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Anomalies and Effective Actions in Two-dimensional Superspace

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Abstract

We derive a chirally split expression for the superdiffeomorphism anomaly in the two-dimensional superplane. The effective action from which this anomaly was obtained is related to the WZ-action associated with the superdiffeomorphism group. Furthermore, we show that the anomalous Ward identities generated by supercoordinate transformations encompass the well-known OPE's for the stress-energy supertensor. In conclusion, the extension to generic super Riemann surfaces is discussed.

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1 Introduction

As is well-known, the Polyakov action for the bosonic string is invariant under Weyl rescalings of the metric and under diffeomorphisms, i.e. general coordinate transformations on the string world-sheet. At the quantum level, not all of these invariances survive and one has a Weyl or a diffeomorphism anomaly (the latter being equivalent to the former in two dimensions [1]). If the two-dimensional metric is parametrized in terms of Beltrami coefficients with respect to some complex coordinates $(z, \bar{z})$ [2], i.e.

$$ds^2 \propto |dz + d\bar{z} \mu(z, \bar{z})|^2, \quad (1)$$

then the *diffeomorphism anomaly* can be obtained in a factorized (chirally split) form [3]:

$$\int_{\mathbb{C}} dz d\bar{z} \mathcal{A}(c; \mu) + \text{c.c.} = \int_{\mathbb{C}} dz d\bar{z} c \partial_z^3 \mu + \text{c.c.} \quad (2)$$

Here, $c = \xi + \mu \bar{\xi}$ parametrizes infinitesimal diffeomorphisms generated by the ghost vector field $\xi(z, \bar{z})\partial_z + \bar{\xi}(z, \bar{z})\partial_{\bar{z}}$ [4]. It should be noted that the two terms in eq.(2) separately satisfy the Wess-Zumino (WZ) consistency condition [5]. The expression (2) is well defined on the complex plane $\mathbb{C}$ and has to be modified on a compact Riemann surface of genus $g \geq 1$ by the inclusion of a projective connection [6]. The local counterterm to the effective action which allows one to shift the Weyl anomaly to the factorized diffeomorphism anomaly on a Riemann surface has been determined quite recently [7]. This result explicitly proves the equivalence of the different anomalies and represents the starting point for a study of holomorphic factorization of partition functions viewed as functionals of Beltrami differentials $\mu, \bar{\mu}$ [7] [8].

The group of Weyl rescalings is Abelian and thereby the WZ-action [5] associated with the Weyl anomaly can be evaluated in a straightforward way, the result being well known. In contrast, general coordinate transformations define a non-commutative group and therefore the construction of the associated WZ-action represents a serious problem (cf. Yang-Mills theories in four dimensions). For the factorized anomaly on the complex plane, i.e. eq.(2), this problem has recently been solved by S.Lazzarini and R.Stora [9], the resulting expression having again a factorized form. In an important special case, their result takes a remarkably simple form and coincides with the effective action (plus its complex conjugate) which was found by Polyakov in the so-called chiral gauge [10]. In section 2, we summarize the corresponding results of reference [9] and we show that the effective action can be given the form of a free scalar field action in two dimensions. Thereby, it can immediately be generalized to $N = 1$ supersymmetry and by considering its BRS variation, we can obtain the superdiffeomorphism anomaly in the superplane in a factorized form (section 3). So far, the latter has only been known in some special cases [11] [12].

The supercomplex structures on $N=1$ super Riemann surfaces (SRS's) are parametrized by odd superfields $H_{\bar{\theta}}^z, H_{\theta}^{\bar{z}}$ (and the complex conjugate fields), see [12] and references therein¹⁾. While $H_{\bar{\theta}}^z$ contains the ordinary Beltrami coefficient and its fermionic partner, the superfield $H_{\theta}^{\bar{z}}$ contains only auxiliary space-time fields. Thus, it seems natural to restrict the superspace geometry by $H_{\theta}^z = 0$ and c.c. (although it is not clear whether this

1) We note that these superfields have first been introduced in a supergravity framework [13]; although the holomorphic factorization of the transformation laws was not realized in that formulation, it has been quite useful in the sequel [13] [14] [15] [11].

leads to a field-independent subgroup of the superdiffeomorphism group [12]). In section 4, we consider this restriction of the geometry and integrate the superspace Ward identities describing the supercoordinate invariance in our local field theory approach. This procedure yields a functional equation which is formally identical to the operator product expansion (OPE) for the stress supertensor that is usually considered as the starting point of superconformal field theory [16]. Actually, this derivation of the OPE provides some extra information, because it shows how the distributions occurring therein can be given a well-defined sense in a way that is compatible with the underlying local field theory.

Comments on the extension of the results to higher genus SRS's and to (p, q) -supersymmetry are gathered in the concluding sections.

2 Bosonic theory

In this section, we briefly summarize some of the results of reference [9] and we cast the effective action into a form which makes its supersymmetric generalization immediate.

Let $(z, \bar{z})$ denote a reference system of holomorphic coordinates on a Riemann surface. Then, the holomorphic coordinates $(Z, \bar{Z})$ corresponding to the complex structure parametrized by the Beltrami coefficient μ are given by the relation

$$dZ = \lambda [dz + \mu d\bar{z}] \quad \text{and c.c.} \quad (3)$$

Here, λ and μ are smooth complex-valued functions of $z, \bar{z}$ which satisfy

$$\begin{aligned} (\bar{\partial} - \mu \partial) Z &= 0 & \text{and c.c.} \\ (\bar{\partial} - \mu \partial) \lambda &= (\partial \mu) \lambda & \text{and c.c.} \end{aligned} \quad (4)$$

with $\partial \equiv \partial_z$ and $\bar{\partial} \equiv \partial_{\bar{z}}$. Locally, Z and $\bar{Z}$ can be given by smooth invertible functions $(z, \bar{z}) \rightarrow (Z(z, \bar{z}), \bar{Z}(z, \bar{z}))$ and $\lambda = \partial Z$, $\mu = (\bar{\partial} Z)/(\partial Z)$.

2.1 Effective actions on the complex plane

The *WZ-action* associated with the diffeomorphism anomaly (2) on the complex plane depends on the “gauge field” μ and the “Goldstone field” φ which takes its values in the group of diffeomorphisms (that are homotopic to the identity) [9] :

$$\Gamma_{WZ}[\varphi; \mu] + \text{c.c.} = \frac{1}{2} \int_{\mathbf{C}} d^2 z \, \mu^\varphi \partial^2 \ln \partial Z^\varphi - \frac{1}{2} \int_{\mathbf{C}} d^2 z \, \mu \partial^2 \ln \partial Z + \text{c.c.} \quad (5)$$

Here, μ^φ and Z^φ denote the fields obtained from μ and Z by the action of the diffeomorphism φ . By construction,

$$s\Gamma_{WZ}[\varphi; \mu] = s \left[-\frac{1}{2} \int_{\mathbf{C}} d^2 z \, \mu \partial^2 \ln \partial Z \right] = \int_{\mathbf{C}} d^2 z \, \mathcal{A}(c; \mu) \quad \text{and c.c.} \quad , \quad (6)$$

where the BRS-operation s is defined by

$$\begin{aligned} sZ &= \lambda c \\ s\lambda &= \partial(\lambda c) \\ s\mu &= (\bar{\partial} - \mu \partial + \partial \mu) c \\ sc &= c \partial c \\ s(\mu^\varphi) &= 0 \end{aligned} \quad (7)$$

(and the complex conjugate expressions). The s -invariance of μ^φ implies $s(Z^\varphi) = 0$ and $s\varphi = (\xi\partial + \bar{\xi}\bar{\partial})\varphi$ [9].

