creation operators go to the left of annihilation operators, and q to the left of p . In the language of conformal field theory $U_\alpha(z)$ is said to be a primary field of weight $\frac{\alpha^2}{2}$. This means that $U_\alpha(z)$ transforms as follows under conformal transformations generated by the Virasoro operators L_n :

$$[L_n, U_\alpha(z)] = z^n \left(z \frac{d}{dz} + \frac{\alpha^2}{2}(n+1) \right) U_\alpha(z). \quad (30)$$

Notice that vertex operators create generalized coherent states, which in view of our first-quantized discussion of Laughlin's wave functions is ideal if we are going to use vertex operators to recover these many-body wave functions from the second-quantized formalism.⁴

A standard calculation which is most easily performed by exploiting the properties of coherent states (see for example Appendix A in [13]) establishes the

⁴In his 1960 treatise on quantum field theory [10], Segal anticipated that “the holomorphic functional representation is the only one that seems useful” when discussing Wick-ordered exponentials, i.e., vertex operators.

anyonic nature of these operators:

$$U_\alpha(z)U_\beta(w) = U_\beta(w)U_\alpha(z)e^{i\alpha\beta\epsilon(\arg z - \arg w)}, \quad (31)$$

where $\epsilon(x)$ is +1 if $x > 0$ and -1 if $x < 0$. Clearly, by judiciously choosing the value of α , $U_\alpha(z)$ can be made to pick up any desired phase when exchanged with an identical operator at a different point w . In particular, if $\alpha^2 \in 2\mathbb{Z}$ then $U_\alpha(z)$ is bosonic, while if $\alpha^2 \in 2\mathbb{Z} + 1$ then $U_\alpha(z)$ is fermionic.

Vertex operators were originally invented for the purpose of *factorizing* string scattering amplitudes, which are completely analogous to (28). So, to summarize: vertex operators create coherent states, have anyonic statistics and factorize matrix elements. They are therefore precisely the kind of field operators we need for constructing Laughlin's wave functions in the second-quantized formalism. This job is greatly simplified by the following remarkable properties of vertex operators, which ultimately follow from our fundamental assumption that the Hall system harbours a hidden scaling symmetry. Note that, in dimensions less than three, conformal invariance is such a powerful constraint that all the following statements are *exact*.

The spectrum of “spin-statistics” labels α_i which actually appear in a specific theory is determined by the dynamics of the problem, and in turn the vertex operators U_{α_i} determine the complete spectrum of “in” states as follows:

$$|\alpha\rangle = \lim_{z \rightarrow 0} U_\alpha(z)|0\rangle. \quad (32)$$

Similarly, from general properties of conformal field theory⁵ it also follows that the adjoint of a vertex operator $U_\alpha(z)$ is

$$U_\alpha^\dagger(z) = U_{-\alpha}\left(\frac{1}{\bar{z}}\right), \quad (33)$$

so that the “out” states are given by:

$$\langle\alpha| = \lim_{z \rightarrow \infty} z^{\alpha^2} \langle 0| U_{-\alpha}(z). \quad (34)$$

The vacuum expectation value of a product of vertex operators $U_{\alpha_i}(z_i)$ ($i = 1, 2, \dots, N$) vanishes unless charge is conserved:

$$\langle 0| U_{\alpha_1}(z_1) U_{\alpha_2}(z_2) \dots U_{\alpha_N}(z_N) |0\rangle = \prod_{i < j}^N (z_i - z_j)^{\alpha_i \alpha_j} \sum_{i=1}^N \alpha_i = 0. \quad (35)$$

⁵For a pedagogical review, see for example [14].

