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***W*-Symmetries: Gauging and Geometry**

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ABSTRACT

We review some aspects of the gauging and of the underlying geometry of *W*-algebras (with finite as well as with infinite numbers of higher-spin generators).

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1. Introduction

The most appealing structure of the *classical* theory of gravity is the underlying geometric principle of invariance under arbitrary (smooth) changes of coordinates, i.e. under diffeomorphisms of the space-time manifold. String theory is aimed to yield a coherent *quantum* theory of gravity. The basic principle of string theory is diffeomorphism invariance of the two-dimensional (2D) string world-sheet or of its Euclidean counterpart, a 2D Riemann surface. Hence, much recent effort has been concentrated on studying quantum theories of *two-dimensional* gravity. The diffeomorphism invariance, like any gauge invariance, has to be gauge fixed. As a result, locally, the 2D metric can be taken to be proportional to the unit matrix. After this gauge choice, there are still residual diffeomorphisms leaving this form invariant. These are the conformal transformations $z \equiv x + iy \rightarrow f(z)$. They are generated by the holomorphic component $T(z) \equiv T_{zz}(z)$ of the energy-momentum tensor.^{*} The original diffeomorphism invariance implies Ward identities for correlation functions with insertions of the energy-momentum tensor. These identities are equivalent to the well-known operator product expansion (OPE) of $T(z)$ with $T(w)$:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \quad (1.1)$$

where c is the central charge. Accordingly the Virasoro generators, which are the Laurent coefficients of T , $T(z) = \sum_n L_n z^{-n-2}$, satisfy the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}. \quad (1.2)$$

Conformal field theory studies the possible representations of this algebra in terms of fields $\Phi(z, \bar{z})$ (called conformal fields) and the properties of correlation functions of these fields. A basic task is to determine the OPE of two conformal fields Φ_1 and Φ_2 :

$$\Phi_1(z, \bar{z})\Phi_2(w, \bar{w}) = \sum_i c_{1,2}^i (z-w)^{\Delta_i - \Delta_1 - \Delta_2} (\bar{z}-\bar{w})^{\bar{\Delta}_i - \bar{\Delta}_1 - \bar{\Delta}_2} \Phi_i(w, \bar{w}) \quad (1.3)$$

Here Δ and $\bar{\Delta}$ denote the (left and right) conformal dimensions of the fields, i.e.

$$T(z)\Phi(w, \bar{w}) = \frac{\Delta\Phi(w, \bar{w})}{(z-w)^2} + \frac{\partial\Phi(w, \bar{w})}{z-w} \quad (1.4)$$

and a similar equation with $T \rightarrow \bar{T}$, $\Delta \rightarrow \bar{\Delta}$ and $\partial \rightarrow \bar{\partial}$. Fields Φ satisfying this OPE with the energy-momentum tensor are called primary fields.

^{*} Of course, the residual diffeomorphisms leaving invariant the metric also include the transformations generated (with a different parameter) by the antiholomorphic component $\bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z})$ of the energy-momentum tensor.

In certain conformal field theories it turns out that some of these conformal fields and the energy-momentum tensor satisfy a *closed* operator product algebra. If furthermore these fields are holomorphic, this algebra is called chiral. A well-known example is given by the superconformal theories where the holomorphic field $G(z)$ and the energy-momentum tensor (and the central charge) form a closed chiral algebra, the superconformal algebra. Other examples of chiral algebras are the current algebra of a WZW model and the Z_N parafermion algebras.

A very interesting class of chiral algebras are the so-called W -algebras [1]. In general, these are *non-linear* algebras: the OPE of two W -generators closes only on normal-ordered products of the other W -generators and of the energy-momentum tensor. As an example, let us write down the simplest case, Zamolodchikov's W_3 algebra [1] with only one W -generator $W(z)$. The OPE of T with W simply tells us that W is a primary field of conformal dimension (spin) 3, i.e. it satisfies equation (1.4) with $\Delta = 3$. The OPE of W with itself is more involved:

$$\begin{aligned} W(z)W(w) = & \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\ & + \frac{3}{10} \frac{\partial^2 T(w)}{(z-w)^2} + \frac{1}{15} \frac{\partial^3 T(w)}{z-w} + \frac{2\beta\mathcal{L}(w)}{(z-w)^2} + \frac{\beta\partial\mathcal{L}(w)}{z-w}, \quad (1.5) \\ \mathcal{L}(z) = & :T(z)T(z): - \frac{3}{10} \partial^2 T(z) \quad , \quad \beta = \frac{16}{22+5c}. \end{aligned}$$

Note that the algebra only closes in the enveloping algebra since $\mathcal{L}$ contains the normal-ordered product of two energy-momentum tensors.[†] More generally, a W_N -algebra contains fields $W^{(k)}$ of conformal spin k with $k = 3, 4, \dots, N$, and the OPE of $W^{(k)}$ with $W^{(l)}$ will contain all sorts of normal-ordered polynomials in $W^{(r)}$'s such that the total (engineering) dimension of the polynomial is at most $k + l - 1$. The OPE (1.5) is equivalent to a (quantum) commutator algebra for the modes W_n . Below, we will be dealing with a classical (Poisson bracket) version of this algebra. There are several "classical" limits of eq. (1.5) one can take. The one we will be discussing below only contains the $T(z)T(z)$ -term and no central term on the r.h.s.

W_N -algebras are associated with the Lie algebra $A_{N-1} = su(N)$. Other W -algebras, the BW_N are associated with $B_N = so(2N+1)$. At the classical level (Poisson brackets instead of commutators or OPE's) there exist W -algebras for all classical Lie algebras A_{N-1}, B_N, C_N, D_N and also for certain exceptional ones.

[†] One usually defines $\mathcal{L}$ to include the $\partial^2 T$ -term in order to make $\mathcal{L}$ as close as possible to a spin-4 primary field.

The existence of a chiral algebra, extending the Virasoro algebra, greatly simplifies the study of a conformal field theory. Indeed, as one sees from equation (1.4) with $\Phi = W^{(k)}$ and $\Delta = k$, L_0 and $W_0^{(k)}$ commute. Since L_0 is the Hamiltonian, the $W_0^{(k)}$ generate symmetries in the usual sense. More generally, the L_n and $W_m^{(k)}$ satisfy a closed (albeit non-linear) algebra and all states of the Hilbert space of the conformal field theory have to fall into representations of this algebra. Hence, the larger is the chiral algebra, the more we know about the theory solely from representation theory without actually solving it. This is particularly true for the conformal field theories with central charge $c > 1$. We know [2] that a modular invariant conformal field theory with $c > 1$ has to include an *infinite* number of Virasoro irreducible representations (primary fields w.r.t. the Virasoro algebra). However, the W_N -algebras have discrete series of unitary representations with central charge*

$$c = (N - 1) \left(1 - \frac{N(N + 1)}{m(m + 1)} \right) < N - 1, \quad m \geq N + 1, \dots \quad (1.6)$$

These admit, also for $c > 1$, modular invariant partition functions with a *finite* number of irreducible representation of the W_N -algebra. Of course, if we reduce these representations with respect to the Virasoro subalgebra, we again obtain an infinite number of Virasoro irreducible representations (if $c > 1$) in accordance with the theorem of ref. 2.

The existence of a W -algebra also simplifies the study of $c < 1$ theories. The classical example [3] is the 3-state Potts model with $c = \frac{4}{5}$ corresponding to $m = 5$ of the standard unitary Virasoro series (eq. (1.6) with $N = 2$), but also to $m = 4$ of the series (1.6) with $N = 3$. The 3-state Potts model corresponds to an exceptional and hence non-diagonal modular invariant of the $m = 5$ Virasoro theory. It was shown in ref. 3 that this exceptional modular invariant is simply a diagonal modular invariant of the W_3 -algebra. It is generally true [4] that certain non-diagonal modular invariants of a chiral algebra are in fact diagonal modular invariants of a bigger chiral algebra. As another example, the exceptional non-diagonal invariant of the $m = 12$ superconformal theory ($c = \frac{7}{10}$) has been shown to be a diagonal invariant of a super W_3 -algebra [5,6].

