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Modular invariance of $N = 2$ minimal models

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Abstract

We prove modular covariance of one-point functions at one loop in the diagonal $N = 2$ minimal superconformal models. We use the recently derived general formalism for computing arbitrary conformal blocks in these models. Our result should be sufficient to guarantee modular covariance at arbitrary genus. It is thus an important check on the general formalism which is not manifestly modular covariant.

1 Introduction

Two-dimensional $N = 2$ superconformal theories have attracted a great deal of attention in recent years. The main reason for this is that such theories may be used to define classical super- and heterotic string vacua with $N = 1$ supersymmetry in spacetime [8, 13]. The most well-known class of $N = 2$ superconformal theories are the $N = 2$ minimal models [9] which were used in [8] to define string vacua.

Perturbative quantum corrections to string amplitudes correspond to correlation functions on higher genus surfaces. Therefore, it becomes an important problem to extend $N = 2$ superconformal theories to arbitrary Riemann surfaces. Recently, in [1] a general sewing formalism for the $N = 2$ minimal models was found. This formalism allows the computation of arbitrary conformal blocks on arbitrary Riemann surfaces. It is far from obvious, however, that the correlation functions defined by these conformal blocks once a set of structure constants are given do not depend on the particular way in which the Riemann surface was obtained, i.e. on the choice of sewing contours.

In this paper we demonstrate by explicit calculations that for the diagonal $N = 2$ minimal models (defined by the structure constants found in [3] by demanding crossing invariance of the four-point functions on the sphere) the one-point correlation functions on the torus are in fact modular covariant and thus independent of the choice of sewing contour.

Then, by extending an argument due to Sonoda [12] for bosonic conformal field theories, one may show that crossing invariance for the four-point functions on the sphere together with modular covariance for the one-point correlators on the torus are sufficient conditions for modular covariance at arbitrary genus.

The paper is organized as follows: in section 2 we define the one-point conformal blocks on the torus and give the explicit expressions for these in terms of (multiple) contour integrals using the results of [1]. In section 3 we compute the modular transformation properties of the conformal blocks and in section 4 we use this result to show that the one-loop one-point correlation functions are modular covariant. Appendix A explains our notation and conventions and Appendix B and C contain some detailed calculations which we couldn't fit into the main sections.

2 The one-point blocks on the torus

The torus is parametrized by the annulus, $\{z \mid |q| \leq |z| \leq 1\}$, with the identification $qz \equiv z$. We shall study the general conformal blocks¹⁾

$$\tilde{J}_{l,m,\lambda}' \left[\begin{matrix} 2\lambda \\ \beta \end{matrix} \right] (\nu, \tau, t) \equiv \text{Tr}_{\mathcal{H}_{N=2}^{l,m,\lambda}} \left(\Psi_{l',0,0}^{(\frac{1}{2}l')} (1) q^{L_0} e^{2\pi i \nu J_0} e^{-2\pi i t k(k+2)} (-1)^{\beta F} \right), \quad (1)$$

where F is the world-sheet fermion number operator and $\beta = 0, 1 \bmod 2$, distinguishes between performing the summation over internal states in the NS or the R sector. The variables, ν, τ , and t parametrize the maximal hypertorus of the $N = 2$ algebra and we have the relation $\tau = \frac{\ln q}{2\pi i}$.

From [1, 2] one finds for the conformal blocks on the annulus,

$$\tilde{J}_{l,m,\lambda}' \left[\begin{matrix} 2\lambda \\ \beta \end{matrix} \right] (\nu, \tau, t) = e^{-2\pi i t k(k+2)} q^{-\frac{1}{4(k+2)}} e^{-i\pi\beta \frac{m+k\lambda}{k+2}} \left(\frac{q^{\frac{1}{24}}}{\eta(\tau)} \right)^3 \prod_{i=1}^{\frac{1}{2}l'} \frac{\partial}{\partial \tilde{y}_i} \bigg|_{\tilde{y}_i=0} \oint_{\mathcal{C}^{\frac{1}{2}l'}} \prod_{i=1}^{\frac{1}{2}l'} dz_i$$

1) See Appendix A for notation and conventions.

$$\begin{aligned}
& \prod_{1 \leq i < j \leq \frac{1}{2}l''} E(z_i, z_j | \tau)^{\frac{2}{k+2}} \prod_{i=1}^{\frac{1}{2}l''} E(1, z_i | \tau)^{\frac{-l'}{k+2}} \prod_{i=1}^{\frac{1}{2}l''} \sigma(z_i | \tau)^{\frac{-2}{k+2}} \\
& \times \prod_{1 \leq i < j \leq \frac{1}{2}l''} e^{-\bar{y}_i \bar{y}_j \partial_{z_i} \partial_{z_j} \ln E(z_i, z_j | \tau)} \sum_{m' \in 2\mathbb{Z}+l} \sum_{n \in \mathbb{Z}} \sum_{s=0}^{\infty} (-1)^s \\
& q^{\frac{((k+2)m' - km + 2k\lambda)^2}{8k(k+2)}} e^{\frac{i\pi}{2(k+2)}(\nu + \beta)((k+2)m' - km + 2k\lambda)} \\
& \times \left[q^{\frac{(k+2)(\frac{l+1}{2(k+2)} + n + \frac{s}{2})^2 - k(\frac{m'}{2k} + \frac{s}{2})^2}{2\pi i}} e^{2\pi i \left\{ (\frac{l}{2(k+2)} + n + \frac{s}{2}) \tilde{u}_1 - (\frac{m'}{2k} + \frac{s}{2}) \tilde{u}_2 \right\}} \right. \\
& \left. - e^{i\pi \frac{\frac{1}{2}l'(l+2)}{k+2}} \times (l \leftrightarrow -l - 2) \right]. \tag{2}
\end{aligned}$$