From (34) and (35) it now immediately follows that

$$\begin{aligned} \langle\alpha| \prod_{i=1}^N U_{\alpha_i}(z_i) |0\rangle &= \lim_{z \rightarrow \infty} z^{\alpha^2} \langle 0| U_{-\alpha}(z) \prod_{i=1}^N U_{\alpha_i}(z_i) |0\rangle \\ &= \lim_{z \rightarrow \infty} z^{\alpha^2} \prod_{i=1}^N (z - z_i)^{-\alpha \alpha_i} \prod_{i < j}^N (z_i - z_j)^{\alpha_i \alpha_j} \\ &= \lim_{z \rightarrow \infty} z^{\alpha^2 - \alpha \sum_{i=1}^N \alpha_i} \prod_{i < j}^N (z_i - z_j)^{\alpha_i \alpha_j} \\ &= \prod_{i < j}^N (z_i - z_j)^{\alpha_i \alpha_j} \sum_{i=1}^N \alpha_i = \alpha. \end{aligned} \quad (36)$$

If the vertex operators are fermionic, say $\alpha_i^2 = 2k+1$ ($k \in \mathbb{Z}$) ($i = 1, 2, \dots, N$), appropriate for electronic wave functions, then (36) immediately reduces to Laughlin's principal wave functions (1).

Together with its “closed string” extension to be discussed next, (36) provides us with an exact master expression for many-body wave functions in the idealized quantum Hall system. The rest of this paper is devoted to discussing some special cases which have appeared in the condensed matter literature.

4 Quasi-Particle Wave Functions

Since the quasi-hole wave function (3) is holomorphic in the z_i 's, we can immediately read off the second-quantized form from (36):

$$\begin{aligned} \lambda_{\text{qh}}^\alpha(z_0, z_1, \dots, z_N) &= \langle U_\alpha(z_0) U_\alpha(z_1) \dots U_\alpha(z_N) \rangle \\ &= \prod_{i=1}^N (z_0 - z_i)^\alpha \lambda^\alpha(z_1, \dots, z_N). \end{aligned} \quad (37)$$

In this section we consider for simplicity only the simplest case where the parameter α is a scalar quantity. In general it may be a vector, which was exploited in [4] to describe the hierarchy.

In order to give a field theoretic representation of the quasi-electron wave functions which are *not* holomorphic, we find it necessary to expand the preceding open string discussion to the closed string or Virasoro-Shapiro (VS) representation [15], which in the language of string theory involves both right (holomorphic)

and left (anti-holomorphic) modes. In our context they appear naturally when considering the probability density of Laughlin's wave functions:

$$\begin{aligned} \rho(z_1, \dots, z_N, \bar{z}_1, \dots, \bar{z}_N) &\equiv \lambda^\alpha(z_1, \dots, z_N) \lambda^\alpha(\bar{z}_1, \dots, \bar{z}_N) \\ &= \prod_{i < j}^N (z_i - z_j)^\alpha (\bar{z}_i - \bar{z}_j)^\alpha. \end{aligned} \quad (38)$$

By introducing two independent commuting sets of vertex operators $U_\alpha(z)$ and $\bar{U}_\alpha(\bar{z})$ we can write the probability density in the second-quantized form:

$$\rho(z_1, \dots, z_N, \bar{z}_1, \dots, \bar{z}_N) = \langle U_\alpha(z_1) \dots U_\alpha(z_N) \bar{U}_\alpha(\bar{z}_1) \dots \bar{U}_\alpha(\bar{z}_N) \rangle. \quad (39)$$

Here we have employed the useful convention that the expectation value $\langle \mathcal{O}_{a,\bar{a}} \rangle$ is always taken to be between the (neutral) Fock vacuum $|0\rangle$ and a "charged vacuum" $|a, \bar{a}\rangle$ which neutralizes the operator in question, i.e., $\langle \mathcal{O}_{a,\bar{a}} \rangle = \langle a, \bar{a} | \mathcal{O}_{a,\bar{a}} | 0 \rangle$.