If φ is restricted by the condition $\mu^\varphi = 0$, the functional (5) becomes the so-called *Wess-Zumino-Polyakov action* [9] :

$$\begin{aligned} -2\Gamma_{WZP}[\mu] &= \int_{\mathbf{C}} d^2z \mu \partial^2 \ln \lambda = \int_{\mathbf{C}} d^2z \frac{\bar{\partial}Z}{\partial Z} \partial^2 \ln \partial Z \\ &= - \int_{\mathbf{C}} d^2z \left[\frac{(\partial^2 Z)(\partial\bar{\partial}Z)}{(\partial Z)^2} - \frac{(\partial^2 Z)^2(\bar{\partial}Z)}{(\partial Z)^3} \right] . \end{aligned} \quad (8)$$

Since λ represents a non-local function of μ by virtue of eq.(4), the expression (8) also has this property. As noted above, it satisfies the anomalous Ward identity $s\Gamma_{WZP}[\mu] = \int_{\mathbf{C}} d^2z \mathcal{A}(c; \mu)$. Moreover, its functional derivative with respect to μ yields the Schwarzian derivative of Z [15] [9] :

$$\frac{\delta\Gamma_{WZP}[\mu]}{\delta\mu(z, \bar{z})} = -S(Z; z) \quad (9)$$

with

$$S(Z; z) \equiv \left[\partial^2 \ln \partial Z - \frac{1}{2} (\partial \ln \partial Z)^2 \right] (z, \bar{z}) . \quad (10)$$

The expression (8) has first been found by Polyakov [10] in a very special case, namely the chiral gauge in the pseudo-Riemannian framework (corresponding to $\bar{\mu} = 0$ and $\bar{\lambda} = 1 = \lambda$).

In order to generalize the WZ-action to the supersymmetric case, we first rewrite Γ_{WZP} in a slightly different way. For this purpose, we disregard complex conjugation for a moment and introduce ‘intermediate’ or ‘tilde’ coordinates [12] :

$$(z, \bar{z}) \longrightarrow (\tilde{z}, \tilde{\bar{z}}) \equiv (Z, \bar{z}) \longrightarrow (Z, \bar{Z}) .$$

(By construction, the tilde coordinates are *not* related by complex conjugation, $\tilde{\bar{z}} \neq \bar{\tilde{z}}$.) Then,

$$d\tilde{z} = \lambda(dz + \mu d\bar{z}) , \quad d\tilde{\bar{z}} = d\bar{z}$$

and

$$\tilde{\partial} = \lambda^{-1} \partial , \quad \tilde{\bar{\partial}} = \bar{\partial} - \mu \partial .$$

Thus, the differential equation (4) for the integrating factor λ reads $\tilde{\bar{\partial}} \ln \lambda = \partial \mu$ and thereby Γ_{WZP} becomes

$$\begin{aligned} -2\Gamma_{WZP}[\mu] &= \int_{\mathbf{C}} d^2z \mu \partial^2 \ln \lambda = - \int_{\mathbf{C}} d^2z (\partial \mu) (\partial \ln \lambda) \\ &= - \int_{\mathbf{C}} d^2\tilde{z} (\partial \mu) (\tilde{\partial} \ln \lambda) = - \int_{\mathbf{C}} d^2\tilde{z} (\tilde{\bar{\partial}} \ln \lambda) (\tilde{\partial} \ln \lambda) . \end{aligned} \quad (11)$$

This expression has the form of a free scalar field action for $\ln \lambda$. Such a functional can easily be generalized to the (1,1) supersymmetric theory for which all necessary ingredients are known. The WZ-action is then obtained by subtracting the same functional depending on the φ -transformed variables (see eq.(5)).

3 Supersymmetric theory

Let us first recall the basic facts which are needed in the sequel [12]. A SRS is locally parametrized by coordinates $(Z, \bar{Z}, \Theta, \bar{\Theta})$. The super 1-forms $e^Z \equiv dZ + \Theta d\bar{\Theta}$, $e^\Theta \equiv d\Theta$ (and c.c.) which span its cotangent space can be expressed with respect to a reference system $e^z \equiv dz + \theta d\bar{\theta}$, $e^\theta \equiv d\theta$ (and c.c.) by

$$\begin{aligned} e^Z &= [e^z + e^{\bar{z}} H_{\bar{z}}^z + e^\theta H_\theta^z + e^{\bar{\theta}} H_{\bar{\theta}}^z] \Lambda \\ e^\Theta &= [e^z + e^{\bar{z}} H_{\bar{z}}^z + e^\theta H_\theta^z + e^{\bar{\theta}} H_{\bar{\theta}}^z] \tau + [e^\theta H_\theta^\theta + e^{\bar{z}} H_{\bar{z}}^\theta + e^{\bar{\theta}} H_{\bar{\theta}}^\theta] \sqrt{\Lambda} \end{aligned} \quad (12)$$

(and the complex conjugate expressions). The structure equations $de^Z + e^\Theta e^\Theta = 0 = de^\Theta$ (and c.c.) imply that the odd superfields $H_{\bar{\theta}}^z, H_\theta^z$ (and c.c.) are the only independent Beltrami coefficients and that all the others depend on them and their derivatives :

$$\begin{aligned} (H_\theta^\theta)^2 &= 1 - (D_\theta - H_\theta^z \partial) H_\theta^z \\ H_\theta^\theta &= -\frac{1}{2H_\theta^\theta} [(D_{\bar{\theta}} - H_{\bar{\theta}}^z \partial) H_\theta^z + (D_\theta - H_\theta^z \partial) H_{\bar{\theta}}^z] \\ H_{\bar{z}}^z &= (D_{\bar{\theta}} - H_{\bar{\theta}}^z \partial) H_{\bar{\theta}}^z + (H_{\bar{\theta}}^\theta)^2 \\ H_{\bar{z}}^\theta &= (D_{\bar{\theta}} - H_{\bar{\theta}}^z \partial) H_{\bar{\theta}}^\theta + \frac{1}{2} H_{\bar{\theta}}^\theta \partial H_{\bar{\theta}}^z \end{aligned} \quad (13)$$

Furthermore, they imply that the integrating factor τ depends on the independent factor Λ through the relation

$$\tau = \frac{1}{(H_\theta^\theta)^2} (D - H_\theta^z \partial) (H_\theta^\theta \sqrt{\Lambda})$$

and that Λ satisfies the differential equation

$$\tilde{D} \ln \Lambda = \partial H_\theta^z - \frac{H_\theta^\theta}{H_\theta^\theta} \partial H_\theta^z, \quad (14)$$

where the linear differential operator $\tilde{D}$ is explicitly given in eq.(21) below. Locally, the coefficients $H_{\bar{\theta}}^z, H_\theta^z$ and the factors Λ, τ take the form

$$H_{\bar{\theta}}^z = \frac{\bar{D}Z - \Theta \bar{D}\Theta}{\partial Z + \Theta \partial \bar{\Theta}}, \quad H_\theta^z = \frac{DZ - \Theta D\Theta}{\partial Z + \Theta \partial \bar{\Theta}} \quad (15)$$

and

$$\Lambda = \partial Z + \Theta \partial \bar{\Theta}, \quad \tau = \partial \bar{\Theta}. \quad (16)$$

The superfield $H_{\bar{\theta}}^z$ admits a component field expansion

$$H_{\bar{\theta}}^z = \sigma_{\bar{\theta}}^z + \theta v^z + \bar{\theta} \mu_{\bar{z}}^z + \theta \bar{\theta} [-i\alpha_{\bar{z}}^\theta], \quad (17)$$

where $\mu_{\bar{z}}^z$ and $\alpha_{\bar{z}}^\theta$ denote the Beltrami coefficient and its fermionic partner, respectively.