Here $\sigma(z|\tau) \equiv 1$ is the $1/2$ -differential with no zeroes and no poles²⁾, and we have defined $\tilde{u}_1 = \frac{i}{\pi} \sum_{i=1}^{\frac{1}{2}l''} \ln z_i$ and $\tilde{u}_2 = \frac{\sqrt{2k}}{2\pi i} \sum_{i=1}^{\frac{1}{2}l''} \frac{\bar{y}_i}{z_i}$. Finally, the prime form is given on the annulus by

$$E(z_i, z_j | \tau) = \frac{\theta_1(\frac{\ln z_1 - \ln z_2}{2\pi i} | \tau)}{\theta'(0 | \tau)} (z_1 z_2)^{\frac{1}{2}}, \tag{3}$$

where θ_1 is the odd theta function $\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the classical theta functions are defined by

$$\theta \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} (w | \tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n + \frac{\epsilon_1}{2})^2} e^{2\pi i (w + \frac{\epsilon_2}{2})(n + \frac{\epsilon_1}{2})}. \tag{4}$$

For the purpose of computing modular transformation properties it is more convenient to parametrize the torus by the parallelogram $\{w | w = x + y\tau, x, y \in [0, 1]\}$, with the identifications $w \equiv w + 1$, $w \equiv w + \tau$. The relation between the two parametrizations is given by $z = e^{2\pi i w}$.

Transforming to the parallelogram we find after a change of integration variables in the second term

$$\begin{aligned}
J_{l,m,\lambda}' \begin{bmatrix} 2\lambda \\ \beta \end{bmatrix} (\nu, \tau, t) &= q^{-\frac{k}{8(k+2)}\tau} \tilde{J}_{l,m,\lambda}' \begin{bmatrix} 2\lambda \\ \beta \end{bmatrix} (\nu, \tau, t) \\
&= (\eta(\tau))^{-3} \prod_{i=1}^{\frac{1}{2}l''} \frac{\partial}{\partial y_i} \Big|_{y_i=0} \int_0^1 \prod_{i=1}^{\frac{1}{2}l''} dw_i \mathcal{E}(\{w_i\}, \{y_i\} | \tau) \Gamma_{l,m}^{2\lambda, \beta}(u_1, u_2 | \nu, \tau, t).
\end{aligned} \tag{5}$$

Here we have defined

$$\begin{aligned}
\mathcal{E}(\{w_i\}, \{y_i\} | \tau) &= \prod_{1 \leq i < j \leq \frac{1}{2}l''} \left(\frac{\theta_1(w_i - w_j | \tau)}{\theta'_1(0 | \tau)} \right)^{\frac{2}{k+2}} \prod_{i=1}^{\frac{1}{2}l''} \left(\frac{\theta_1(-w_i | \tau)}{\theta'_1(0 | \tau)} \right)^{\frac{-l'}{k+2}} \\
&\times \prod_{1 \leq i < j \leq \frac{1}{2}l''} e^{-y_i y_j \partial_{w_i} \partial_{w_j} \ln \theta_1(w_i - w_j | \tau)},
\end{aligned} \tag{6}$$

and

$$\Gamma_{l,m}^{\epsilon_1, \epsilon_2}(u_1, u_2 | \nu, \tau, t) = e^{-2\pi i t k(k+2)} \sum_{m' \in 2\mathbb{Z}+l} \sum_{n \in \mathbb{Z}} \sum_{s=0}^{\infty} \left[\mathcal{S}_{m',n,s}^{l,m,\epsilon_1,\epsilon_2}(u_1, u_2 | w, \tau) - \mathcal{S}_{m',-n,-s}^{l,m,\epsilon_1,\epsilon_2}(u_1, u_2 | w, \tau) \right], \tag{7}$$

2) On the covering space, i.e. the Riemann sphere, σ has just a simple pole at $z = \infty$.

where the summand is given by

$$\begin{aligned} \mathcal{S}_{m',n,s}^{l,m,\epsilon_1,\epsilon_2}(u_1, u_2|w, \tau) = & e^{-i\pi\epsilon_2 \frac{m+\frac{1}{2}k\epsilon_1}{k+2}} (-1)^s q^{\frac{((k+2)m'-km+k\epsilon_1)^2}{8k(k+2)}} e^{\frac{i\pi}{2(k+2)}(w+\epsilon_2)((k+2)m'-km+k\epsilon_1)} \\ & \times q^{(k+2)(\frac{l+1}{2(k+2)}+n+\frac{s}{2})^2 - k(\frac{m'}{2k}+\frac{s}{2})^2} e^{2\pi i \left\{ (\frac{l+1}{2(k+2)}+n+\frac{s}{2})u_1 - (\frac{m'}{2k}+\frac{s}{2})u_2 \right\}}, \end{aligned} \quad (8)$$

and where $u_1 = -2 \sum_{i=1}^{\frac{1}{2}l'} w_i$ and $u_2 = \sqrt{2k} \sum_{i=1}^{\frac{1}{2}l'} y_i$.