If desired we can also express the wave functions directly in the closed string representation:

$$\lambda^\alpha(z_1, \dots, z_N) = \langle U_\alpha(z_1) \dots U_\alpha(z_N) : \bar{U}_\alpha(\bar{z}_1) \dots \bar{U}_\alpha(\bar{z}_N) : \rangle, \quad (40)$$

where the ordering ($b^\dagger, q$ to the left of b, p) is applied to the product of all $\bar{U}_\alpha$ vertex operators.

Eqs.(2) and (40) do of course represent the same object, so at first sight (40) looks like an unnecessary complication. However, the doubling of operators in (40) allows us to obtain Laughlin's representation (5) for the quasi-electron excitations from the second-quantized matrix element

$$\langle \bar{U}_\alpha(\bar{z}_0) U_\alpha(z_1) \dots U_\alpha(z_N) : \bar{U}_\alpha(\bar{z}_1) \dots \bar{U}_\alpha(\bar{z}_N) : \rangle = \prod_{i=1}^N (\bar{z}_0 - \bar{z}_i) \lambda^\alpha(z_1, \dots, z_N). \quad (41)$$

This expression is *weakly equivalent* to Laughlin's wave function λ_{qe}^α , which means that while they are not (necessarily) identical they do yield the same matrix elements (and hence the same physics) when projected onto the lowest Landau level. Girvin and Jach [5] showed that we in this case can exchange $\bar{z}_i$ for $\bar{z}_i$ and *vice versa* in the wave functions (and similarly for $\bar{z}_i$ and z_i), thus establishing the effective equivalence of (41) and (5).

Since $U_\alpha(z)$ represents a quasi-hole and $\bar{U}_\alpha(\bar{z})$ represents a quasi-electron, particle-hole duality is manifest in this second-quantized formulation, as it should be. We conclude that the closed string or VS-representation indeed is the appropriate framework in which to discuss quasi-particle excitations. Further evidence for this is provided by the anyonic properties of the VS-vertex operators.

The most general vertex operator

$$W_{\alpha,\bar{\alpha}}(z, \bar{z}) = U_\alpha(z) \otimes \bar{U}_{\bar{\alpha}}(\bar{z}) \quad (42)$$

acts on a doubled Hilbert space which is built from independent "right-moving" (holomorphic) and "left-moving" (anti-holomorphic) operators U and $\bar{U}$. In the language of conformal field theory there are two independent Virasoro algebras, generated by L_n and $\bar{L}_n$ ($n \in \mathbb{Z}$), acting on U and $\bar{U}$, respectively.

The rotation operator

$$\Lambda = L_0 - \bar{L}_0 \quad (43)$$

corresponds to the transformation

$$z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} = i \frac{\partial}{\partial \theta}, \quad (44)$$

with

$$2i\theta = \log z - \log \bar{z} \quad (45)$$

so that we have

$$[\Lambda, W_{\alpha,\bar{\alpha}}(z, \bar{z})] = (i \frac{\partial}{\partial \theta} + \frac{m}{2}) W_{\alpha,\bar{\alpha}}(z, \bar{z}), \quad (46)$$

where

$$m = \alpha^2 - \bar{\alpha}^2. \quad (47)$$

Consequently, under a full 2π rotation the operator W acquires a phase πm :

$$e^{2\pi i \Lambda} W_{\alpha,\bar{\alpha}}(z, \bar{z}) e^{-2\pi i \Lambda} = W_{\alpha,\bar{\alpha}}(z, \bar{z}) e^{\pi i m}. \quad (48)$$

The general relation between spin and statistics is confirmed by performing the permutation:

$$W_{\alpha,\bar{\alpha}}(z, \bar{z}) W_{\alpha,\bar{\alpha}}(w, \bar{w}) = W_{\alpha,\bar{\alpha}}(w, \bar{w}) W_{\alpha,\bar{\alpha}}(z, \bar{z}) e^{i\pi m \epsilon(\psi)}, \quad (49)$$

where ψ is the relative phase between z and w : $\psi = \arg z - \arg w = \arg \bar{z} - \arg \bar{w}$. Notice that U and $\bar{U}$ rotate in opposite directions.