In conclusion of our brief summary, we note that the superspace BRS-operation s is defined by

$$\begin{aligned} sH_{\bar{\theta}}^z &= (D_{\bar{\theta}} - H_{\bar{\theta}}^z \partial) C^z + (\partial H_{\bar{\theta}}^z) C^z - 2H_{\bar{\theta}}^\theta C^\theta \\ sH_\theta^z &= (D_\theta - H_\theta^z \partial) C^z + (\partial H_\theta^z) C^z - 2H_\theta^\theta C^\theta \\ s\Lambda &= \partial(C^z \Lambda) + 2C^\theta \tau \sqrt{\Lambda} \\ s\tau &= \partial(C^z \tau + C^\theta \sqrt{\Lambda}) \\ sC^z &= -[C^z \partial C^z + C^\theta C^\theta] \\ sC^\theta &= -\left[C^z \partial C^\theta + \frac{1}{2} C^\theta (\partial C^z)\right], \end{aligned} \quad (18)$$

where

$$\begin{aligned} C^z &= \Xi^z + \Xi^{\bar{z}} H_{\bar{z}}^z + \Xi^\theta H_\theta^z + \Xi^{\bar{\theta}} H_{\bar{\theta}}^z \\ C^\theta &= \Xi^\theta H_\theta^\theta + \Xi^{\bar{z}} H_{\bar{z}}^\theta + \Xi^{\bar{\theta}} H_{\bar{\theta}}^\theta \end{aligned} \quad (19)$$

parametrize superdiffeomorphisms generated by the ghost vector field

$$\Xi \cdot \partial \equiv \Xi^z(z, \bar{z}, \theta, \bar{\theta}) \partial_z + \Xi^{\bar{z}}(z, \bar{z}, \theta, \bar{\theta}) \partial_{\bar{z}} + \Xi^\theta(z, \bar{z}, \theta, \bar{\theta}) D_\theta + \Xi^{\bar{\theta}}(z, \bar{z}, \theta, \bar{\theta}) D_{\bar{\theta}} \quad .$$

3.1 Anomaly and effective actions on the superplane

The non-local effective action (11) can immediately be generalized to the superplane **SC** by considering the tilde supercoordinates [12] :

$$(z, \bar{z}, \theta, \bar{\theta}) \longrightarrow (\tilde{z}, \tilde{\bar{z}}, \tilde{\theta}, \tilde{\bar{\theta}}) \equiv (Z, \bar{Z}, \Theta, \bar{\Theta}) \longrightarrow (Z, \bar{Z}, \Theta, \bar{\Theta}) \quad .$$

This generalization depends on the super Beltrami coefficients $H_\theta^z, H_{\bar{\theta}}^{\bar{z}}$ (and c.c.) which parametrize $N = 1$ supercomplex structures :

$$\Gamma_{WZP}[H_\theta^z, H_{\bar{\theta}}^{\bar{z}}] + \text{c.c.} = \int_{\text{SC}} d\tilde{z} d\tilde{\bar{z}} d\tilde{\theta} d\tilde{\bar{\theta}} (\tilde{D} \ln \Lambda) (\tilde{\bar{D}} \ln \Lambda) + \text{c.c.} \quad . \quad (20)$$

Here, $\tilde{D} = \partial_{\tilde{\theta}} + \tilde{\theta} \partial_{\tilde{z}}, \tilde{\bar{D}} = \partial_{\tilde{\bar{\theta}}} + \tilde{\bar{\theta}} \partial_{\tilde{\bar{z}}}$ and the tilde-derivatives have the explicit form

$$\begin{aligned} \tilde{\partial} &= \frac{1}{\Lambda} \left[\partial - \frac{\tau}{\sqrt{\Lambda} H_\theta^\theta} (D - H_\theta^z \partial) \right] \\ \tilde{\bar{\partial}} &= (\bar{\partial} - H_{\bar{z}}^{\bar{\theta}} \partial) - \frac{H_{\bar{z}}^\theta}{H_\theta^\theta} (D - H_\theta^z \partial) \\ \tilde{D} &= \frac{1}{\sqrt{\Lambda} H_\theta^\theta} (D - H_\theta^z \partial) \\ \tilde{\bar{D}} &= (\bar{D} - H_{\bar{\theta}}^{\bar{z}} \partial) - \frac{H_{\bar{\theta}}^\theta}{H_\theta^\theta} (D - H_\theta^z \partial) \quad , \end{aligned} \quad (21)$$

which implies

$$\text{sdet} \frac{\partial(\tilde{z}, \tilde{\bar{z}}, \tilde{\theta}, \tilde{\bar{\theta}})}{\partial(z, \bar{z}, \theta, \bar{\theta})} = \frac{\sqrt{\Lambda}}{H_\theta^\theta} \quad . \quad (22)$$

The integrating factor Λ is locally given by eq.(16) and satisfies the differential equation (14).

By substituting eqs.(14),(21) and (22) into the functional (20), we get

$$\Gamma_{WZP}[H_\theta^z, H_{\bar{\theta}}^{\bar{z}}] = - \int_{\text{SC}} d^4 z \frac{1}{(H_\theta^\theta)^2} \left(\partial H_\theta^z - \frac{H_{\bar{\theta}}^\theta}{H_\theta^\theta} \partial H_{\bar{\theta}}^{\bar{z}} \right) (D - H_\theta^z \partial) \ln \Lambda \quad . \quad (23)$$

In section 4.2, we will see that, for $H_\theta^z = 0$, this quantity reduces to the expression which was previously obtained in the superchiral gauge of the metric approach [15].

In appendix A, we show that application of the BRS-operation s to (23) yields a local functional :

$$s\Gamma_{WZP}[H_\theta^z, H_{\bar{\theta}}^{\bar{z}}] = -2 \int_{\text{SC}} d^4 z \frac{1}{(H_\theta^\theta)^2} \left(\partial C^z - C^\theta \frac{\partial H_\theta^z}{H_\theta^\theta} \right) (D - H_\theta^z \partial) \left(\partial H_{\bar{\theta}}^{\bar{z}} - \frac{H_{\bar{\theta}}^\theta}{H_\theta^\theta} \partial H_\theta^z \right) . \quad (24)$$

Since the s -operator is nilpotent, the r.h.s. of eq.(24) satisfies the WZ-consistency condition; this expression (plus its c.c.) represents the *factorized form of the superdiffeomorphism anomaly on the $N=1$ superplane*. Taking into account the component field expansion (17), we see that it contains the bosonic theory expression (2). The generalization of the anomaly to generic SRS's will be discussed in section 5.

The WZ-action $\Gamma_{WZ}[\varphi; H_{\bar{\theta}}^z, H_{\theta}^z] + \text{c.c.}$ is obtained from (23) by subtracting the same expression with Z, Θ and $H_{\bar{\theta}}^z, H_{\theta}^z$ replaced by $Z^\varphi, \Theta^\varphi$ and $(H_{\bar{\theta}}^z)^\varphi, (H_{\theta}^z)^\varphi$. Here, the Goldstone field φ takes its values in the superdiffeomorphism group and it transforms like a scalar, i.e. $s\varphi = \Xi \cdot \partial \varphi$.