In fact, we shall have to be a little more careful with the definition of the “zero mode contribution”, $\Gamma_{l,m}^{2\lambda,\beta}(u_1, u_2|\nu, \tau, t)$. As it stands Eq. (7) is not absolutely convergent. However, from the cohomology analysis in [2] (Eq. (22)) we know that changing the summation range of s in Eq. (7) from the non-negative integers to the negative integers gives the same result up to an overall factor of -1 . This allows us to rewrite Eq. (7) as a sum of four absolutely convergent series. In fact, let

$$\begin{aligned} A &= -\frac{l+1-m+\epsilon_1}{2(k+2)} - \frac{\text{Im}(u_1 - \frac{k+2}{k}u_2 + \nu + \epsilon_2)}{2\tau_2(k+2)} \\ B &= -\frac{-l-1-m+\epsilon_1}{2(k+2)} - \frac{\text{Im}(-u_1 - \frac{k+2}{k}u_2 + \nu + \epsilon_2)}{2\tau_2(k+2)}, \end{aligned} \quad (9)$$

where $\tau_2 = \text{Im } \tau$. Then one finds

$$\begin{aligned} \Gamma_{l,m}^{\epsilon_1,\epsilon_2}(u_1, u_2|\nu, \tau, t) = & e^{-2\pi i t k(k+2)} \sum_{m' \in 2\mathbb{Z}+l} \sum_{n \geq A} \sum_{s=0}^{\infty} \mathcal{S}_{m',n,s}^{l,m,\epsilon_1,\epsilon_2}(u_1, u_2|\nu, \tau) - \sum_{m' \in 2\mathbb{Z}+l} \sum_{n < A} \sum_{s < 0} \mathcal{S}_{m',n,s}^{l,m,\epsilon_1,\epsilon_2}(u_1, u_2|\nu, \tau) \\ & - \sum_{m' \in 2\mathbb{Z}+l} \sum_{n \geq B} \sum_{s=0}^{\infty} \mathcal{S}_{m',-n,-s}^{l,m,\epsilon_1,\epsilon_2}(u_1, u_2|\nu, \tau) + \sum_{m' \in 2\mathbb{Z}+l} \sum_{n < B} \sum_{s < 0} \mathcal{S}_{m',-n,-s}^{l,m,\epsilon_1,\epsilon_2}(u_1, u_2|\nu, \tau). \end{aligned} \quad (10)$$

It is possible to simplify Eq. (10) somewhat. First, using $l-m \in 2\mathbb{Z}$, put $m' = 2n' + m$, $n' \in \mathbb{Z}$. Then, since all four sums are now absolutely convergent, we may change summation variable by introducing $n' = \tilde{n} + s$. This causes the quadratic terms in s in the exponents to vanish [7, 5]. Thus the summation over s may be carried out explicitly. Once this summation has been done one may write Eq. (10) as a sum of two terms each with a summation over n running over all integers. The final expression is

$$\begin{aligned} \Gamma_{l,m}^{2\lambda,\beta}(u_1, u_2|\nu, \tau, t) = & e^{-2\pi i t k(k+2)} e^{-i\pi\beta\lambda} q^{-\frac{(m-2\lambda)^2}{4(k+2)}} e^{2\pi i(m-2\lambda)(\frac{\nu}{2(k+2)} - \frac{u_2}{2k})} \\ & \times \theta \left[\begin{matrix} 2\lambda \\ \beta \end{matrix} \right] \left(\frac{\nu}{2} - \frac{u_2}{k} \right) \hat{\Gamma}_{l,m}^{2\lambda,\beta}(u_1, u_2|\nu, \tau), \end{aligned} \quad (11)$$

where we defined

$$\begin{aligned} \hat{\Gamma}_{l,m}^{\epsilon_1,\epsilon_2}(u_1, u_2|w, \tau) = & \sum_{n \in \mathbb{Z}} q^{(k+2)(n+\frac{l+1}{2(k+2)})^2} e^{2\pi i u_1(n+\frac{l+1}{2(k+2)})} \\ & \times \left[\left(1 + q^{(k+2)n+\frac{l+1-m+\epsilon_1}{2}} e^{2\pi i \left\{ \frac{1}{2}u_1 - \frac{k+2}{2k}u_2 + \frac{1}{2}w + \frac{1}{2}\epsilon_2 \right\}} \right)^{-1} \right. \\ & \left. - \left(1 + q^{-(k+2)n+\frac{-l-1-m+\epsilon_1}{2}} e^{2\pi i \left\{ -\frac{1}{2}u_1 - \frac{k+2}{2k}u_2 + \frac{1}{2}w + \frac{1}{2}\epsilon_2 \right\}} \right)^{-1} \right]. \end{aligned} \quad (12)$$

Our expression Eq. (11) for the zero mode sum is different from the one given in [1] where this was given (in direct correspondence to the factorization of the $N = 2$ Hilbert space) as a finite sum of terms each of which was a product of a contribution from ϕ_0 (a classical theta function) and from the PF-sector (a ‘generalized Kac-Peterson string function’). For the *characters*—i.e. the case $l' = 0$ —the contribution from the PF sector reduces to the ordinary string functions whose modular transformation properties are known [4] (and may in fact be calculated from the explicit expression resulting from the free field approach). However, in the case of general conformal blocks the PF-contribution is more complicated and its modular transformation properties are much more difficult to establish by direct calculation. The point of rewriting the whole $N = 2$ zero mode sum as above is that, as we shall see below, the modular transformation properties of Eq. (11) can be computed. With this result it is then a straightforward matter to extract the transformation properties of the general PF-contributions (although we shall not need to do that here) using the well-known transformation properties of theta functions.

3 Modular transformations of conformal blocks

The modular group is generated by two elements [4]: $T: \tau \rightarrow \tau + 1, \nu \rightarrow \nu, t \rightarrow t$, and $S: \tau \rightarrow -\frac{1}{\tau}, \nu \rightarrow \frac{\nu}{\tau}, t \rightarrow t + \frac{k}{8(k+2)} \frac{\nu^2}{\tau}$. We consider the transformation properties of the conformal blocks under these two transformations in turn.