The preceding remarks already show the importance of W -algebras in conformal field theories. But there are many other areas of two-dimensional physics where W -algebras are relevant. We will only briefly describe some of them. First, let us mention non-linear integrable differential equations. In the same way as the KdV-hierarchy is related to the Virasoro algebra [7] and the super-KdV-hierarchy to the super-Virasoro

* For $N = 2$ one has of course the standard unitary Virasoro series.

algebra [8], the various reductions of the KP-hierarchy (see e.g. ref. 9 for a review) are related to the W_N -algebras [10]. A field theoretic (Lagrangian) description is given in terms of Toda theories based on (non-affine) $su(N)$ [11]. These are two-dimensional integrable theories of $N - 1$ bosonic fields with exponential interactions. They naturally lead to the basic differential operator of order N in terms of which the $(N - 1)^{\text{st}}$ reduction of the KP-hierarchy is formulated, and thus to a realisation of the W_N -algebra [11]. The latter is of the Feigin-Fuchs type which in particular makes it possible to compute the braiding and fusion properties of the conformal blocks [12]. W -algebras have also shown up in the recent studies of 2D gravity by means of matrix models. The partition function of the $(N - 1)$ matrix model is closely related to the τ -function of the $(N - 1)^{\text{st}}$ reduction of the KP-hierarchy, and it is annihilated by certain constraint operators which satisfy a W_N -algebra [13].

Thus, W -algebras are a rather universal structure extending the Virasoro algebra. However, so far, it is a *chiral* algebra in *two* dimensions. Of course, the same is true for the Virasoro algebra, but, as discussed above, we know that the Virasoro algebra occurs upon gauge-fixing the two-dimensional diffeomorphism invariance. The latter is neither chiral nor restricted to two dimensions. It is a natural question to ask what replaces the diffeomorphisms when we extend the Virasoro to a W -algebra. In other words: what is W -geometry, or as a dynamical question: what is W -gravity?

In this brief review, we cannot discuss these topics exhaustively, and we will only give a brief account of some aspects we have been involved in. Also, we will only deal with classical (i.e. Poisson bracket) W -algebras. In section 2, we will discuss a recent attempt [14] by one of us to describe the geometry that might underlie the W_N -algebras (for finite N), just as the Virasoro algebra is related to diffeomorphisms, i.e. Riemannian geometry of a two-dimensional manifold. For other approaches to identify the geometry underlying the W -algebras see references 15-17. An interesting related paper is also ref. 18 although, at the time it was written, W -algebras were not mentioned explicitly. In section 3, we turn to the dynamical aspects, i.e. W -gravity. This (and the fourth) section can be read independently from section 2. We will first discuss a field theoretic (nonlinear) realisation of chiral algebras. We will then review the nonchiral and covariant gauging of these algebras. The former will appear as the gauge-fixed version of the latter. We will first show how this works in the more familiar case of the Virasoro algebra and two-dimensional conformal gravity. The same technique is then applied to the w_∞ algebra, which can be considered as a certain $N \rightarrow \infty$ limit of the (classical) W_N -algebras. In section 4, we will discuss how the nonlinear realisations of W -symmetries considered in section 3 are related to linear ones.

2. Aspects of W_N -geometry

In this section, we will try to give an answer to the question: what is W -geometry? We would like to know what geometric principle leads upon gauge-fixing and in two dimensions to the W_N -algebras, in the same way as the diffeomorphisms lead to the Virasoro algebra. In the next two sections we will deal with the corresponding *dynamical* question: what is W -gravity? At the present state of the art, we are not able to directly relate these two approaches, and the following two sections can be read independently from the present one.

Following ref. 14, we will exhibit a set of transformations of *geometric* origin that a) are covariant, b) can be formulated not only in $D = 2$, and c) when restricted to $D = 2$ and chiral transformation parameters, form an algebra which shows a close resemblance to the W_N -algebras. However, some questions related to the nonlinear structure of the W_N -algebras remain open. This approach is probably best described as “Kaluza-Klein on group manifolds” : we will consider the geometry of a D -dimensional manifold resulting from a dimensional reduction from a $\dim G$ -dimensional group-manifold.* As usual in Kaluza-Klein theories, the basic idea is the following : simple geometric transformations (diffeomorphisms) on the group-manifold give rise to different transformations (i.e. geometry) on the D -dimensional manifold. Some of them are simply diffeomorphisms again, but others are more complicated and have a striking resemblance to W -transformations. Contrary to usual Kaluza-Klein approaches, we obtain *higher-spin* (W -type) transformations from *lower-spin* transformations (i.e. diffeomorphisms) by a dimensional reduction. This is possible due to the presence of the group-invariant tensors $d^{a_1 \dots a_k}$ associated with the Casimir invariants of order k .

Consider a Lie group G with Lie algebra $\mathcal{G}$. $\mathcal{G}$ is the set of left-invariant vector fields on G . Its dual $\mathcal{G}^*$ can be identified with the left-invariant 1-forms. We may choose a basis $\{e^a\}$ ($a = 1, \dots, \dim G$) in $\mathcal{G}^*$ obeying the Cartan-Maurer equations

$$de^a = -\frac{1}{2}f^{abc}e^b \wedge e^c, \quad (2.1)$$

where the f^{abc} are the (completely antisymmetric) structure constants of $\mathcal{G}$. Choosing explicitly coordinates θ^μ on the group manifold, we write $e^a = e_\mu^a d\theta^\mu$ and the Cartan-

* It is known (see e.g. ref. 19) that there is a close connection between the three-dimensional Chern-Simons theory with gauge group $SL(2, R)$, the non-compact version of $SU(2)$, and the Virasoro algebra. This has motivated us to think about the Virasoro algebra as arising by a dimensional reduction from a three-dimensional space which could be the $SU(2)$ group manifold.

Maurer equations read in components

$$\partial_\mu e_\nu^a - \partial_\nu e_\mu^a + f^{abc} e_\mu^b e_\nu^c = 0 . \quad (2.2)$$

e_μ^a is a $\dim G$ -bein and the (constant curvature) metric on the group manifold is

$$g_{\mu\nu} = \delta_{ab} e_\mu^a e_\nu^b . \quad (2.3)$$

We can also think of e_μ^a as a $\mathcal{G}$ -gauge field living on the group-manifold. The Cartan-Maurer equations (2.1) or (2.2) then simply state that the associated field strength vanishes, i.e. e_μ^a is pure gauge (we assume G to be a simply-connected covering group). Let us consider the effect of gauge transformations with parameters ξ^a

$$\delta_\xi e_\mu^a = \partial_\mu \xi^a + f^{abc} e_\mu^b \xi^c . \quad (2.4)$$

Obviously, they leave invariant the Cartan-Maurer equations, but induce a change of the metric (2.3)

$$\delta_\xi g_{\mu\nu} = e_\nu^a \partial_\mu \xi^a + e_\mu^a \partial_\nu \xi^a . \quad (2.5)$$

The $\dim G$ -bein e_ρ^a allows us to trade the flat index a for an index ρ of a vector field on the group manifold :

$$\xi^a = e_\rho^a \zeta^\rho . \quad (2.6)$$

Let us stress that the e_ρ^a is invertible and hence the mapping (2.6) is one to one: there are as many parameters ξ^a as there are ζ^ρ . Inserting this equation (2.6) into (2.5) and using the Cartan-Maurer equations (2.2) we get for the metric and for the $\dim G$ -bein

$$\begin{aligned} \delta_\zeta g_{\mu\nu} &\equiv \delta_\xi g_{\mu\nu} = \zeta^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu \zeta^\rho + g_{\nu\rho} \partial_\mu \zeta^\rho \\ \delta_\zeta e_\mu^a &\equiv \delta_\xi e_\mu^a = \zeta^\rho \partial_\rho e_\mu^a + e_\rho^a \partial_\mu \zeta^\rho . \end{aligned} \quad (2.7)$$

The gauge transformation (2.4) plus the trading of a Lie algebra index for a vector index result in a diffeomorphism on the $\dim G$ -dimensional group manifold.

Let us note that eq. (2.6), though natural if we consider e_ρ^a as a $\dim G$ -bein, is rather unnatural from the point of view of gauge theory : the gauge transformation parameter ξ^a is a combination of the gauge field e_ρ^a and a vector field ζ^ρ . In particular, this implies that ξ^a transforms itself non-trivially under gauge transformations. As a consequence, the commutator of two such gauge transformations with parameters

$\xi^a = e_\rho^a \zeta^\rho$ and $\tilde{\xi}^a = e_\rho^a \tilde{\zeta}^\rho$ is a gauge transformation *not* with parameter $f^{abc} \xi^b \tilde{\xi}^c$ but instead with parameter

$$\xi'^a = e_\rho^a \zeta'^\rho \quad , \quad \zeta'^\rho = \{\tilde{\zeta}, \zeta\}^\rho = \tilde{\zeta}^\nu \partial_\nu \zeta^\rho - \zeta^\nu \partial_\nu \tilde{\zeta}^\rho . \quad (2.8)$$

Here $\{\tilde{\zeta}, \zeta\}$ is the Lie-bracket of $\tilde{\zeta}$ and ζ . This is of course what one expects for the commutator of two diffeomorphisms generated by the vector fields ζ and $\tilde{\zeta}$.