As usual, the expression for the (integrated) anomaly is only determined up to s -exact terms. Remarkably enough, it is possible to obtain an equivalent expression which only depends on C^θ (and not on C^z) by adding some local counterterms to the effective action. In fact, by substituting Λ by $\Lambda(H_\theta^\theta)^2$ in the functional (20), we find

$$\begin{aligned} \Gamma'_{WZP}[H_{\bar{\theta}}^z, H_{\theta}^z] &= \Gamma_{WZP}[H_{\bar{\theta}}^z, H_{\theta}^z] + 2 \int_{\text{SC}} d^4 \tilde{z} (\tilde{D} \ln (H_\theta^\theta)^2) (\tilde{D} \ln \Lambda) \\ &+ \int_{\text{SC}} d^4 \tilde{z} (\tilde{D} \ln (H_\theta^\theta)^2) (\tilde{D} \ln (H_\theta^\theta)^2) \end{aligned} \quad (25)$$

and

$$s\Gamma'_{WZP}[H_{\bar{\theta}}^z, H_{\theta}^z] = -4 \int_{\text{SC}} d^4 z \frac{C^\theta}{H_\theta^\theta} \partial \tilde{D} \ln [\Lambda (H_\theta^\theta)^2] \quad . \quad (26)$$

Note that, by virtue of eq.(14), the expression $\tilde{D} \ln \Lambda$ does not depend on the (non-local) integrating factor Λ , but only on the Beltrami coefficients and their derivatives. The derivation of equation (26) follows the lines of appendix A and makes repeated use of the relation $\tau = \frac{1}{2} \Lambda \tilde{D} \ln [\Lambda (H_\theta^\theta)^2]$. For $H_\theta^z = 0$, the result (26) reduces to the same expression as (24), since $H_\theta^\theta = 1$ (and $C^\theta = \frac{1}{2} DC^z$) in this case. Thereby, the expression (26) also contains the bosonic theory anomaly. However, it cannot be expected to be generalizable to generic SRS's, because the superconformal transformation of C^θ depends on C^z (see reference [12] and section 5).

An obvious question to ask is whether or not the variation of the WZP-action with respect to $H_{\bar{\theta}}^z$ yields the (super) Schwarzian derivative in analogy to the $N = 0$ result, eq.(9). Clearly, this can only be true for a particular choice of the local functional which may be freely added to Γ_{WZP} . Our choices (23) and (25) do not have this property²⁾. However, we will show in section 4.1 that the variation with respect to $H_{\bar{\theta}}^z$ reproduces the super Schwarzian derivative in the special case where $H_\theta^z = 0$.

4 Supersymmetry with $H_\theta^z = 0$

In this chapter, we concentrate on the restriction of the geometry where $H_\theta^z = 0$ (and c.c.).

For $H_\theta^z = 0$ and

$$0 = sH_\theta^z = D_\theta C^z - 2C^\theta \quad , \quad (27)$$

2) Eventually, the super Schwarzian derivative can be reproduced by varying $\Gamma_{WZP}[H_{\bar{\theta}}^z, H_{\theta}^z]$ with respect to a function of $H_{\bar{\theta}}^z$ and H_{θ}^z whose superconformal transformation law is independent of H_θ^z (e.g. $H_{\bar{\theta}}^z - (H_\theta^\theta)^{-1} H_{\bar{\theta}}^\theta H_\theta^z$), but we did not succeed in proving such a result.

the BRS-operation (18) becomes

$$\begin{aligned} sH_{\bar{\theta}}^z &= \left[\bar{D} - H_{\bar{\theta}}^z \partial + \frac{1}{2} (DH_{\bar{\theta}}^z) D \right] C^z + (\partial H_{\bar{\theta}}^z) C^z \\ sC^z &= - \left[C^z \partial C^z + \frac{1}{4} (DC^z) (DC^z) \right] \\ s\Lambda &= C^z \partial \Lambda + \frac{1}{2} (DC^z) D\Lambda + \Lambda \partial C^z . \end{aligned} \quad (28)$$

Furthermore, the integrating factor equation (IFEQ), eq.(14), takes the simple form

$$\left[D_{\bar{\theta}} - H_{\bar{\theta}}^z \partial + \frac{1}{2} (DH_{\bar{\theta}}^z) D \right] \Lambda = (\partial H_{\bar{\theta}}^z) \Lambda . \quad (29)$$

From the explicit form of $H_{\bar{\theta}}^z$, eq.(15), we deduce

$$\begin{aligned} DZ &= \Theta D\Theta \\ (D\Theta)^2 &= \partial Z + \Theta \partial \Theta , \end{aligned} \quad (30)$$

where the second equation follows from the first one by application of D . It should be emphasized that both Z and Θ still depend on z, θ and $\bar{z}, \bar{\theta}$.

For consistency, we can only consider superdiffeomorphisms

$$(z, \bar{z}, \theta, \bar{\theta}) \longrightarrow (z'(z, \bar{z}, \theta, \bar{\theta}), \bar{z}'(z, \bar{z}, \theta, \bar{\theta}), \theta'(z, \bar{z}, \theta, \bar{\theta}), \bar{\theta}'(z, \bar{z}, \theta, \bar{\theta}))$$

which respect the condition $H_{\bar{\theta}}^z = 0$, i.e. [12] satisfy a finite version of eq.(27) :

$$0 = [Dz' - \theta' D\theta'] + [D\bar{z}' - \bar{\theta}' D\bar{\theta}'] H_{\bar{z}'}^{z'} + [D\bar{\theta}'] H_{\bar{\theta}'}^{z'} . \quad (31)$$

4.1 Anomaly and effective action on SC

For $H_{\bar{\theta}}^z = 0$, the WZP-functional (23) reduces to

$$\Gamma_{WZP}[H_{\bar{\theta}}^z] = \int_{\text{SC}} d^4z H_{\bar{\theta}}^z \partial D \ln \Lambda , \quad (32)$$

with

$$\Lambda \equiv \partial Z + \Theta \partial \Theta = (D\Theta)^2 \quad (33)$$

and $H_{\bar{\theta}}^z = 1/\Lambda (\bar{D}Z - \Theta \bar{D}\Theta)$. Its BRS-variation yields the known expression [12] for the anomaly in this special case,

$$s\Gamma_{WZP}[H_{\bar{\theta}}^z] = 2 \int_{\text{SC}} d^4z C^z \partial^2 D H_{\bar{\theta}}^z , \quad (34)$$

a result which is easy to verify directly by use of the variations (28) and the IFEQ (29).

The expression for Γ_{WZP} in eq.(32) represents a very compact notation for the superspace generalization [15] of Polyakov's chiral gauge action (8). The latter corresponds to the gauge choice $H_{\bar{\theta}}^z = 0 = H_{\bar{\theta}}^{\bar{z}}$ and $H_{\theta}^{\bar{z}} = 0$, $\bar{\Lambda} = 1 = \Lambda$ in the metric approach. It is explicitly given by [15]

$$4 \int_{\text{SC}} d^4z \left[\frac{(D^2\Theta)(\bar{D}D\Theta)}{(D\Theta)^2} - H_{\bar{\theta}}^z \frac{(D^2\Theta)(D^3\Theta)}{(D\Theta)^2} \right] = 4 \int_{\text{SC}} d^4z \frac{(\partial\Theta)}{(D\Theta)^2} (\bar{D} - H_{\bar{\theta}}^z \partial) D\Theta \quad (35)$$

and coincides with (32) by virtue of equations (33) and (29).