From (48) and (49) we see that a pure quasi-hole corresponds to a positive statistical phase and a pure quasi-electron to a negative phase.

5 Conclusions

We have seen that the “open string” operator formalism, which describes Laughlin’s holomorphic ground states [3, 4], naturally generalizes to a “closed string” formalism which is needed in order to represent both kinds of fundamental excitations. Independently of the specific form of (3) and (5), the doubling of operators is required if one wants to deal with anyons with statistical phases of both sign. The main features of this approach can be summarized as follows:

- (i) The basic operator $U_1(z) =: e^{Q(z)}$ represents an elementary fermion (ψ_F) à la Skyrme [16], the zero-mode p being related to the particle number [17].
- (ii) Composition of operators is naturally accommodated by vertex operators: the product $\psi_F \phi_B \phi_{B_1} \dots \phi_{B_k}$ of a fermion with k bosons, for example, can be represented by the vertex operator $U_{\vec{\alpha}}(z)$ where $\vec{\alpha}$ is a $(k+1)$ -dimensional vector:

$$\vec{\alpha} = \begin{pmatrix} 1 \\ \sqrt{2} \\ \vdots \\ \sqrt{2} \end{pmatrix}. \quad (50)$$

In this way Laughlin’s wave function can be viewed as a “superfermion” made out of one elementary fermion and k bosons.

- (iii) Quasi-holes and quasi-electrons are given by “anyonic” operators obtained by the transformations

$$\vec{\alpha} \rightarrow \frac{\vec{\alpha}}{\alpha^2} \quad \vec{\alpha} \rightarrow \frac{\vec{\alpha}}{\alpha^2}. \quad (51)$$

- (iv) The fundamental quantity encoding the rotation-permutation properties of $W_{\vec{\alpha}, \vec{\beta}}$ is the spin-statistics label $m = \vec{\alpha}^2 - \vec{\beta}^2$.

Since the repeated use of (i)–(iv) generates values of m which are fractions with increasing denominator, this method can be applied to describe the hierarchical model (in the form due to Halperin [18]) which has been used to explain the higher fractions in the FQHE.

As an example let us consider the $\vec{\alpha}^2 = 3$ ground state which is related to the vertex $U_{\vec{\alpha}}(z)$ with

$$\vec{\alpha} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}. \quad (52)$$

The quasi-hole and quasi-electron are represented by $\vec{\alpha}_{qh} = \vec{\alpha}_{qe} = 0$ and

$$\vec{\alpha}_{qh} = \vec{\alpha}_{qe} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{3} \end{pmatrix}, \quad (53)$$

so that $m_{qh} = m_{qe} = \frac{1}{3}$. In analogy to what was done in [4] we can introduce “superanyons” by multiplying the elementary anyonic operators by the boson vertex $U_{\sqrt{2}}(z)$. In this way we get new anyonic vertex operators $W_{\vec{\alpha}, \vec{\beta}}^{qh}$ and $W_{\vec{\alpha}, \vec{\beta}}^{qe}$ with $\vec{\alpha}_{qh}' = 0$, $\vec{\alpha}_{qh}' = \sqrt{2}$ and

$$\vec{\alpha}_{qh}' = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{3} \end{pmatrix} \quad \vec{\alpha}_{qe}' = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{3} \end{pmatrix}. \quad (54)$$

The new quasi-hole therefore has spin-statistics label $m_{qh}' = \frac{7}{3}$, while the new quasi-electron has $m_{qe}' = \frac{5}{3}$, which are indeed the fractions appearing at the next level in the hierarchy.

The results obtained here and in [3, 4] show that the operator formalism of dual models can be used to give a faithful second-quantized representation of the theory of the FQHE. The origin, or indeed existence, of the conformal symmetry implicit in the use of the string formalism needs elucidation, and will be the subject of future work.

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