All preceding equations remain true if we perform a dimensional reduction and consider a D -dimensional submanifold of the group-manifold. Then, all Greek indices only run from 1 to D whereas Latin indices still run from 1 to $\dim G$. In particular, the e_μ^a are no longer invertible. Equation (2.4) to eq. (2.6) now generate D -dimensional diffeomorphisms but they only constitute a (D parameter) subset of all possible gauge transformations. What other gauge transformations can we write down? Rewriting eq. (2.6) as $\xi^a = \delta^{ab} e_\rho^b \zeta^\rho$, it is suggestive to replace the G -invariant tensor δ^{ab} (associated with the second-order Casimir invariant) by other G -invariant tensors. To be specific, consider $G = SU(N)$ with invariant (completely symmetric) tensors $d^{a_1 a_2 \dots a_i}$, $i = 3, \dots, N$, corresponding to the higher order Casimir invariants. Equation (2.6) can then be replaced by

$$\begin{aligned} \xi^a(\zeta) &= e_\rho^a \zeta^\rho , \\ \xi^a(\Lambda) &= d^{abc} e_\rho^b e_\sigma^c \Lambda^{\rho\sigma} , \\ &\dots \\ \xi^a(\eta) &= d^{ab_1 \dots b_k} e_{\rho_1}^{b_1} \dots e_{\rho_k}^{b_k} \eta^{\rho_1 \dots \rho_k} , \\ &\dots \\ \xi^a(\omega) &= d^{ab_1 \dots b_{N-1}} e_{\rho_1}^{b_1} \dots e_{\rho_{N-1}}^{b_{N-1}} \omega^{\rho_1 \dots \rho_{N-1}} , \end{aligned} \quad (2.9)$$

where $\zeta, \Lambda, \dots, \eta, \dots, \omega$ are completely symmetric tensors. Altogether they have $\sum_{k=1}^{N-1} \sum_{l=1}^k \binom{k-1}{l-1} \binom{D}{l} = \binom{N+D-1}{D} - 1$ independent components. For $D = 2$, e.g., these are $\frac{1}{2}N(N+1) - 1$ components, whereas $\dim SU(N) = N^2 - 1$. Hence, in $D = 2$, they all generate independent transformations but there are always some gauge transformations that cannot be generated this way.* Let us stress that it is the existence of the invariant tensors $d^{ab_1 \dots b_k}$ that allows us to trade the index a for

* For $D > 2$, the situation is different. For $D = 3$, e.g. there are $\frac{1}{6}N(N+1)(N+2) - 1$ independent components. Hence, the transformations (2.9) contain too many parameters for all N (except for $N = 2$ where $D = \dim G$), i.e. the corresponding gauge transformations are not all independent. This is the generic situation for $D \geq 3$. If we restrict the tensors to be also traceless then, for $D = 3$, we always have precisely $N^2 - 1$ parameters which is the dimension of $SU(N)$, and equation (2.9) defines a one to one map between the parameters.

a rank- k symmetric tensor $\eta^{\rho_1 \dots \rho_k}$. This is different from standard Kaluza-Klein where a vector only yields a vector and scalars.

It is straightforward to write down the transformation laws of e_μ^a under these gauge transformations. Using the Cartan-Maurer equations (2.2) and the Jacobi identities $f^{ab_1c}d^{cb_2 \dots b_{k+1}} + (\text{cyclic permutations of the } b_i) = 0$, we obtain for the transformation involving e.g. the rank k symmetric tensor $\eta^{\rho_1 \dots \rho_k}$

$$\delta_{\xi(\eta)} e_\mu^a = d^{ab_1 \dots b_k} \left(e_{\rho_1}^{b_1} \dots e_{\rho_k}^{b_k} \partial_\mu \eta^{\rho_1 \dots \rho_k} + k \partial_{\rho_1} e_\mu^{b_1} e_{\rho_2}^{b_2} \dots e_{\rho_k}^{b_k} \eta^{\rho_1 \dots \rho_k} \right). \quad (2.10)$$

For $k = 1$ these are simply D -dimensional diffeomorphisms.

The commutator of two such transformations, one involving a rank k tensor $\eta^{\rho_1 \dots \rho_k}$ and the other a vector ζ^ρ (i.e. a diffeomorphism) is easily computed. In doing this, we assume that the parameters ζ^ρ , $\eta^{\rho_1 \dots \rho_k}$, etc. are inert under the gauge transformations which only act on the gauge field e_μ^a as in (2.10). We obtain:

$$\left[\delta_{\xi(\zeta)}, \delta_{\xi(\eta)} \right] e_\mu^a = \delta_{\xi(\eta')} e_\mu^a \quad (2.11)$$

where

$$\begin{aligned} \eta'^{\rho_1 \dots \rho_k} &= \{\eta, \zeta\}^{\rho_1 \dots \rho_k} \\ &\equiv \eta^{\nu \rho_2 \dots \rho_k} \partial_\nu \zeta^{\rho_1} + (\text{cyclic perm. of } \rho_1 \dots \rho_k) - \zeta^\nu \partial_\nu \eta^{\rho_1 \dots \rho_k}. \end{aligned} \quad (2.12)$$

For later reference we note that if all indices take the same value we simply have

$$\eta'^{\rho \dots \rho} = k \eta^{\nu \rho \dots \rho} \partial_\nu \zeta^\rho - \zeta^\nu \partial_\nu \eta^{\rho \dots \rho}. \quad (2.13)$$

How do we interpret this commutation relation? Commuting a diffeomorphism with the rank- k tensor transformation gives back another rank- k tensor transformation. This commutator is very reminiscent of the OPE of the energy-momentum tensor T and a spin- $(k+1)$ field $W^{(k+1)}$ in 2D conformal field theory. Let us be more explicit. The OPE (1.4) with $\Phi = W^{(k+1)}$ and $\Delta = k+1$ implies

$$\left[\oint dz \zeta(z) T(z), \oint dz' \eta(z') W^{(k+1)}(z') \right] = \oint dz \eta'(z) W^{(k+1)}(z) \quad (2.14)$$

where

$$\eta' = k \eta \partial_z \zeta - \zeta \partial_z \eta \quad (2.15)$$

The analogy between eqs. (2.11), (2.12) and (2.14), (2.15) is obvious. More precisely, if we consider eqs. (2.11), (2.12) for $D = 2$ and restrict ourselves to chiral transformation parameters, i.e. only $\eta^{z \dots z} \equiv \eta$ and $\zeta^z \equiv \zeta$ are non-zero, the composition laws

(2.13) and (2.15) are identical. This suggests identifying the (gauge) transformation $\delta_{\xi(\eta)}$ containing the rank- k symmetric tensor $\eta^{\rho_1 \dots \rho_k}$ with a covariant generalisation of the (chiral) $W^{(k+1)}$ transformation generated by $\oint dz \eta W^{(k+1)}$. Equations (2.11), (2.12) however not only hold in $D = 2$, and, a priori, do not correspond to a conformal field theory.

Let us now consider the commutator of two of our generalised W -candidates $\delta_{\xi(\eta)}$ when acting on e_μ^a . The computations are slightly lengthy and we refer the reader to ref. 14 for details. Let us only note that the crucial identity one has to use is the one expressing the product of two d -tensors of $SU(N)$ in terms of products of lower-order d -tensors and structure constants. Here we only give the resulting commutator of two $W^{(3)}$ -candidates $\delta_{\xi(\omega)}$ and $\delta_{\xi(\Lambda)}$ (with rank-two tensors ω and Λ) for $D = 2$ and with chiral transformation parameters, i.e. we suppose that $\omega^{zz} \equiv \omega$ and $\Lambda^{zz} \equiv \Lambda$ are the only non-zero components. In this case, only terms proportional to $\psi^{zzz} = \partial_z \omega \Lambda - \omega \partial_z \Lambda$ survive and the general equation simplifies to yield

$$\left[\delta_{\xi(\omega)}, \delta_{\xi(\Lambda)} \right]_{\text{chiral}} e_\mu^a = 8\alpha \delta_{\xi(\psi^{zzz})} e_\mu^a - \frac{8(N^2 - 4)}{N(N^2 + 1)} \left(e_z^a e_z^b e_z^b \partial_\mu \psi^{zzz} + (\partial_z e_\mu^a e_z^b e_z^b + 2\partial_z e_\mu^b e_z^b e_z^a) \psi^{zzz} \right) \quad (2.16)$$