In appendix B, we prove that the variation of Γ_{WZP} with respect to $H_{\bar{\theta}}^z$ is given by [15]

$$\frac{\delta\Gamma_{WZP}[H_{\bar{\theta}}^z]}{\delta H_{\bar{\theta}}^z} = -4 \mathcal{S}(Z, \Theta; z, \theta) \quad , \quad (36)$$

where

$$\mathcal{S}(Z, \Theta; z, \theta) \equiv \frac{\partial^2 \Theta}{D\Theta} - 2 \frac{(\partial\Theta)(\partial D\Theta)}{(D\Theta)^2} \quad (37)$$

denotes the *super Schwarzian derivative* [16].

4.2 Ward identities and OPE's on SC

The procedure we use in the following was introduced in the purely bosonic framework by C.Becchi [4] (for more details concerning that case, see [9] or [12]).

For $H_{\theta}^z = 0 = H_{\bar{\theta}}^{\bar{z}}$, the Ward operators acting on functionals $Z_c[H_{\bar{\theta}}^z, H_{\theta}^{\bar{z}}]$ have the form

$$\mathcal{W}(\Xi \cdot \partial) = \int_{\text{SC}} d^4 z \left[\delta_{\Xi} H_{\bar{\theta}}^z \frac{\delta}{\delta H_{\bar{\theta}}^z} + \text{c.c.} \right] \quad .$$

Here, $\delta_{\Xi} H_{\bar{\theta}}^z$ coincides with the s -variation (28), in which $\Xi^z, \Xi^{\bar{z}}, \Xi^{\theta}, \Xi^{\bar{\theta}}$ are now to be viewed as symmetry parameters and not as ghost fields. The *anomalous Ward identity*

$$\mathcal{W}(\Xi \cdot \partial) Z_c \propto \int_{\text{SC}} d^4 z \mathcal{A}(C^z; H_{\bar{\theta}}^z) + \text{c.c.}$$

takes the explicit form

$$\left[\bar{D} - H_{\bar{\theta}}^z \partial + \frac{1}{2} (DH_{\bar{\theta}}^z) D - \frac{3}{2} (\partial H_{\bar{\theta}}^z) \right] \frac{\delta Z_c}{\delta H_{\bar{\theta}}^z} = -k \partial^2 DH_{\bar{\theta}}^z \quad \text{and c.c.} \quad (38)$$

where k denotes a constant.

Following the proof of the distributional relation

$$\pi \delta^{(2)}(z - z') = \partial_{\bar{z}} \partial_z \ln |z - z'|^2 = \partial_{\bar{z}} \left[\frac{1}{z - z'} \right] \quad \text{in } \mathbf{C} \quad , \quad (39)$$

one derives a similar superspace identity [16],

$$-\pi \delta^{(4)}(z - z') = D_{\bar{\theta}} D_{\theta} \ln |z - z' - \theta \theta'|^2 = D_{\bar{\theta}} \left[\frac{\theta - \theta'}{z - z' - \theta \theta'} \right] \quad \text{in } \mathbf{SC} \quad , \quad (40)$$

where $\delta^{(4)}(z - z') \equiv \delta^{(2)}(z - z') (\theta - \theta') (\bar{\theta} - \bar{\theta}')$. Using this relation, eq.(38) can now be integrated in the superplane with the following result :

$$-2\pi i \frac{\delta Z_c}{\delta H_2} = \int_{\text{SC}} d^4 z_1 H_1 \left\{ k \partial_1^2 D_1 \left(\frac{\theta_{12}}{z_{12}} \right) + \left[-\frac{3}{2} \partial_1 \left(\frac{\theta_{12}}{z_{12}} \right) - \frac{\theta_{12}}{z_{12}} \partial_1 + \frac{1}{2} D_1 \left(\frac{\theta_{12}}{z_{12}} \right) D_1 \right] \frac{\delta Z_c}{\delta H_1} \right\} \quad (41)$$

where

$$\theta_{12} \equiv \theta_1 - \theta_2 \quad , \quad z_{12} \equiv z_1 - z_2 - \theta_1 \theta_2 \quad \text{and} \quad H_1 \equiv H_{\bar{\theta}_1}^{z_1} \quad , \quad H_2 \equiv H_{\bar{\theta}_2}^{z_2} \quad .$$

Henceforth,

$$\begin{aligned} \pi \frac{\delta^2 Z_c}{\delta H_1 \delta H_2} \Big|_{H=0} &= k \partial_1^2 D_1 \left(\frac{\theta_{12}}{z_{12}} \right) + \left[-\frac{3}{2} \partial_1 \left(\frac{\theta_{12}}{z_{12}} \right) - \frac{\theta_{12}}{z_{12}} \partial_1 + \frac{1}{2} D_1 \left(\frac{\theta_{12}}{z_{12}} \right) D_1 \right] \frac{\delta Z_c}{\delta H_1} \Big|_{H=0} \\ &= \frac{2k}{(z_{12})^3} + \left[\frac{3}{2} \frac{\theta_{12}}{(z_{12})^2} - \frac{\theta_{12}}{z_{12}} \partial_1 + \frac{1}{2} \frac{1}{z_{12}} D_1 \right] \frac{\delta Z_c}{\delta H_1} \Big|_{H=0} \quad , \end{aligned} \quad (42)$$

which is to be compared (and is in agreement) with the *operator product expansion* [16] for the component $\mathcal{T}_{\theta z} = \delta Z_c / \delta H_{\bar{\theta}}^z$ of the stress supertensor :

$$\mathcal{T}(\mathcal{Z}_1) \mathcal{T}(\mathcal{Z}_2) = \frac{c/4}{(z_{12})^3} \mathbf{1} + \left[\frac{3}{2} \frac{\theta_{12}}{(z_{12})^2} + \frac{\theta_{12}}{z_{12}} \partial_2 + \frac{1}{2} \frac{1}{z_{12}} D_2 \right] \mathcal{T}(\mathcal{Z}_2) + \text{regular terms} \quad . \quad (43)$$

Here, $\mathcal{Z} \equiv (z, \bar{z}, \theta, \bar{\theta})$ and in the *WZ-gauge* [12] where $H_{\bar{\theta}}^z$ has only space-time components $\mu_{\bar{z}}^z, \alpha_{\bar{z}}^{\theta}$, one has

$$\mathcal{T}_{\theta z} \equiv S_{\theta z} + \theta T_{zz} = i \frac{\delta Z_c}{\delta \alpha_{\bar{z}}^{\theta}} + \theta \frac{\delta Z_c}{\delta \mu_{\bar{z}}^z} \quad .$$