3.1 The T transformation

One has the well-known formulae

$$\begin{aligned} \eta(\tau + 1) &= e^{\frac{i\pi}{12}} \eta(\tau) \\ \theta \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} (w|\tau + 1) &= e^{-\frac{i\pi}{4} \epsilon_1(\epsilon_1 + 2)} \theta \begin{bmatrix} \epsilon_1 \\ \epsilon_1 + \epsilon_2 + 1 \end{bmatrix} (w|\tau) \end{aligned} \quad (13)$$

and a straightforward calculation gives

$$\Gamma_{l,m}^{\epsilon_1, \epsilon_2}(u_1, u_2|\nu, \tau + 1, t) = e^{\frac{i\pi}{2(k+2)}} e^{2\pi i h(l, m, \frac{1}{2}\epsilon_1)} \Gamma_{l,m}^{\epsilon_1, \epsilon_1 + \epsilon_2 + 1}(u_1, u_2|\nu, \tau, t). \quad (14)$$

Thus altogether one finds

$$J_{l,m,\lambda}^{l'} \begin{bmatrix} 2\lambda \\ \beta \end{bmatrix} (\nu, \tau + 1, t) = e^{-i\pi \frac{k}{4(k+2)}} e^{2\pi i h(l, m, \lambda)} J_{l,m,\lambda}^{l'} \begin{bmatrix} 2\lambda \\ \beta + 2\lambda + 1 \end{bmatrix} (\nu, \tau, t). \quad (15)$$

3.2 The S transformation

Here one has

$$\begin{aligned} \eta\left(\frac{-1}{\tau}\right) &= (-i\tau)^{\frac{1}{2}} \eta(\tau) \\ \theta \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} \left(\frac{w}{\tau} \middle| \frac{-1}{\tau}\right) &= (-i\tau)^{\frac{1}{2}} e^{i\pi \frac{\epsilon_1 \epsilon_2}{2}} e^{i\pi \frac{w^2}{\tau}} \theta \begin{bmatrix} \epsilon_2 \\ -\epsilon_1 \end{bmatrix} (w|\tau), \end{aligned} \quad (16)$$

which gives

$$\mathcal{E}\left(\left\{\frac{w_i}{\tau}\right\}, \left\{\frac{y_i}{\tau}\right\} \middle| \frac{-1}{\tau}\right) = \tau^{h(l', 0, 0)} q'^{-\frac{u_1^2}{4(k+2)}} q'^{-\frac{u_2^2}{4k} + \frac{1}{2}} \sum y_i^2 \mathcal{E}(\{w_i\}, \{y_i\}|\tau), \quad (17)$$

where we introduced $q' = e^{\frac{-2\pi i}{\tau}}$. Using Eq. (51) in Appendix B we find

$$\begin{aligned} \Gamma_{l,m}^{2\lambda,\beta}\left(\frac{u_1}{\tau}, \frac{u_2}{\tau} \middle| \frac{\nu}{\tau}, \frac{-1}{\tau}, t + \frac{k}{8(k+2)} \frac{\nu^2}{\tau}\right) &= (-i\tau)^{1/2} \tau (k+2)^{-1} e^{-i\pi\beta\lambda} q'^{\frac{1}{4(k+2)}u_1^2 + \frac{1}{4k}u_2^2} \\ &\times \sum_{\tilde{l}=0}^{2k+3} \sum_{\substack{\tilde{m}=0 \\ \tilde{m}-\tilde{l} \in 2\mathbb{Z}}}^{k+1} e^{i\pi \frac{(\tilde{m}-\beta)(m-2\lambda)}{k+2} - i\pi \frac{(\tilde{l}+1)(l+1)}{k+2}} \Gamma_{\tilde{l},\tilde{m}}^{\beta,-2\lambda}(u_1, u_2 | \nu, \tau, t). \end{aligned} \quad (18)$$

Thus we have

$$\begin{aligned} J_{l,m,\lambda}^{l''} \left[\begin{array}{c} 2\lambda \\ \beta \end{array} \right] \left(\frac{\nu}{\tau}, \frac{-1}{\tau}, t + \frac{k}{8(k+2)} \frac{\nu^2}{\tau} \right) &= (\eta(\frac{-1}{\tau}))^{-3} \prod_{i=1}^{\frac{1}{2}l''} \frac{\partial}{\partial y_i} \bigg|_{y_i=0} \int_0^1 \prod_{i=1}^{\frac{1}{2}l''} dw_i \mathcal{E}(\{w_i\}, \{y_i\} | \frac{-1}{\tau}) \Gamma_{l,m}^{2\lambda,\beta}(u_1, u_2 | \frac{\nu}{\tau}, \frac{-1}{\tau}, t + \frac{k}{8(k+2)} \frac{\nu^2}{\tau}) \\ &= (\eta(\frac{-1}{\tau}))^{-3} \prod_{i=1}^{\frac{1}{2}l''} \frac{\partial}{\partial y_i} \bigg|_{y_i=0} \int_0^\tau \prod_{i=1}^{\frac{1}{2}l''} dw_i \mathcal{E}(\{w_i\}, \{y_i\} | \frac{-1}{\tau}) \Gamma_{l,m}^{2\lambda,\beta}(\frac{u_1}{\tau}, \frac{u_2}{\tau} | \frac{\nu}{\tau}, \frac{-1}{\tau}, t + \frac{k}{8(k+2)} \frac{\nu^2}{\tau}) \\ &= \frac{i\tau^{h(l'',0,0)}}{k+2} e^{-i\pi\beta\lambda} \sum_{\tilde{l}=0}^{2k+3} \sum_{\substack{\tilde{m}=0 \\ \tilde{m}-\tilde{l} \in 2\mathbb{Z}}}^{k+1} e^{i\pi \frac{(\tilde{m}-\beta)(m-2\lambda)}{k+2} - i\pi \frac{(\tilde{l}+1)(l+1)}{k+2}} I_{\tilde{l},\tilde{m},\frac{1}{2}\beta}^{l''} \left[\begin{array}{c} \beta \\ -2\lambda \end{array} \right] (\nu, \tau, t), \end{aligned} \quad (19)$$