The first term on the r.h.s. is a generalised $W^{(4)}$ -candidate acting on e_μ^a . Here α is a constant depending on the normalisation of the four-index tensor d^{abcd} (see ref. 14). Note that for $N = 2$ the coefficient of the second term on the r.h.s. vanishes as it should since in this case there is no W -algebra and $W^{(3)} = W^{(4)} = 0$. Let us compare (2.16) with the corresponding commutator of the W_N -algebra. The OPE is

$$W^{(3)}(z)W^{(3)}(z') \sim \left(\frac{1}{(z - z')^2} + \frac{\partial}{z - z'} \right) \left(xW^{(4)}(z') + yT^2(z') \right) + \dots \quad (2.17)$$

where $+\dots$ indicates (Virasoro-) descendent fields of the identity (like $T, \partial T, \partial^2 T$) or the identity itself (i.e. the central term). These latter fields are present in the (quantum) OPE (cf. eq. (1.5)) but are expected to be absent in the present classical computation. x and y are normalisation-dependent constants that can be set equal to 1. Hence

$$\left[\oint dz \omega(z) W^{(3)}(z), \oint dz' \Lambda(z') W^{(3)}(z') \right] = \oint dz' \psi(z') \left(xW^{(4)}(z') + yT^2(z') \right) + \dots \quad (2.18)$$

where

$$\psi = \partial \omega \Lambda - \omega \partial \Lambda \equiv \psi^{zzz}. \quad (2.19)$$

The analogy with eq. (2.16) is striking. As already mentioned, the first term on the

r.h.s. of (2.16) is a generalised $W^{(4)}$ -transformation (with the rank-3 symmetric tensor ψ as parameter) acting on e_μ^a . What about the T^2 -term ? The second term in eq. (2.16) is almost a diffeomorphism acting on e_μ^a but with field-dependent parameter. This statement becomes exact if we concentrate on $\mu = z$. Then

$$\left[\delta_{\xi(\omega)} , \delta_{\xi(\Lambda)} \right]_{\text{chiral}} e_z^a = 8\alpha \delta_{\xi(\psi)} e_z^a - \frac{8(N^2 - 4)}{N(N^2 + 1)} \delta_{\xi(\zeta)} e_z^a \quad (2.20)$$

where the field-dependent parameter ζ of the diffeomorphism is

$$\zeta^z = e_z^b e_z^b \psi^{zzz} . \quad (2.21)$$

At the classical level, we can interpret the non-linear T^2 -term in eq. (2.18) as a diffeomorphism with parameter $T_{zz}\psi^{zzz}$. The analogy between eqs. (2.18), (2.19) and (2.20), (2.21) would be complete if we could identify $e_z^b e_z^b$, which is the metric g_{zz} of our 2D manifold inherited by the dimensional reduction from the group manifold, with its conjugate quantity, the energy-momentum tensor T_{zz} . We will not explore under which circumstances such an identification could be justified. This point touches upon the intrinsic non-linear structure of the W_N -algebras and clearly needs a better understanding. Nevertheless, starting with very simple geometric transformations, we have come amazingly close to the W_N -algebra.

In conclusion, it thus seems that the W_N -algebras have a natural geometric interpretation as remnants of the (bigger) diffeomorphism symmetry on the higher-dimensional group manifold. However, more work is required in order to turn this into a precise statement.

3. Gauging

In this and in the next section we turn to the dynamical question of W -gravity. In particular, we will discuss a field theoretic realization of the W -algebras in which all W -symmetries are gauged by introducing corresponding gauge fields. For other recent reviews on the gauging of W -symmetries, see [20, 21, 22]. The discussion of this section is based upon the work of [23, 24].

In the case of the Virasoro algebra, the gauge field of the Virasoro symmetries is the zweibein field. As is well-known, the nonlinear interactions of the zweibein field can be understood in terms of an underlying Riemannian geometry. A similar geometric understanding of the higher spin gauge fields is not complete; one approach however was discussed in the preceding section.

Our starting point is the Lagrangian for a free scalar field φ that takes its values in the Lie algebra $SU(N)$:

$$\mathcal{L} = \frac{1}{2} \text{tr}(\partial_+ \varphi \partial_- \varphi), \quad (3.1)$$

with $\partial_+ = \partial/\partial z$, $\partial_- = \partial/\partial \bar{z}$ and $z, \bar{z} = x^0 \pm ix^1$ and where x^0, x^1 are the coordinates of a two-dimensional space-time. This Lagrangian changes by a total derivative and the corresponding action is invariant under the conformal transformations $\delta\varphi = k\partial_+\varphi$, where the parameter k is a function of z only, i.e. $\partial_- k = 0$. These conformal symmetries are generated by the $(++)$ -component of the energy-momentum tensor $T = \frac{1}{2} \text{tr} \partial_+ \varphi \partial_+ \varphi$, which is a conserved spin-2 current. One can check that the spin-2 currents close among themselves under the equal $\bar{z}$ Poisson brackets $[\varphi^{ab}(z, \bar{z}), \partial_+ \varphi^{cd}(w, \bar{z})] = \delta(z-w) \delta^{ad} \delta^{bc}$. Here we have indicated the $SU(N)$ indices $a = 1, \dots, N$ explicitly. By expanding the energy-momentum tensor as a Laurent series $T = \sum_m L_m z^{-m-2}$ one recovers the Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n}. \quad (3.2)$$

Of course, the action corresponding to (3.1) has many more rigid symmetries realising different infinite-dimensional algebras. In particular one can realise the non-linear W_N algebras [1] and the linear w_∞ algebra [25]. Both kinds of algebras contain the Virasoro algebra as a subalgebra. The easiest way to see how the Virasoro symmetry of (1) can be extended to W_N or w_∞ is to consider the conserved currents which generate these symmetries. Similar to the energy-momentum tensor one can consider the general conserved spin- $(\ell+2)$ current $V^\ell = \frac{1}{\ell+2} \text{tr}(\partial_+ \varphi)^{\ell+2}$. One can check that for $SU(N)$ the currents $\{V^\ell\}$ ($\ell = 0, 1, \dots, N-1$) close among themselves to form the nonlinear W_N algebra. This is essentially the classical version of the quantum analysis of ref. 3 where the W -generators are constructed as the Casimirs of the affine $SU(N)$ current algebra. For instance, for $N = 3$ the spin-2 currents $\frac{1}{2} \text{tr}(\partial_+ \varphi)^2$ and spin-3 currents $W(z) = \frac{1}{3} \text{tr}(\partial_+ \varphi)^3$ lead to the W_3 algebra [3, 26]:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} \\ [L_m, W_n] &= (2m-n)W_{m+n} \\ [W_m, W_n] &= 2(m-n)(LL)_{m+n}, \end{aligned} \quad (3.3)$$

where we have used the notation $(LL)_m = \sum_n (L_{m-n} L_n)$ and the brackets indicate Poisson brackets. Strictly speaking, for finite N , only the currents V^ℓ with $\ell = 0, \dots, N-1$ are independent. Using special properties of $SU(N)$, the algebra of this set of currents can be shown to be closed in the enveloping algebra, to yield the W_N algebras, an

example of which is eq. (3.3). Without using these special properties of $SU(N)$, one can still obtain a closed algebra by introducing the set of currents containing all spins $\{V^\ell\}$ ($\ell = 0, 1, \dots, \infty$). This leads to the w_∞ algebra:^{*}

$$[v_m^j, v_n^\ell] = [(\ell + 1)m - (j + 1)n]v_{m+n}^{j+\ell} \quad j, \ell \geq 0. \quad (3.4)$$

The w_∞ algebra may be extended to include the spin-1 current V^{-1} as well. The resulting algebra is denoted $w_{1+\infty}$. The transformation of the scalar φ under the symmetry generated by the current V^ℓ is given by the Poisson brackets $\delta\varphi = [k_\ell V^\ell, \varphi]$, where the parameters k_ℓ are functions of z only.

So far, we have introduced the Virasoro, W_N and w_∞ algebras together with their realisation in terms of conserved currents. These currents generate symmetries of the free action corresponding to (3.1) which are characterised by a $\bar{z}$ independent parameter. The w_∞ algebra arises as a special limit as $N \rightarrow \infty$ of the W_N algebra [27]. This limiting procedure is not unique. Different limits lead to different algebras (see e.g. ref. 28).

We should note that one can consider more general W algebras containing an infinite number of higher-spin generators, which we generically denote as W_∞ -type algebras. They distinguish themselves from the w_∞ algebra in the sense that the r.h.s. of the commutation relations (3.4) also involves generators of lower spin. An interesting feature of these algebras is that in contradistinction to the w_∞ algebra they allow central extensions in the commutator of two higher-spin generators. The explicit expression of two such algebras, called W_∞ and $W_{1+\infty}$, has been discussed in the work of [29] (for a recent review see [22]). The notation indicates that the $W_{1+\infty}$ algebra contains an additional spin-1 generator.