In this gauge, the superspace equation (43) is equivalent to the ordinary space relations [16]

$$\begin{aligned} S(z_1, \bar{z}_1) S(z_2, \bar{z}_2) &= \frac{c/4}{(z_1 - z_2)^3} \mathbf{1} + \frac{1/2}{z_1 - z_2} T(z_2, \bar{z}_2) + \text{r.t.} \\ T(z_1, \bar{z}_1) S(z_2, \bar{z}_2) &= \frac{3/2}{(z_1 - z_2)^2} S(z_2, \bar{z}_2) + \frac{1}{z_1 - z_2} \partial_2 S(z_2, \bar{z}_2) + \text{r.t.} \\ S(z_1, \bar{z}_1) T(z_2, \bar{z}_2) &= \frac{3/2}{(z_1 - z_2)^2} S(z_2, \bar{z}_2) + \frac{1/2}{z_1 - z_2} \partial_2 S(z_2, \bar{z}_2) + \text{r.t.} \\ T(z_1, \bar{z}_1) T(z_2, \bar{z}_2) &= \frac{3c/4}{(z_1 - z_2)^4} \mathbf{1} + \frac{2}{(z_1 - z_2)^2} T(z_2, \bar{z}_2) + \frac{1}{z_1 - z_2} \partial_2 T(z_2, \bar{z}_2) + \text{r.t.} \end{aligned} \quad (44)$$

These relations can also be reproduced directly from the anomalous Ward identities generated by the symmetry transformations in the *WZ-gauge*, i.e. infinitesimal diffeomorphisms with parameter $c^z = \xi^z + \xi^{\bar{z}} \mu_{\bar{z}}^z$ and supersymmetry variations with parameter $\epsilon^{\theta} = \xi^{\theta} + \xi^{\bar{z}} \alpha_{\bar{z}}^{\theta}$:

$$\int_{\mathbf{C}} d^2 z \left[\delta \mu \frac{\delta}{\delta \mu} + \delta \alpha \frac{\delta}{\delta \alpha} + \text{c.c.} \right] Z_c[\mu, \bar{\mu}, \alpha, \bar{\alpha}] = k \int_{\mathbf{C}} d^2 z \left[c \partial^3 \mu - \epsilon \partial^2 \alpha + \text{c.c.} \right] \quad (45)$$

with

$$\begin{aligned} \delta \mu &= \left(\bar{\partial} - \mu \partial + \partial \mu \right) c + \frac{1}{2} \alpha \epsilon && \text{and c.c.} \\ \delta \alpha &= \left(\bar{\partial} - \mu \partial + \frac{1}{2} \partial \mu \right) \epsilon + c \partial \alpha - \frac{1}{2} \alpha \partial c && \text{and c.c.} \end{aligned}$$

The coefficients of c and ϵ in eq.(45) give rise to the *WZ-gauge Ward identities*

$$\begin{aligned} 0 &= \left[\bar{\partial} - \mu \partial - 2 \partial \mu \right] \frac{\delta Z_c}{\delta \mu} - \left[\frac{1}{2} \alpha \partial + \frac{3}{2} \partial \alpha \right] \frac{\delta Z_c}{\delta \alpha} + k \partial^3 \mu && \text{and c.c.} \\ 0 &= \left[\bar{\partial} - \mu \partial - \frac{3}{2} \partial \mu \right] \frac{\delta Z_c}{\delta \alpha} + \frac{1}{2} \alpha \frac{\delta Z_c}{\delta \mu} - k \partial^2 \alpha && \text{and c.c.} \end{aligned} \quad (46)$$

and use of the distributional relation (39) leads to the OPE's (44).

Eq.(43) represents an expansion with respect to (operator-valued) distributions in superspace. If one requires the OPE (43) to be compatible with the Ward identity of the underlying field theory, then the distributions occurring therein have to be defined as derivatives of the regular distribution θ_{12}/z_{12} according to the first line in eq.(42). Similarly, the ordinary space distributions occurring in the OPE's (44) are to be defined as derivatives of the regular distribution $(z_1 - z_2)^{-1}$. (Here, 'regular' means that one obtains a well-defined integral after smearing out with a smooth test-function [17].)

5 Generalization to generic (super) Riemann surfaces

Let us first discuss the purely bosonic theory [6] [9]. The integral $\int d^2z c \partial^3 \mu$ is not well defined on a generic (compact) Riemann surface Σ due to the fact that the integrand does not transform with the Jacobian upon passage from one coordinate chart to another. To be more precise, under a conformal change of coordinates $z \rightarrow z'(z)$, the variables c and μ transform according to

$$c^z = c^{z'} \frac{\partial z}{\partial z'} \quad , \quad \mu_{\bar{z}}^z = \mu_{\bar{z}'}^{z'} \frac{\partial z}{\partial z'} \frac{\partial \bar{z}'}{\partial \bar{z}} \quad ,$$

which implies

$$(c \partial^3 \mu)' = \frac{\partial z}{\partial z'} \frac{\partial \bar{z}}{\partial \bar{z}'} c (\partial^3 + 2S \partial + (\partial S)) \mu \quad . \quad (47)$$

Here, $S \equiv S(z'; z)$ denotes the Schwarzian derivative introduced in eq.(10).

The S -dependent terms in the transformation law (47) can be eliminated [6] by the introduction of a projective projection. By definition [18], a *projective* (or *Schwarzian*) *connection* is locally represented by a field R_{zz} which is locally holomorphic, i.e. $\partial_{\bar{z}} R_{zz} = 0$, and which transforms under conformal transformations $z \rightarrow z'(z)$ according to

$$R_{z'z'}(z') = \left(\frac{\partial z}{\partial z'} \right)^2 [R_{zz}(z) - S(z'; z)] \quad . \quad (48)$$

Therefore, the modified *anomaly* [6] [9] [7]

$$\tilde{\mathcal{A}}(c; \mu) \equiv c (\partial^3 + 2R \partial + (\partial R)) \mu \quad (49)$$

transforms properly,

$$\tilde{\mathcal{A}}(c; \mu)' = \frac{\partial z}{\partial z'} \frac{\partial \bar{z}}{\partial \bar{z}'} \tilde{\mathcal{A}}(c; \mu) \quad (50)$$

and $sR \equiv 0$ implies that it yields a solution of the WZ-consistency condition on the Riemann surface Σ ,

$$s \int_{\Sigma} d^2z \tilde{\mathcal{A}}(c; \mu) = 0 \quad .$$

Since the expressions for the anomaly on the plane and on a generic Riemann surface Σ only differ by connection-dependent terms, one might wonder whether it is possible to obtain an effective action on Σ by adding similar terms to the action for the plane. In fact, the s -variation of the functional

$$\int_{\Sigma} d^2z \left\{ -\frac{1}{2} \mu \partial^2 \ln \lambda + \mu R \right\} \quad (51)$$

reproduces the expression (49) for the anomaly on Σ . Furthermore, the integrand in (51) transforms properly under a conformal change of coordinates, $z \rightarrow z'(z)$. However, λ is locally given by ∂Z and (51) is not well defined under a conformal change of the capital coordinates, $Z \rightarrow Z'(Z)$.

So far, a proper generalization of the actions (5) and (8) to higher genus Riemann surfaces has not been found except for the case of the torus [9]. The existence of such an extension in the general case follows from a theorem on the holomorphic factorization on the space of Beltrami differentials [8]; concrete expressions are expected to depend on the model under consideration.