where $I_{l,m,\lambda}^{l''} \left[\begin{array}{c} 2\lambda \\ \beta \end{array} \right] (\nu, \tau, t)$ is defined just as $J_{l,m,\lambda}^{l''} \left[\begin{array}{c} 2\lambda \\ \beta \end{array} \right] (\nu, \tau, t)$, except that the screening integrations run from 0 to τ , i.e. along the b -cycle.

To complete the calculation of the transformation of the blocks under S we must now deform the contours of I back onto the contours of J . We do this one contour at a time using the same analytic continuation paths as in [11] (see fig.1 of that paper.) We now define

$$\begin{aligned} I_{l,m,\lambda}^{l''[r]} \left[\begin{array}{c} 2\lambda \\ \beta \end{array} \right] (\nu, \tau, t) &= \\ (\eta(\tau))^{-3} \prod_{i=1}^{\frac{1}{2}l''} \frac{\partial}{\partial y_i} \bigg|_{y_i=0} \int_0^\tau \prod_{i=1}^r dw_i \int_0^1 \prod_{i=r+1}^{\frac{1}{2}l''} dw_i \mathcal{E}(\{w_i\}, \{y_i\} | \tau) \Gamma_{l,m}^{2\lambda,\beta}(u_1, u_2 | \nu, \tau, t), \end{aligned} \quad (20)$$

and the idea is to find a recursion relation between the $I^{[r]}$'s which will enable us to express $I = I^{[\frac{1}{2}l']}$ in terms of $J = I^{[0]}$. For this we shall need to know the relevant analytic continuation of fractional powers of θ_1 as well as the behaviour of $\Gamma_{l,m}^{2\lambda,\beta}(u_1, u_2 | \nu, \tau, t)$ under shifts of u_1 .

The analytic continuation of fractional powers of θ_1 is given for $w = x + y\tau$, $x, y \in \mathbb{R}$, by [11]

$$\begin{aligned} \theta_1(w+1|\tau)^\alpha &= e^{i\alpha\phi_a} \theta(w|\tau)^\alpha, \text{ where } \phi_a = \begin{cases} -\pi, & 2n < y < 2n+1, \\ \pi, & 2n-1 < y < 2n; \end{cases} \\ \theta_1(w+\tau|\tau)^\alpha &= e^{i\alpha\phi_b} e^{-i\pi\alpha\tau-2\pi i\alpha w} \theta(w|\tau)^\alpha, \text{ where } \phi_b = \begin{cases} \pi, & 2m < y < 2m+1, \\ -\pi, & 2m-1 < y < 2m, \end{cases} \end{aligned} \quad (21)$$

where $n, m \in \mathbb{Z}$. Using the identity $\theta_1(-w-1|\tau) = \theta_1(w|\tau)$ one has also

$$\theta_1(-w|\tau)^\alpha = e^{i\alpha\phi_a} \theta(w|\tau)^\alpha, \text{ where } \phi_a = \begin{cases} \pi, & 2n < y < 2n+1, \\ -\pi, & 2n-1 < y < 2n. \end{cases} \quad (22)$$

The shift properties of Γ are straightforward to obtain:

$$\begin{aligned}\Gamma_{l,m}^{2\lambda,\beta}(u_1 - 2, u_2 | \nu, \tau, t) &= e^{-2\pi i \frac{l+1}{k+2}} \Gamma_{l,m}^{2\lambda,\beta}(u_1, u_2 | \nu, \tau, t) \\ \Gamma_{l,m}^{2\lambda,\beta}(u_1 - 2\tau, u_2 | \nu, \tau, t) &= e^{-2\pi i \frac{\tau}{k+2}} e^{2\pi i \frac{u_1}{k+2}} \Gamma_{l-2,m}^{2\lambda,\beta}(u_1, u_2 | \nu, \tau, t).\end{aligned}\quad (23)$$

Using Eqns. (21) and (22) one finds

$$\begin{aligned}\mathcal{E}(w_1, \dots, w_r + 1, \dots, w_{\frac{1}{2}l'}; \{y_i\} | \tau) &= e^{2\pi i \frac{2r-1}{k+2}} \mathcal{E}(\{w_i\}, \{y_i\} | \tau) \\ \mathcal{E}(w_1, \dots, w_r + \tau, \dots, w_{\frac{1}{2}l'}; \{y_i\} | \tau) &= e^{2\pi i \frac{2r-l'-1}{k+2}} e^{2\pi i \frac{\tau}{k+2}} e^{-2\pi i \frac{u_1}{k+2}} \mathcal{E}(\{w_i\}, \{y_i\} | \tau).\end{aligned}\quad (24)$$