The following results are known about the supersymmetric generalisations of the above algebras. First, as is well-known, there exists the super-Virasoro algebra which is generated by a spin-2 and spin- $\frac{3}{2}$ current. At the classical (Poisson bracket) level we are considering here, this algebra exists for arbitrary central charge, and in particular for $c = 0$. Similar results for the super- W_N algebras (for finite N) do not exist.[†] There exists a super- w_∞ algebra [31] but a realisation for it in terms of currents is not known. A superalgebra of the W_∞ -type containing $W_\infty \oplus W_{1+\infty}$ as its bosonic subalgebra has been constructed in [32]. Recently, another superalgebra of the W_∞ -type containing

^{*} Note that the $SU(N)$ -properties of φ are now irrelevant, and one could just as well take φ to be a single real scalar.

[†] For results on super W_N algebras at the quantum level with specific non-zero central extensions, see e.g. [5, 6, 30] and references therein.

$W_{1+\infty} \oplus W_{1+\infty}$ as its bosonic subalgebra has been given in [33]. In the latter work the algebra was formulated in terms of a one-parameter family of bases. The arbitrary parameter allows a systematic discussion of the different truncations of the algebra.

We would like to stress that most of what we will discuss about gauging applies to any algebra of symmetries that can be realised in terms of conserved currents. In this section however we will only discuss the gauging of the bosonic Virasoro, W_N and w_∞ algebra.* The Virasoro algebra and the w_∞ algebras can be considered as the two extreme cases $N = 2$ and $N = \infty$, respectively, of the general W_N algebras. These two extreme cases distinguish themselves from the general case in the sense that they are linear algebras. We thus have the following situation:

$$\begin{array}{lll} \text{Virasoro} & : & \text{linear} \\ W_N & : & \text{nonlinear} \\ w_\infty & : & \text{linear} \end{array} \quad (3.5)$$

Because of the linearity property, the Virasoro and the w_∞ algebras are in several respects easier to deal with than the nonlinear W_N algebras. It is therefore easier to consider the gauging of the w_∞ algebra. Moreover, it turns out that, having gauged w_∞ , one can obtain results for gauged W_N by truncation [23]. This truncation then automatically leads to the nonlinearities inherent to W_N . An important role in this truncation procedure is played by certain shift symmetries which are present in the gauged w_∞ theory. At the field theory level we are considering here, one could therefore say that the W_N algebras are contained in the direct sum of the w_∞ algebra and the shift symmetries:

$$W_N \subset w_\infty \oplus \text{shift symmetries} \quad (3.6)$$

It is not yet clear to what the above inclusion relation corresponds to at the algebraic level. To understand this one should first find out what the algebraic analogue of the shift symmetries are.

We will now discuss the gauging of w_∞ in 3 steps of increasing complexity. We will denote these steps by *chiral*, *nonchiral* and *covariant* gauging, respectively. In each step we will use the gauging of the Virasoro algebra to explain the main idea. To simplify our formulae we will from now on take $N = 1$, i.e. φ is a single real scalar.

* For a discussion of the gauging of the W_∞ -type algebras of [29, 32], see [34]. For recent results on W-supergravity, see [20, 35]

1. *Chiral Gauging.* We first gauge only one (left-moving) copy of the w_∞ algebra. Chiral gaugings of this type were first discussed in [26]. Effectively we are dealing with a one-dimensional system and the gauging is achieved by introducing a gauge field A_ℓ for every current V^ℓ and by adding to the free Lagrangian (1) terms of the form $A_\ell V^\ell$, i.e. gauge field $\times$ current terms. We thus end up with the following Lagrangian [23]

$$\mathcal{L}^{\text{chiral}} = \frac{1}{2}(\partial_+ \varphi \partial_- \varphi) - \sum_0^\infty \frac{1}{\ell+2} A_\ell (\partial_+ \varphi)^{\ell+2} \quad (3.7)$$

and transformation rules

$$\begin{aligned} \delta \varphi &= k_\ell (\partial_+ \varphi)^{\ell+1} \\ \delta A_\ell &= \partial_- k_\ell - \sum_j [(j+1) A_j \partial_+ k_{\ell-j} - (\ell-j+1) k_{\ell-j} \partial_+ A_j]. \end{aligned} \quad (3.8)$$

Note that the transformation rules of the gauge fields are of the standard form $\delta A^I \sim \bar{\partial}_+ k^I + f_{JK}^I k^J A^K$ with the structure constants f_{JK}^I given by eq. (3.4). The parameters k_ℓ are now arbitrary functions of z and $\bar{z}$, in particular $\bar{\partial}_+ k_\ell \neq 0$. This generalises the result of [26] for chiral W_3 . A similar result exists for the Virasoro algebra. In that case one introduces only one gauge field $A_0 = h_{--}^\dagger$ corresponding to the spin-2 current $(\partial_+ \varphi)^2$. This gauge field can be viewed as a particular component of the zweibein field e_μ^a . The Lagrangian and transformation rules for gauging the Virasoro algebra can be obtained from the w_∞ results given in eqs. (3.7) and (3.8) by setting $A_\ell = 0$ ($\ell = 1, 2, \dots, \infty$) everywhere. The resulting Lagrangian is the same as that of a chiral boson [36].

In addition to the local w_∞ symmetries that have been demanded by construction in (3.7), this Lagrangian also displays the local “shift symmetries” [23] referred to earlier (see eq. (3.6)). These shift symmetries are akin to the Stückelberg transformation of the extra field that can be introduced into massive Maxwell theory to make the mass term gauge invariant. In the present case, the local shift transformations, with parameters $\alpha_\ell(x^+, x^-)$, are

$$\begin{aligned} \delta A_0 &= - \sum_{\ell \geq 1} \frac{2}{\ell+2} \alpha_\ell (\partial_+ \varphi)^\ell \\ \delta A_\ell &= \alpha_\ell \quad \ell \geq 1. \end{aligned} \quad (3.9)$$

In addition, the Lagrangian (3.7) displays further local symmetries, with parameters

[‡] In the following we will use a notation where $V_+ = V_z = V^0 + iV^1$ and $V_- = V_{\bar{z}} = V^0 - iV^1$ for any vector V . In a two-dimensional Minkowski space $V_\pm$ would correspond to $V_\pm = V^0 \pm V^1$.

$\beta_\ell(x^+, x^-)$, $\gamma_\ell(x^+, x^-)$, that involve A_0 alone:

$$\delta A_0 = \frac{\ell}{\ell+2} \partial_- \beta_\ell (\partial_+ \varphi)^\ell + \beta_\ell \partial_- (\partial_+ \varphi)^\ell + \frac{\ell}{\ell+2} \partial_+ \gamma_\ell (\partial_- \varphi)^\ell + \gamma_\ell \partial_+ (\partial_- \varphi)^\ell. \quad (3.10)$$

The shift symmetries can be used to gauge away all the gauge fields beyond the lowest one by making the gauge choice

$$A_\ell = 0 \quad \ell \geq 1. \quad (3.11)$$

In order to maintain this gauge when making w_∞ transformations, the w_∞ transformations need to be combined with compensating gauge transformations. If one includes also specially chosen β and γ symmetries, one finds that in the shift symmetry gauge (3.11) the w_∞ transformations are realised as a set of *field-dependent Virasoro transformations*

$$\begin{aligned} \delta \varphi &= k \partial_+ \varphi \\ \delta A_0 &= \partial_- k + k \partial_+ A_0 - A_0 \partial_+ k, \end{aligned} \quad (3.12)$$

where $k = \sum_\ell k_\ell (\partial_+ \varphi)^\ell$.

The above “telescoping” of the w_∞ gauge fields down to a single spin-two field for the Virasoro algebra occurs only for the particularly simple example that we have been considering with a single scalar field φ . More generally [23], if one considers a matrix-valued multiplet of scalars corresponding to some Lie algebra $\mathcal{G}$, then there will still be shift symmetries, but in that case one will be able to gauge away only the A_ℓ that couple to powers of the matrix $\partial_+ \varphi$ that can be factorised. As a result, one is left with just R gauge fields A_ℓ , where R is the rank of $\mathcal{G}$. For example, picking $\mathcal{G} = su(3)$, one can gauge away all but A_0 and A_1 . Instead of just the field-dependent Virasoro transformations that we had in the simple single-scalar model above, the remaining fields in the $SU(3)$ case display the local W_3 symmetry of ref. [26]. In this way, the finite component models with local w_∞ can be viewed equally as W_N models; the non-linearities in the W_N algebra arise as a result of the included shift, β and γ symmetries. In this way, we can also understand from our field-theoretic construction the association of the W_N algebra with $su(N)$ that was mentioned in the introduction, and extensively used in section 2.