The supersymmetric theory is plagued with the same global problems. In this case, one encounters the additional complication that the superconformal transformation law of one of the independent Beltrami superfields ($H_{\bar{\theta}}^z$) depends on the other one (H_{θ}^z) [12]. For $H_{\theta}^z = 0$, the projective connection $R_{zz}(z)$ can be generalized to a superfield $\mathcal{R}_{z\theta}(z, \theta)$ and then the modified *anomaly*

$$\tilde{A}(C^z; H_{\bar{\theta}}^z) \equiv C^z \left(\partial^2 D + 3 \mathcal{R}_{z\theta} \partial + (D \mathcal{R}_{z\theta}) D + 2 (\partial \mathcal{R}_{z\theta}) \right) H_{\bar{\theta}}^z \quad (52)$$

simply transforms with the Jacobian under superconformal transformations. For more details concerning this result, we refer to [19]. Here, we only wish to point out that the expression (52) provides a solution of the WZ-consistency condition, $\oint_{\Sigma} d^4 z \tilde{A}(C^z; H_{\bar{\theta}}^z) = 0$. Moreover, we note that we did not succeed in extending this result to the general form of the anomaly. The failure to do so is due to the difficulty in reproducing the super Schwarzian derivative upon superconformal transformation of the non-integrated anomaly (a kind of problem we encountered already before, see the comments at the end of section 3.1).

6 Concluding comments

In summary, we have derived general expressions for the chirally split superdiffeomorphism anomaly and the associated effective action on the $N = 1$ superplane. Eventually, the WZ-action can be explicitly constructed from the known expression of the anomaly [5] [20] [9] by using the Maurer-Cartan forms associated with the superdiffeomorphism group [21].

For $H_{\theta}^z = 0$, we found that the effective action reacts by the super Schwarzian derivative to an infinitesimal change of the supercomplex structure. Moreover, it was shown in this case that the anomalous Ward identities of local Lagrangian field theory lead to the familiar OPE's of superconformal field theory, the distributions occurring therein having been further specified. The global issues were briefly discussed for the bosonic theory and their supersymmetric generalization will be presented in more detail elsewhere [19].

Obviously, the particular (scalar field) form we found for the effective action in the bosonic case can be generalized to (p, q) supersymmetry once the corresponding superconformal structures have been parametrized in terms of Beltrami superfields. Furthermore, the superspace derivations presented here should also extend to that case. The $(1, 0)$ theory is briefly addressed in appendix C, but the geometric framework of extended supersymmetry is more involved [22] and deserves a separate study.

In conclusion, we would like to mention a very recent work [23] whose authors derived superspace expressions for the factorized superdiffeomorphism anomaly together with a local compensating action.

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A Derivation of the anomaly on SC

We consider the form (20) of Γ_{WZP} . The BRS-variations are nothing but infinitesimal supercoordinate transformations which act passively on the reference coordinates, i.e. $0 = s(z, \bar{z}, \theta, \bar{\theta})$, and actively on the capital coordinates [12] :

$$\begin{aligned} s\tilde{z} &= sZ = C^z \Lambda - \Theta s\Theta \\ s\tilde{\theta} &= s\Theta = C^z \tau + C^\theta \sqrt{\Lambda} \\ s\tilde{\bar{z}} &= s\bar{z} = 0 \\ s\tilde{\bar{\theta}} &= s\bar{\theta} = 0 \end{aligned} \quad (53)$$

A simple method to determine the variations of partial derivatives $\partial_i \equiv \partial/\partial x^i$ induced by an infinitesimal change of coordinates, $x^i \rightarrow x^i + \delta x^i$, is to require the invariance of the relations $\partial_i x^j = \delta_i^j$:

$$(s\partial_i) x^j + \partial_i (s x^j) = 0 \quad .$$

The variation of ∂_i is necessarily a linear combination of ∂_j , i.e. $s\partial_i = \lambda_i^j \partial_j$ and substitution of this ansatz into the last equation yields

$$s\partial_i = -\partial_i (s x^j) \partial_j \quad .$$

By specifying this formula to the coordinate system $(\tilde{z}, \tilde{\bar{z}}, \tilde{\theta}, \tilde{\bar{\theta}})$, we obtain

$$\begin{aligned} s\tilde{\partial} &= -\tilde{\partial}(s\tilde{z}) \tilde{\partial} - \tilde{\partial}(s\tilde{\theta}) \partial_{\tilde{\theta}} \\ s\partial_{\tilde{\theta}} &= -\partial_{\tilde{\theta}}(s\tilde{z}) \tilde{\partial} - \partial_{\tilde{\theta}}(s\tilde{\theta}) \partial_{\tilde{\theta}} \\ s\tilde{D} &= s\partial_{\tilde{\theta}} + \tilde{\theta} s\tilde{\partial} + (s\tilde{\theta}) \tilde{\partial} \end{aligned}$$

and similarly for $\tilde{\partial}, \tilde{D}$. Substitution of the explicit expressions (53) into these equations leads to

$$\begin{aligned} s\tilde{\partial} &= -\tilde{\partial}(C^z \Lambda) \tilde{\partial} - \tilde{\partial}(C^z \tau + C^\theta \sqrt{\Lambda}) \tilde{D} \\ s\tilde{D} &= -\left[\tilde{D}(C^z \Lambda) - 2(C^z \tau + C^\theta \sqrt{\Lambda}) \right] \tilde{\partial} - \tilde{D}(C^z \tau + C^\theta \sqrt{\Lambda}) \tilde{D} \\ s\tilde{\bar{D}} &= -\tilde{\bar{D}}(C^z \Lambda) \tilde{\partial} - \tilde{\bar{D}}(C^z \tau + C^\theta \sqrt{\Lambda}) \tilde{D} \end{aligned} \quad (54)$$

Furthermore, from the general formula

$$s(d^m x) = (d^m x) \cdot |\text{str}((\partial_i s x^j))| \quad ,$$

where ‘str’ denotes the supertrace, it follows that

$$\begin{aligned} s(d^4 \tilde{z}) &= d^4 \tilde{z} \left[\tilde{\partial}(s\tilde{z}) - \partial_{\tilde{\theta}}(s\tilde{\theta}) \right] \\ &= d^4 \tilde{z} \left[\tilde{\partial}(C^z \Lambda) - \tilde{D}(C^z \tau + C^\theta \sqrt{\Lambda}) \right] \end{aligned} \quad (55)$$

Finally, eq.(18) for $s\Lambda$ implies

$$\begin{aligned} s(\ln \Lambda) &= \Lambda^{-1} \partial(C^z \Lambda) + 2C^\theta \tau \Lambda^{-1/2} \\ &= \left(\tilde{\partial} + \frac{\tau}{\Lambda} \tilde{D} \right) (C^z \Lambda) + 2C^\theta \tau \Lambda^{-1/2} \end{aligned} \quad (56)$$

where we used the relation $\tilde{\partial} + \Lambda^{-1} \tau \tilde{D} = \Lambda^{-1} \partial$ to pass to the last line.

In conclusion, we insert all previous expressions into $s\Gamma_{WZP}$ and perform some partial integrations to obtain

$$s\Gamma_{WZP} = \int_{\text{SC}} d^4\tilde{z} \left[2C^z (\Lambda\tilde{\partial} + \tau\tilde{D}) \ln \Lambda - \frac{2}{\Lambda} \partial(C^z \Lambda) + 2C^\theta \sqrt{\Lambda} (\tilde{D} \ln \Lambda - \frac{2}{\Lambda} \tau) \right] \tilde{D} \tilde{D} \ln \Lambda .$$

Now, using again $\Lambda\tilde{\partial} + \tau\tilde{D} = \partial$ as well as

$$\tilde{D} \ln \Lambda - \frac{2}{\Lambda} \tau = \frac{1}{\sqrt{\Lambda} H_\theta^z} \partial H_\theta^z$$

leads to

$$s\Gamma_{WZP} = -2 \int_{\text{SC}} d^4\tilde{z} (\partial C^z - C^\theta \frac{\partial H_\theta^z}{H_\theta^z}) \tilde{D} \tilde{D} \ln \Lambda \quad (57)$$

and substitution of the integrating factor equation (14) yields the final result (24).