By considering figs.2 and 3 of [11] one derives the recursion relation

$$\begin{aligned}I_{l,m,\lambda}^{[r]} \left[\begin{matrix} 2\lambda \\ \beta \end{matrix} \right] (\nu, \tau, t) &= I_{l,m,\lambda}^{[r-1]} \left[\begin{matrix} 2\lambda \\ \beta \end{matrix} \right] (\nu, \tau, t) + e^{i\pi \frac{2r-2l-2}{k+2}} I_{l,m,\lambda}^{[r]} \left[\begin{matrix} 2\lambda \\ \beta \end{matrix} \right] (\nu, \tau, t) \\ &\quad - e^{i\pi \frac{2r-l'-2}{k+2}} I_{l-2,m,\lambda}^{[r-1]} \left[\begin{matrix} 2\lambda \\ \beta \end{matrix} \right] (\nu, \tau, t),\end{aligned}\quad (25)$$

where we included in the second and third terms on the right hand side the phase factors, $e^{-i\pi \frac{2(r-1)}{k+2}}$ and $e^{+i\pi \frac{l'-2r}{k+2}}$, respectively, which arise when we deform the shifted contours to the standard ones.

We now have all we need to write down the transformation of the blocks under S . First we simplify Eq. (25) somewhat by introducing the quantities

$$i_{l-r,m,\lambda}^{[r]}(\nu, \tau, t) = i^r e^{i\pi \frac{(l+1)l'+r(r-2l-1)}{2(k+2)}} I_{l,m,\lambda}^{[r]} \left[\begin{matrix} 2\lambda \\ \beta \end{matrix} \right] (\nu, \tau, t). \quad (26)$$

Now Eq. (25) reads

$$i_{l,m,\lambda}^{[r]}(\nu, \tau, t) = \sum_j A_{lj} i_{j,m,\lambda}^{[r-1]}(\nu, \tau, t), \quad (27)$$

with the matrix A given by

$$A_{lj} = \left[2 \sin \left(\pi \frac{l+1}{k+2} \right) \right]^{-1} (\delta_{l,j-1} - \delta_{l,j+1}). \quad (28)$$

At this point we pause to verify that our explicit form for the conformal block, Eq. (5), satisfies the fusion rules, i.e. that $J_{l,m,\lambda}^{l'}$ vanishes unless $\frac{1}{2}l' \leq l \leq k - \frac{1}{2}l'$. First, by changing integration variables, $w_i \rightarrow 1 - w_{\frac{1}{2}l'-i+1}$, one may derive the identity

$$J_{-l-2,m,\lambda}^{l'} = -e^{i\pi l' \frac{l+1}{k+2}} J_{l,m,\lambda}^{l'}. \quad (29)$$

On the other hand, from Eqs. (27) and (28) follows by induction after l and $k-l$, respectively, the relations

$$\left. \begin{aligned} i_l^{[r]} &= i_{-l-2}^{[r]} \\ i_{k-l}^{[r]} &= i_{k+l+2}^{[r]} \end{aligned} \right\} \text{ for } 0 \leq r \leq \frac{1}{2}l' - l - 1. \quad (30)$$

By using Eq. (26) we obtain as a special case of Eq. (30) the identities³⁾

$$\left. \begin{aligned} J_{-l-2}^{l'} &= e^{i\pi l' \frac{l+1}{k+2}} J_l^{l'} \\ J_{-k+l-2}^{l'} &= e^{i\pi l' \frac{k-l+1}{k+2}} J_{k-l}^{l'} \end{aligned} \right\} \text{ for } l < \frac{1}{2}l'. \quad (31)$$

3) To obtain the second relation we use the periodicity of J under $l \rightarrow l + 2n(k+2)$, $n \in \mathbb{Z}$.

By combining these identities with Eq. (29) we see that the fusion rules are indeed satisfied.

By iterating Eq. (27) one finds

$$i_{l,m,\lambda}^{[\frac{1}{2}l']}(\nu, \tau, t) = \sum_j \left(A^{\frac{1}{2}l'} \right)_{lj} i_{j,m,\lambda}^{[0]}(\nu, \tau, t). \quad (32)$$

The explicit form of $A^{\frac{1}{2}l'}$ was worked out in [11]. Define

$$\begin{aligned} [j] &= 2 \sin \left(\frac{\pi j}{k+2} \right), \\ \begin{bmatrix} l \\ j \end{bmatrix} &= \frac{[l][l-1] \cdots [l-j+1]}{[j][j-1] \cdots [1]}. \end{aligned} \quad (33)$$

Then $A^{\frac{1}{2}l'}$ is given by

$$\left(A^{\frac{1}{2}l'} \right)_{\tilde{l}\tilde{l}} = (-1)^{\frac{1}{2}(\tilde{l}-\frac{1}{2}l'-\tilde{l})} \left[\frac{1}{2} \left(\hat{l} + \frac{1}{2}l' - \tilde{l} \right) \right]_{j=\frac{1}{2}(\tilde{l}+\tilde{l}-\frac{1}{2}l')+1}^{\frac{1}{2}(\hat{l}+\tilde{l}+\frac{1}{2}l')+1} [j]^{-1}, \quad (34)$$

if $|\hat{l} - \tilde{l}| \leq \frac{1}{2}l'$, $\hat{l} + \tilde{l} + \frac{1}{2}l' \in 2\mathbb{Z}$, and vanishes otherwise.