2. Nonchiral Gauging. We next consider the gauging of two copies (left-moving and right-moving) of the w_∞ algebra. To distinguish between the two we denote the

parameters and gauge fields of the left-moving (right-moving) copy by $k_{-\ell}$ and $A_{-\ell}$ ($k_{+\ell}$ and $A_{+\ell}$). It turns out that the nonchiral gauging *cannot* be achieved by adding gauge field $\times$ current terms for the left-moving and right-moving copies separately: terms of higher order in $A_{-\ell}$ and $A_{+\ell}$ occur [26]. In [37] a nice trick was found to describe the polynomial dependence of the nonchiral Lagrangian on $A_{-\ell}$ and $A_{+\ell}$ involving the definition of a set of “nested covariant derivatives”.

We first illustrate the idea of the “nested covariant derivatives” using the Virasoro algebra. The standard result for the nonchiral gauged Virasoro algebra is given in terms of a zweibein field e_μ^a

$$\mathcal{L}_1^{\text{nonchiral}} = \frac{1}{4} e \eta^{ab} E_a^\mu E_b^\nu \partial_\mu \varphi \partial_\nu \varphi, \quad (3.13)$$

where E_a^μ is the inverse zweibein. The two gauge fields $A_{-0} = h_{--}$ and $A_{+0} = h_{++}$ arise as the following particular components of e_μ^a :

$$h_{--} = e_-^+ / e_+^+ \quad \quad h_{++} = e_+^- / e_-^-. \quad (3.14)$$

Indeed, writing (3.13) out in components and using (3.14) the nonchiral Lagrangian can be written as

$$\mathcal{L}_1^{\text{nonchiral}} = \frac{1}{2} (1 - h)^{-1} (\partial_+ \varphi - h_{++} \partial_- \varphi) (\bar{\partial}_+ \varphi - h_{--} \partial_- \varphi), \quad (3.15)$$

with $h \equiv h_{++} h_{--}$. We see that indeed the gauge fields h_{++} and h_{--} do not merely occur as gauge field $\times$ current terms. Since the Lagrangian (3.13) is the standard result for describing the coupling of a scalar to gravity we will call this the “gravicovariant” basis.

It turns out that using the gravicovariant basis one cannot treat the spin-2 gauge field on the same footing as the higher-spin gauge fields: there are no higher-spin zweibein fields with a similar geometric interpretation as the spin-2 zweibein field. However, in two dimensions there exists an alternative formulation to describe the coupling of gravity to matter which includes two auxiliary fields J_+ and J_- [23]. This alternative formulation does allow a natural extension to include the higher-spin gauge fields on the same footing as the spin-2 gauge field. The Lagrangian is given by

$$\begin{aligned} \mathcal{L}_2^{\text{nonchiral}} = & -\frac{1}{2} \partial_+ \varphi \partial_- \varphi + J_+ \bar{\partial}_+ \varphi + J_- \partial_+ \varphi - J_+ J_- \\ & -\frac{1}{2} h_{--} J_+^2 - \frac{1}{2} h_{++} J_-^2. \end{aligned} \quad (3.16)$$

Since this Lagrangian corresponds to an unconventional way of coupling gravity to a scalar we will refer to this alternative formulation as the “noncovariant” basis. The

noncovariant basis is related to the gravicovariant basis as follows. The equations of motion for the auxiliary fields $J_{\pm}$ give

$$\begin{aligned} J_+ &= \partial_+ \varphi - h_{++} J_- \\ J_- &= \partial_- \varphi - h_{--} J_+. \end{aligned} \quad (3.17)$$

These equations define a set of “nested covariant derivatives” [37]. They can be solved for $J_{\pm}$ iteratively to give

$$\begin{aligned} J_+ &= (1-h)^{-1}(\partial_+ \varphi - h_{++} \partial_- \varphi) \\ J_- &= (1-h)^{-1}(\bar{\partial}_+ \varphi - h_{--} \partial_+ \varphi). \end{aligned} \quad (3.18)$$

Substituting this solution back into $\mathcal{L}_2^{\text{nonchiral}}$ one recovers the Lagrangian $\mathcal{L}_1^{\text{nonchiral}}$.

In order to relate the noncovariant basis to the gravicovariant basis it is not essential to first eliminate the auxiliary fields $J_{\pm}$. Alternatively, one can make the following redefinitions [24]

$$F_+ = \alpha(J_+ + h_{++} J_- - \partial_+ \varphi) + \beta(J_- + h_{--} J_+ - \partial_- \varphi)h_{++}, \quad (3.19)$$

where $\alpha^2 = (1+\sqrt{1-h})/(1-h)$ and $\beta^2 = (1-\sqrt{1-h})/(h(1-h))$. A similar redefinition involving F_- (with $+$ and $-$ interchanged everywhere) is also made. In terms of the new auxiliary variables $F_{\pm}$ the Lagrangian reads

$$\begin{aligned} \mathcal{L}_1^{\text{nonchiral}} &= \frac{1}{2}(1-h)^{-1}(\partial_+ \varphi - h_{++} \partial_- \varphi)(\partial_- \varphi - h_{--} \partial_+ \varphi) \\ &\quad - (F_+ - h_{++} F_-)(F_- - h_{--} F_+) \\ &= \frac{1}{4}e\eta^{ab}\{E_a^{\mu}E_b^{\nu}\partial_{\mu}\varphi\partial_{\nu}\varphi - F_a F_b\}. \end{aligned} \quad (3.20)$$

Gravity now is represented in the gravicovariant basis and the auxiliary fields $F_{\pm}$ can be solved for trivially.

We are now in a position to discuss the nonchiral gauging of w_{∞} .^{*} It is convenient to start from the nonchiral Virasoro gauge theory in the noncovariant basis (cf. (3.16)). In that basis the gauge fields h_{--} and h_{++} only occur in the Lagrangian as gauge field $\times$ current terms, as they did in the chiral Lagrangian (3.7). This makes the extension to w_{∞} straightforward. Indeed, the nonchiral gauging of w_{∞} is obtained by modifying

^{*} The corresponding nonchiral gauging of W_3 was first discussed in [37].

the Lagrangian $\mathcal{L}_2^{\text{nonchiral}}$ of the Virasoro gauge theory only in the gauge field $\times$ current terms, so as to include all gauge fields and all currents. In this formulation, all gauge fields occur on an equal footing. The result for the nonchiral gauged w_∞ theory in the noncovariant basis is therefore given by [23]

$$\begin{aligned} \mathcal{L}^{\text{nonchiral}} = & -\frac{1}{2}\partial_+\varphi\partial_-\varphi + J_+\partial_-\varphi + J_-\partial_+\varphi - J_+J_- \\ & - \sum_{\ell=0} \frac{1}{\ell+2} A_{-\ell}(J_+)^{\ell+2} - \sum_{\ell=0} \frac{1}{\ell+2} A_{+\ell}(J_-)^{\ell+2}. \end{aligned} \quad (3.21)$$

This generalises the result of [37] for nonchiral W_3 . Of course, one could also use a gravicovariant basis. In that case the nonchiral Lagrangian is given by [38]

$$\begin{aligned} \mathcal{L}^{\text{nonchiral}} = & e\eta^{ab} \left\{ -\frac{1}{4}E_a^\mu E_b^\nu \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}F_a F_b + F_a E_b^\mu \partial_\mu\varphi \right\} \\ & - e \sum_{j \geq 3} \frac{1}{j} A^{(a_1 \dots a_j)} F_{a_1} \dots F_{a_j}. \end{aligned} \quad (3.22)$$

In (3.22) we have combined the $+, -$ indices into a vector index $a : a = (+, -)$. The round brackets in (3.22) indicate that the enclosed Lorentz indices are symmetrised and made traceless. For such a representation only the extreme helicity components are nonzero. We furthermore identify $A^{(a_1 \dots a_{\ell+2})} = \{A_{+\ell}, A_{-\ell}\}$ ($\ell \geq 3$). As in the Virasoro case the two Lagrangians (3.21) and (3.22) are related to each other by field redefinitions. Note that the Lagrangian (3.16) for the nonchiral Virasoro gauge theory can be obtained from the Lagrangian (3.21) for the nonchiral w_∞ gauge theory by setting $A_{\pm\ell} = 0$ ($\ell = 1, 2, \dots, \infty$) everywhere, since for the non-chiral case we also have shift symmetries analogous to (3.9). The nonchiral transformation rules corresponding to (3.21) are given by [23]

$$\begin{aligned} \delta\varphi &= k_{-\ell}(\partial_+\varphi)^{\ell+1} + k_{+\ell}(\partial_-\varphi)^{\ell+1} \\ \delta A_{-\ell} &= \partial_- k_{-\ell} - \sum_j [(j+1)A_{-j}\partial_+ k_{-\ell-j} - (\ell-j+1)k_{-\ell-j}\partial_+ A_{-j}] \\ \delta A_{+\ell} &= \partial_+ k_{+\ell} - \sum_j [(j+1)A_{+j}\partial_- k_{+\ell-j} - (\ell-j+1)k_{+\ell-j}\partial_- A_{+j}] \\ \delta J_+ &= \partial_+(k_{-\ell}(J_+)^{\ell+1}) & \delta J_- &= \partial_-(k_{+\ell}(J_-)^{\ell+1}). \end{aligned} \quad (3.23)$$

3. Covariant Gauging. We finally consider the covariant gauging of w_∞ . To outline what one might mean by “covariantising” the nonchiral w_∞ gauge theory considered

above it is instructive to first consider the Virasoro algebra. From eq. (3.15) we see that the nonchiral gauging of the Virasoro algebra can be described entirely in terms of only two independent gauge fields h_{++} and h_{--} . The Lagrangian given in eq. (3.15) is perfectly invariant under general coordinate transformations. On the other hand the coupling of matter fields to gravity is usually described in terms of the four components of a zweibein field as given in eq. (3.13). Corresponding to these two extra gauge field components, the Lagrangian given in (3.13) has two extra gauge symmetries as compared to the one given in (3.15): Lorentz rotations and dilatations. Usually it is said that the Lagrangians (3.13) and (3.15) are gauge equivalent to each other.