B Γ_{WZP} and the Schwarzian derivative

This appendix is devoted to a proof of eq.(36) and proceeds along the lines of the bosonic theory [9].

For an infinitesimal variation $H_{\bar{\theta}}^z \rightarrow H_{\bar{\theta}}^z + \delta H_{\bar{\theta}}^z$, we have

$$\delta \int_{\text{SC}} d^4z H_{\bar{\theta}}^z \partial D \ln \Lambda = \int_{\text{SC}} d^4z [\delta H_{\bar{\theta}}^z \partial D \ln \Lambda - (\partial D H_{\bar{\theta}}^z) \delta \ln \Lambda] . \quad (58)$$

To evaluate the second term, we consider the relations obtained from the IFEQ (29), respectively, by application of D , i.e.

$$D \left[\bar{D} - H_{\bar{\theta}}^z \partial + \frac{1}{2} (D H_{\bar{\theta}}^z) D \right] \ln \Lambda = \partial D H_{\bar{\theta}}^z , \quad (59)$$

and of δ , i.e.

$$\left[\bar{D} - H_{\bar{\theta}}^z \partial + \frac{1}{2} (D H_{\bar{\theta}}^z) D \right] \delta \ln \Lambda = \partial \delta H_{\bar{\theta}}^z + (\delta H_{\bar{\theta}}^z) \partial \ln \Lambda - \frac{1}{2} (D \delta H_{\bar{\theta}}^z) D \ln \Lambda . \quad (60)$$

By substituting (59) into the second term and integrating by parts, we get

$$\begin{aligned} - \int_{\text{SC}} d^4z (\partial D H_{\bar{\theta}}^z) \delta \ln \Lambda &= \int_{\text{SC}} d^4z (D \ln \Lambda) \left[\bar{D} - H_{\bar{\theta}}^z \partial + \frac{1}{2} (D H_{\bar{\theta}}^z) D \right] \delta \ln \Lambda \\ &= \int_{\text{SC}} d^4z \delta H_{\bar{\theta}}^z [\partial D \ln \Lambda - (D \ln \Lambda) (\partial \ln \Lambda)] , \end{aligned}$$

where we used (60) and another integration by parts to pass to the last line. Henceforth,

$$\delta \int_{\text{SC}} d^4z H_{\bar{\theta}}^z \partial D \ln \Lambda = 2 \int_{\text{SC}} d^4z \delta H_{\bar{\theta}}^z \left[\partial D \ln \Lambda - \frac{1}{2} (D \ln \Lambda) (\partial \ln \Lambda) \right]$$

or

$$\frac{\delta}{\delta H_{\bar{\theta}}^z} \int_{\text{SC}} d^4z H_{\bar{\theta}}^z \partial D \ln \Lambda = 4 \mathcal{S}(Z, \Theta; z, \theta)$$

by virtue of eq.(33).

C Comments on (1,0) supersymmetry

The (1,0) geometry is fairly similar to that of the (1,1) theory considered so far [24] [12]. In this case, the superconformal structures are parametrized by two odd and one even Beltrami superfields, $H_{\theta}^{\bar{z}}, H_{\theta}^z$ and $H_{\bar{z}}^z$ [12]. For $H_{\theta}^z = 0$, the WZP-action in the z -sector reads

$$\begin{aligned}\Gamma_{WZP}[H_{\bar{z}}^z] &= - \int_{\text{SC}} d^3z H_{\bar{z}}^z \partial D \ln \Lambda \\ s \Gamma_{WZP}[H_{\bar{z}}^z] &= 2 \int_{\text{SC}} d^3z C^z \partial^2 D H_{\bar{z}}^z .\end{aligned}\tag{61}$$

As in the (1,1) theory, it can be rewritten in a more complicated way as

$$\begin{aligned}\Gamma_{WZP}[H_{\bar{z}}^z] &= 4 \int_{\text{SC}} d^3z \frac{\partial \Theta}{(D\Theta)^2} (\bar{\partial} - H_{\bar{z}}^z \partial) D\Theta \\ &= 4 \int_{\text{SC}} d^3z \frac{\partial D\Theta}{(D\Theta)^2} (\bar{\partial} - H_{\bar{z}}^z \partial) \Theta .\end{aligned}$$

Here, the last line follows from the identity $2(\bar{\partial}\Theta)(D\Theta) = D[\bar{\partial}Z + \Theta\bar{\partial}\Theta]$ which is a direct consequence of $DZ = \Theta D\Theta$ [25] (or, equivalently, $H_{\theta}^z = 0$). These expressions again coincide with those previously found [26] [25] in the chiral gauge (corresponding to the supergravity gauge choice $H_{\theta}^z = 0, H_{\theta}^{\bar{z}} = 0$ and $\Lambda = 1 = \bar{\Lambda}$).

For $H_{\theta}^z = 0$, the *anomalous Ward identities* on SC in the z - and $\bar{z}$ -sectors read, respectively,

$$\left[\bar{\partial} - H_{\bar{z}}^z \partial - \frac{1}{2} (DH_{\bar{z}}^z) D - \frac{3}{2} (\partial H_{\bar{z}}^z) \right] \frac{\delta Z_c}{\delta H_{\bar{z}}^z} = -k \partial^2 D H_{\bar{z}}^z \tag{62}$$

and

$$\left[D - H_{\theta}^{\bar{z}} \bar{\partial} - 2(\bar{\partial} H_{\theta}^{\bar{z}}) \right] \frac{\delta Z_c}{\delta H_{\theta}^{\bar{z}}} = -\hat{k} \bar{\partial}^3 H_{\theta}^{\bar{z}} . \tag{63}$$

By using the distributional superspace identities

$$\begin{aligned}\pi \delta^{(3)}(z - z') &= \partial_{\bar{z}} D_{\theta} \ln(z - z' - \theta\theta')(\bar{z} - \bar{z}') = \partial_{\bar{z}} \left[\frac{\theta - \theta'}{z - z' - \theta\theta'} \right] \\ &= D_{\theta} \left[\frac{1}{\bar{z} - \bar{z}'} \right] ,\end{aligned}\tag{64}$$

one can reproduce the OPE's for the stress supertensors³⁾

$$\mathcal{T}_{\theta z}(\mathcal{Z}) = \frac{\delta Z_c}{\delta H_{\bar{z}}^z} , \quad \mathcal{T}_{\bar{z}\bar{z}}(\mathcal{Z}) = \frac{\delta Z_c}{\delta H_{\theta}^{\bar{z}}} ,$$

where $\mathcal{Z} \equiv (z, \bar{z}, \theta)$. For the OPE of $\mathcal{T}_{\theta z}$, one finds the same result as in the z -sector of the (1,1)- theory, i.e. eq.(43) and, for the OPE of $\mathcal{T}_{\bar{z}\bar{z}}$, one recovers a superspace expression which has the same form as the one for $\bar{T}_{\bar{z}\bar{z}}(z, \bar{z})$ in ordinary space-time, see eq.(44).

3) Concerning the assignment of the $\mathcal{T}$ -indices, we note that a (1,0)-superspace Lagrangian carries a θ -index.

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