As a consequence of Eqs. (26) and (32) we may rewrite Eq. (19) as

$$\begin{aligned} J_{l,m,\lambda}^{l'} \begin{bmatrix} 2\lambda \\ \beta \end{bmatrix} \left(\frac{\nu}{\tau}, \frac{-1}{\tau}, t + \frac{k}{8(k+2)} \frac{\nu^2}{\tau} \right) &= \frac{i^{\frac{1}{2}l'+1} (-i\tau)^{h(l',0,0)}}{k+2} e^{-i\pi\beta\lambda} \\ &\cdot \sum_{\tilde{l}=0}^{2k+3} \sum_{\substack{\tilde{m}=0 \\ \tilde{m}-\tilde{l} \in 2\mathbb{Z}}}^{k+1} S_{l,m;\tilde{l},\tilde{m}} J_{\tilde{l},\tilde{m},\frac{1}{2}\beta}^{l'} \begin{bmatrix} \beta \\ 2\lambda \end{bmatrix} (\nu, \tau, t), \end{aligned} \quad (35)$$

where the sum runs over those $\tilde{l}$ for which the fusion rules are satisfied. The S -matrix is given by

$$S_{l,m;\tilde{l},\tilde{m}} = e^{i\pi \frac{(\tilde{m}-\beta)(m-2\lambda)}{k+2}} \hat{S}_{l\tilde{l}}; \quad \hat{S}_{l\tilde{l}} = \sum_{j=0}^{2k+3} e^{-i\pi \frac{(j+1)(l+1)}{k+2}} \left(A^{\frac{1}{2}l'} \right)_{j\tilde{l}}. \quad (36)$$

4 Modular covariance of one-loop one-point correlation functions

The one-point correlation functions of primary fields are⁴⁾

$$\langle \Psi_{l',0,0}(0) \rangle_\tau = \frac{1}{Z(\tau)} \sum_{l,m,\lambda,\beta} D_{l';l,m,\lambda}^{l,m,\lambda} J_{l,m,\lambda}^{l'} \begin{bmatrix} 2\lambda \\ \beta \end{bmatrix} (0, \tau, 0) J_{l,m,\lambda}^{l'} \begin{bmatrix} 2\lambda \\ \beta \end{bmatrix} (0, -\bar{\tau}, 0), \quad (37)$$

where, in fact, the D -coefficients [16] do not depend on λ and are precisely the same as in the $SU(2)_k$ WZNW model. From Appendix C we have, in the notation introduced above,

$$D_{l',0;l,m}^{l,m} = C_{l'} \prod_{\substack{j=l+1-\frac{1}{2}l' \\ j \neq l+1}}^{l+1+\frac{1}{2}l'} [j]^{-1}, \quad (38)$$

4) In fact, the proof of modular covariance goes through in exactly the same way without the specialization $\nu = t = 0$.

where the factor $C_{l'}$ is independent of l . The partition function is given by

$$Z(\tau) = C_0 \sum_{l,m,\lambda,\beta} J_{l,m,\lambda}^0 \left[\begin{matrix} 2\lambda \\ \beta \end{matrix} \right] (0, \tau, 0) J_{l,m,\lambda}^0 \left[\begin{matrix} 2\lambda \\ \beta \end{matrix} \right] (0, -\bar{\tau}, 0), \quad (39)$$

and is easily seen from Eqs. (15) and (35) to be modular *invariant*.

Modular covariance of one-point functions means that

$$\begin{aligned} \langle \Psi_{l',0,0}(0) \rangle_{\tau+1} &= \langle \Psi_{l',0,0}(0) \rangle_{\tau} \\ \langle \Psi_{l',0,0}(0) \rangle_{-1/\tau} &= |\tau|^{2h(l',0,0)} \langle \Psi_{l',0,0}(0) \rangle_{\tau}. \end{aligned} \quad (40)$$

It is easy to see using Eq. (15) that the first of these equations is satisfied. We now proceed to a proof of the second one. In matrix notation it reads $S^T D S = D$, where S is the S -matrix of section 3 and D is the diagonal matrix with matrix elements given by Eq. (38). Using the identity $S^2 = 1$ we may rewrite this matrix equation as $(S D^{-1})^T = S D^{-1}$. Since the matrix elements of D depend only on l we may replace S by $\hat{S}$ in this equation. This reduces the calculation of $S D^{-1}$ to the case considered in [11]⁵⁾ where it was shown to be symmetric. This concludes the proof of modular covariance.

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5) Or more precisely: to half the case considered there!

A Conventions and notation

Let ϕ_i , $i = 1, 2, 3$, be free scalar bosons with propagators $\phi_i(z)\phi_j(w) \sim +\log(z-w)$. Then the operators

$$\psi^\pm(z) = \frac{1}{\sqrt{2}} e^{\pm i\sqrt{\frac{2}{k}}\phi_2(z)} \left(\sqrt{\frac{k+2}{k}} \partial\phi_1(z) \pm i\partial\phi_2(z) \right) \quad (41)$$

are the generators of a Z_k PF algebra [17]. The corresponding stress-energy tensor is given by

$$T_{PF}(z) = \frac{1}{2} [(\partial\phi_1(z))^2 + (\partial\phi_2(z))^2] - \frac{1}{\sqrt{2(k+2)}} \partial^2\phi_1(z). \quad (42)$$

The principal fields of the PF theory are

$$\phi_m^l(z) = \left(\frac{l}{\frac{l-m}{2}} \right)^{1/2} e^{\frac{l}{\sqrt{2(k+2)}}\phi_1(z) + \frac{i m}{\sqrt{2k}}\phi_2(z)}, \quad -l \leq m \leq l, \quad (43)$$

where $\left(\frac{l}{\frac{l-m}{2}} \right)$ is a binomial coefficient⁶⁾. The operator product expansions of the screening operator

$$S(z) = -\frac{1}{\sqrt{2}} \left(\sqrt{k+2} \partial\phi_1(z) + i\sqrt{k} \partial\phi_2(z) \right) e^{-\sqrt{\frac{2}{k+2}}\phi_1(z)} \quad (44)$$

with the PF generators and stress-energy tensor are total derivatives.