To illustrate the notion of gauge equivalence, consider a toy model Lagrangian $\mathcal{L}(A)$ which depends on only one field A . From this Lagrangian one can construct a new Lagrangian $\mathcal{L}(\tilde{A}, B)$ by introducing two new fields $\tilde{A}$ and B via the field redefinition $A = \tilde{A} + B$. Since the r.h.s. of this field redefinition is invariant under the gauge transformations $\delta\tilde{A} = \alpha, \delta B = -\alpha$, the new Lagrangian $\mathcal{L}(\tilde{A}, B)$ has the same gauge invariance too. This extra gauge invariance enables one to impose the gauge $B = 0$ after which $A = \tilde{A}$ and the two Lagrangians coincide. It is in this sense that the two Lagrangians $\mathcal{L}(A)$ and $\mathcal{L}(\tilde{A}, B)$ are gauge equivalent to each other. A similar relation exists between the Lagrangians given in eqs. (3.13) and (3.15) with $A = \{\varphi, h_{++}, h_{--}\}$ and $\{\tilde{A}, B\} = \{\varphi, e_\mu^a\}$.

Although the Lagrangians (3.13) and (3.15) are classically gauge equivalent to each other, at the quantum level this is not necessarily the case. For instance, to calculate possible quantum anomalies one first needs to regularise the theory, e.g. by using a Pauli-Villars mass regularization. However, such a mass term is not conformally invariant and therefore one is forced to introduce extra gauge field components beyond h_{++} and h_{--} .

We now turn our attention to the covariant gauging of w_∞ . Analogous to the Virasoro algebra we expect that additional components for all higher-spin gauge fields should be introduced. A priori we do not know how many components should be added. However, it seems reasonable to expect that a spin- ℓ symmetry should be gauged by a vector gauge field, of the form $e_\mu^{a_1 \dots a_{\ell-1}}$ (symmetric and traceless in the indices $a_1 \dots a_{\ell-1}$), generalising the zweibein field e_μ^a . This is indeed the proposal of [38] which extends the results of [39, 21] for the covariant gauging of the nonlinear W_3 algebra[★]. This also is consistent with equation (2.9) where we argued that the parameter for a spin- ℓ symmetry should be a rank- $(\ell - 1)$ symmetric[†] tensor. Upon

★ For a more general discussion on the gauging of nonlinear algebras, see [40].

† Indeed we could also have considered these parameters traceless.

using the zweibeins this can be converted into a scalar with $\ell - 1$ Lorentz indices. The corresponding gauge field should have the same Lorentz indices together with an additional vector index. In [38] an expression for the covariant Lagrangian was derived by gauging the w_∞ algebra in a similar way as has been done for the Virasoro algebra in [41]. Here we will give an alternative derivation of the same Lagrangian [24].

Comparing with the above toy model, we start with the nonchiral Lagrangian given in eq. (3.22). Since we want a covariant gauging it is more natural to use the gravicovariant basis. The original set of fields present in the Lagrangian $\mathcal{L}^{\text{nonchiral}}(A)$ is given by $A = \{\varphi, e_\mu^a, A^{(a_1 \dots a_j)} (j \geq 3), F_a\}$. We first write $A^{a_1 \dots a_j}$ as $\eta^{a_1 b} E_b^\mu e_\mu^{a_2 \dots a_j}$. We then introduce two extra gauge field components^{*} $e_+^{+(j-1)}$ and $e_-^{-(j-1)}$, by making redefinitions for each spin $j \geq 3$ such that we have four gauge fields that combine as follows:

$$e_b^{a_1 \dots a_{j-1}} = \begin{pmatrix} e_+^{+(j-1)} & e_-^{+(j-1)} \\ e_+^{-(j-1)} & e_-^{-(j-1)} \end{pmatrix}, \quad (3.24)$$

where $e_\pm \equiv E_\pm^\mu e_\mu$. Note that in this notation, $e_b^a = \delta_b^a$. The new set of fields present in the covariant theory defined by the Lagrangian $\mathcal{L}^{\text{covariant}}(\tilde{A}, B)$ is therefore given by $\{\tilde{A}, B\} = \{\varphi, e_\mu^a, e_b^{a_1 \dots a_{j-1}} (j \geq 3), G_a\}$.

The particular redefinitions relating the set of fields A to the set of fields $\{\tilde{A}, B\}$ are given by [24]:

$$F_a = G_a + \sum_{j \geq 3} e_{(a}^{a_2 \dots a_j} G_{a_2} \dots G_{a_j})$$

$$\sum_{\ell \geq 3} A^{(a a_1 \dots a_{\ell-1})} F_{a_1} \dots F_{a_{\ell-1}} = \sum_{\ell \geq 3} e^{(a a_1 \dots a_{\ell-1})} G_{a_1} \dots G_{a_{\ell-1}}. \quad (3.25)$$

Using these redefinitions the covariant Lagrangian is then given by

$$\mathcal{L}^{\text{covariant}}(\varphi, e_\mu^a, e_b^{a_1 \dots a_{j-1}} (j \geq 3), G_a) = \mathcal{L}^{\text{nonchiral}}(\varphi, e_\mu^a, A^{(a_1 \dots a_j)} (j \geq 3), F_a). \quad (3.26)$$

The r.h.s of the redefinitions (3.25) is invariant under the following generalised Lorentz and Weyl transformations [39,21]

$$\delta e_\mu^{a_1 \dots a_{j-1}} = - \sum_{\ell=2}^j (j - \ell + 1) \lambda^{(a_1 \dots a_{\ell-2}} e_\mu^{a_{\ell-1} \dots a_{j-1})}$$

^{*} We use here the notation where $e_+^{\pm(j-1)}$ is a field with one lower index $+$ and $j - 1$ upper indices $\pm$, etc.

$$\delta G_a = \sum_{j \geq 2} \lambda_{(a}^{a_1 \dots a_{j-1}} G_{a_1} \dots G_{a_{j-1}}). \quad (3.27)$$

By construction the covariant Lagrangian given in (3.26) is invariant under the same transformations. For each spin we have two extra gauge transformations. These gauge transformations can be fixed by setting two gauge field components for each spin equal to zero. The covariant Lagrangian then reduces to the nonchiral Lagrangian. We thus see that the covariant and nonchiral Lagrangians are gauge equivalent to each other.

In conclusion, we should stress that the process of adding extra components to the gauge fields in a W -gravity theory does not change the theory at the classical level. It is only at the quantum level that the covariant and nonchiral formulations of the theory can become inequivalent. For instance, one could try to calculate possible generalised Lorentz and conformal anomalies and see which extra gauge field components are needed in order to regularise the theory[†]. Without doing such quantum calculations it is not clear which formulation of the theory is preferable. Experience with string theory has taught us that it is useful to introduce Lorentz and Weyl invariance as additional symmetries at the classical level. It would be interesting to see whether the higher-spin Lorentz and Weyl symmetries given in eq. (3.27) will play a similar rôle in the quantum theory of W -gravity.

4. Linear versus Nonlinear realisations

The gauged realisation of w_∞ described in section 3 is a nonlinear realisation of this symmetry in terms of a single scalar field. We now shall discuss how this is related to linear realisations of the algebra. This discussion is based upon the work of ref. [23].

As in the previous sections, our considerations here will be purely classical, in the sense that we are looking for realisations of the commutation relations (3.4), where the central term c has been set to zero. Moreover, as we already saw in section 3, the w_∞ algebra may be extended to include a spin-1 generator; this is effected in the commutation relations (3.4) simply by letting the j and ℓ indices run from -1 to ∞ . The resulting algebra is denoted $w_{1+\infty}$; it may also be obtained by contraction from the $W_{1+\infty}$ algebra of [29].

Both w_∞ and $w_{1+\infty}$ can be linearly realised in terms of functions defined on a two-dimensional cylinder (w, y) . The algebra (3.4) may be realised in this case as the

[†] For a recent discussion on anomalies in W -gravity, see [42].

algebra of area-preserving diffeomorphisms on the cylinder [29]. An appropriate choice of basis functions for this realisation is

$$v_m^\ell = -ie^{imw}y^{\ell+1}, \quad (4.1)$$

which form a complete set of non-singular functions on $S^1 \times R$ for $\ell \geq -1$, the full range corresponding to $w_{1+\infty}$. In this realisation, the commutator of two elements of the algebra generated by the functions $f(w, y)$ and $g(w, y)$ is represented by the Poisson bracket

$$\{f, g\} = \partial_w f \partial_y g - \partial_y f \partial_w g. \quad (4.2)$$

Alternatively, one can represent the algebra in terms of Hamiltonian vector fields defined by

$$\hat{v}_m^\ell = \{v_m^\ell, \bullet\} \quad (4.3)$$

where any function on which the vector field operates is to be placed at the location of the bullet $\bullet$. For the basis (4.1), these vector fields are given explicitly by

$$\hat{v}_m^\ell = e^{imw}(my^{\ell+1}\partial_y + i(\ell+1)y^\ell\partial_w). \quad (4.3)$$

When written in terms of the hamiltonian vector fields, the commutators of w -transformations are given by ordinary commutators of the vector field differential operators.

The $w_{1+\infty}$ algebra differs from w_∞ only in the inclusion of the extra basis functions

$$v_m^{-1} = -ie^{imw}, \quad (4.5)$$

corresponding to particular area-preserving diffeomorphisms on the cylinder that are generated by y -independent functions $v(w) = \sum_m c_m v_m^{-1}$. The corresponding hamiltonian vector fields are

$$\hat{v}_m^{-1} = me^{imw}\partial_y. \quad (4.6)$$

As we have seen in section 3, the “global” transformations corresponding to a left-handed chiral w_∞ symmetry realised on a single scalar field $\varphi(x^+)$ are given by

$$\delta\varphi = k_\ell(\partial_+\varphi)^{\ell+1}, \quad (4.7)$$

where $k(x^+)$ may be considered to be a “global” parameter since it does not depend

on the “time” x^- . The extra transformations appearing in $w_{1+\infty}$ are simply *shifts*

$$\delta\varphi = k_{-1}(x^+). \quad (4.8)$$

These non-linear transformations can be related to the linear realisation given by the area-preserving diffeomorphisms on the cylinder only through a construction in which a quotient is taken between two very large algebras, for the above linear realisation expanded in terms of functions of one variable would comprise an infinite-component “multiplet”. In order to reduce this to a single $\varphi(x^+)$, the “denominator” group in the non-linear construction must itself have a linear realisation in terms of an infinite set of fields over a one-dimensional space. This in turn implies that the non-linear realisation will have some unfamiliar features, in particular that the coset involved does not form a linear realisation of the denominator group.

Since the only two relevant infinite-dimensional algebras that have linear realisations on infinite sets of fields over one dimension are $w_{1+\infty}$ and w_∞ , it is natural for these purposes to explore the quotient $w_{1+\infty}/w_\infty$. In order to construct the explicit non-linear transformations corresponding to this coset, it is necessary first to learn how to express a finite w -transformation. In the following, we shall view the transformations *actively*. In the Poisson bracket language, the infinitesimal transformation of a function $f(w, y)$ on the cylinder is given by

$$f(w, y) \rightarrow \tilde{f}(w, y) = f + \{\Lambda, f\}, \quad (4.9)$$

where $\Lambda(w, y)$ is the parameter of the transformation. For Λ non-infinitesimal, one has

$$\tilde{f} = e^{\text{Ad}_\Lambda} f \equiv f + \{\Lambda, f\} + \frac{1}{2!}\{\Lambda, \{\Lambda, f\}\} + \dots \quad (4.10)$$

Indeed, one may check that if one rewrites this transformation as an Einstein-style transformation for a scalar $f(w, y) \rightarrow \tilde{f}(\tilde{w}, \tilde{y}) = f(w, y)$, then the area-preserving condition is satisfied:

$$\det \left(\frac{\partial(\tilde{w}, \tilde{y})}{\partial(w, y)} \right) = 1. \quad (4.11)$$

In terms of the hamiltonian vector fields, one may write the above finite transformations using the differential generator

$$\vec{T} = \left(\overleftarrow{\partial}_w \overrightarrow{\partial}_y - \overleftarrow{\partial}_y \overrightarrow{\partial}_w \right), \quad (4.12)$$

in terms of which the transformation of a function $f(y, z)$ is given by

$$\tilde{f}(w, y) = \exp \left(\Lambda \cdot \overleftrightarrow{T} \right) f. \quad (4.13)$$

We are now ready to write down the non-linear realisation of the w -transformations. We can parametrise the $w_{1+\infty}/w_\infty$ coset by the “Goldstone field” $\varphi(x^+, x^-)$, where x^+ is a circular coordinate, and can thus be identified with the w coordinate of the linear realisation, while x^- is a “time” coordinate that does not bear any special relation to the left-handed w symmetry that we are discussing. A coset element can now be written

$$\exp \left(\varphi \cdot \overleftrightarrow{T} \right) = \exp \left(\varphi' \partial_y \right), \quad (4.14)$$

where $\varphi' \equiv \partial\varphi/\partial x^+$. The nonlinear realisation on the $w_{1+\infty}/w_\infty$ coset is now given for a finite $w_{1+\infty}$ transformation g by $g : \varphi \rightarrow \tilde{\varphi}$, where

$$g e^{\varphi' \partial_y} = e^{\tilde{\varphi}' \partial_y} \tilde{h}, \quad (4.15)$$

in which $\tilde{h}$ is an element of the “denominator” group, i.e. it is a finite w_∞ transformation.

The “non-linearity” of the above $\varphi \rightarrow \tilde{\varphi}$ transformation is of a somewhat unfamiliar nature. For transformations g belonging to the coset $w_{1+\infty}/w_\infty$, i.e. $g = e^{\Lambda'(x^+) \partial_y}$, eq. (4.15) produces a simple shift $\varphi(x^+) \rightarrow \tilde{\varphi} = \varphi + \Lambda(x^+)$. This is inhomogeneous and hence non-linear, as one would expect for a coset transformation; indeed, we have just reproduced the $w_{1+\infty}$ transformation (4.8). A more surprising nonlinearity appears in the transformations corresponding to the w_∞ denominator. Consider an infinitesimal transformation $g = 1 + \hat{\lambda}$ and make the standard rearrangement of (4.15) into

$$e^{-\varphi' \partial_y} \hat{\lambda} e^{\varphi' \partial_y} = \delta\varphi' + \tilde{\hat{\lambda}}, \quad (4.16)$$

where $\delta\varphi' = \tilde{\varphi}' - \varphi'$. The projection into the coset necessary for working out $\delta\varphi'$ is effected by setting $y \rightarrow 0$. Then, for $\lambda = k_\ell(x^+) y^{\ell+1}$, one finds that $\delta\varphi' = (k_\ell(\varphi')^{\ell+1})'$, and hence

$$\delta\varphi = k_\ell(\partial_+ \varphi)^{\ell+1}, \quad (4.17)$$

which is the correct w_∞ transformation of φ agreeing with (4.7). The unfamiliar situation of having the transformations of the *denominator* group realised nonlinearly arises

because the coset $w_{1+\infty}/w_\infty$ cannot fill out a linear representation of the denominator group w_∞ . Being a single scalar function of x^+ (and remembering that x^- is irrelevant to this realisation), φ is not enough to transform linearly under w_∞ , which would require an infinite number of functions of x^+ .

The unusual aspects of the $w_{1+\infty}/w_\infty$ construction are reminiscent of the realisation of conformal symmetry on a single scalar field in four dimensions discussed long ago by Mack and Salam [43]. In four dimensions, one can realise the conformal group $SO(4,2)$ on the coset space constructed from $SO(4,2)$ divided by the Poincaré group times the proper conformal boosts. In that case, the realisation on a single scalar could be understood another way, by a more conventional coset space construction of $SO(4,2)$ divided by the Lorentz group. This produces a five-dimensional coset, but four of the five Goldstone fields parametrising the coset may be eliminated by covariant constraints, in what has been termed the “inverse Higgs mechanism” [44]. In the present w_∞ realisation, a similar construction may also be made, dividing first by the Virasoro algebra and then imposing covariant constraints to eliminate excess Goldstone fields [45].

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