The generators of the $N = 2$ superconformal algebra are

$$\begin{aligned} J(z) &= \sqrt{\frac{k}{k+2}} \partial\phi_0(z) \\ G^\pm(z) &= \sqrt{\frac{2k}{k+2}} e^{\pm\sqrt{\frac{k+2}{k}}\phi_0(z)} \psi^\pm(z) \\ T(z) &= T_{PF}(z) + \frac{1}{2} (\partial\phi_0(z))^2 \end{aligned} \quad (45)$$

and the irreducible unitary highest weight representations in the NS and R sectors are parametrized by two integers l and m , satisfying $0 \leq l \leq k$, $m = -l, -l+2, \dots, l$, and by λ which is zero ($\pm\frac{1}{2}$) in the NS (R) sector. We denote these representations by $\mathcal{H}_{N=2}^{l,m,\lambda}$. The highest weight states are characterized by their conformal weight and $U(1)$ charge given by [9]

$$h(l, m, \lambda) = \frac{l(l+2)}{4(k+2)} - \frac{m^2}{4k} + \frac{(m+\lambda k)^2}{2k(k+2)} \quad (46)$$

and

$$q(m, \lambda) = \frac{m+\lambda k}{k+2}, \quad (47)$$

respectively. The primary fields are

$$\Psi_{l,m,\lambda}(z) = \frac{1}{\sqrt{2}} e^{\frac{m+\lambda k}{\sqrt{k(k+2)}}\phi_0(z)} \phi_m^l(z). \quad (48)$$

6) This normalization was left out in [1, 2] for notational convenience.

A representative of the chiral vertex operator $\Phi_{l,m,\lambda;l_a,m_a,\lambda_a}^{l_b,m_b,\lambda_b} : \mathcal{H}_{N=2}^{l_a,m_a,\lambda_a} \rightarrow \mathcal{H}_{N=2}^{l_b,m_b,\lambda_b}$ is given for $r = (l_a + l - l_b)/2$ by the screened vertex operator

$$\Psi_{l,m,\lambda}^{(r)}(z) = \oint_{\mathcal{P}^r} \Psi_{l,m,\lambda}(z) S(v_1) \cdots S(v_r). \quad (49)$$

The integration paths, $\mathcal{P}^r$, are in general as in ref.[2], but for the purposes of computing one-loop conformal blocks we may take them to be just the standard nested contours $\mathcal{C}^r$ of [10]. Further, by explicit evaluation one finds that the screened vertex operators are consistently normalizable to the chiral vertex operators precisely when these are allowed by the fusion rules. I.e. the normalizations

$$\mathcal{N}_{l_b,m_b,\lambda_b;l_a,m_a,\lambda_a}^{l_c,m_c,\lambda_c} = \langle l_c, m_c, \lambda_c | \Psi_{l_b,m_b,\lambda_b}^{(r)}(1) | l_a, m_a, \lambda_a \rangle \quad (50)$$

are non-zero (and finite) iff the fusion rules are satisfied.

B A modular transformation property of $\hat{\Gamma}_{l,m}^{\epsilon_1,\epsilon_2}(u_1, u_2|w, \tau)$

In this appendix we show that

$$\begin{aligned} \hat{\Gamma}_{l,m}^{\epsilon_1,\epsilon_2}\left(\frac{u_1}{\tau}, \frac{u_2}{\tau} \middle| \frac{w}{\tau}, \frac{-1}{\tau}\right) &= \frac{\tau}{k+2} q'^{\frac{-u_1^2}{4(k+2)}} q'^{\frac{1}{4(k+2)}(m-\epsilon_1-\frac{k+2}{k}u_2+w)^2} \\ &\times \sum_{\tilde{l}=0}^{2k+3} \sum_{\substack{\tilde{m}=0 \\ \tilde{m}-\tilde{l} \in 2\mathbb{Z}}}^{k+1} e^{i\pi \frac{(\tilde{m}-\epsilon_2)(m-\epsilon_1)}{k+2} - i\pi \frac{(\tilde{l}+1)(l+1)}{k+2} + 2\pi i(\tilde{m}-\epsilon_2)(\frac{w}{2(k+2)} - \frac{u_2}{2k})} \\ &\times q^{-\frac{(\tilde{m}-\epsilon_2)^2}{4(k+2)}} \hat{\Gamma}_{\tilde{l},\tilde{m}}^{\epsilon_2,-\epsilon_1}(u_1, u_2|w, \tau). \end{aligned} \quad (51)$$

Define the function

$$\begin{aligned} f(x) &= q'^{(k+2)(x+\frac{l+1}{2(k+2)})^2} e^{2\pi i \frac{u_1}{\tau}(x+\frac{l+1}{2(k+2)})} \\ &\times \left[\left(1 + q'^{(k+2)x+\frac{l+1-m+\epsilon_1}{2}} e^{2\pi i \left\{ \frac{1}{2} \frac{u_1}{\tau} - \frac{k+2}{2k} \frac{u_2}{\tau} + \frac{1}{2} \frac{w}{\tau} + \frac{1}{2} \epsilon_2 \right\}} \right)^{-1} \right. \\ &\left. - \left(1 + q'^{-(k+2)x+\frac{-l-1-m+\epsilon_1}{2}} e^{2\pi i \left\{ -\frac{1}{2} \frac{u_1}{\tau} - \frac{k+2}{2k} \frac{u_2}{\tau} + \frac{1}{2} \frac{w}{\tau} + \frac{1}{2} \epsilon_2 \right\}} \right)^{-1} \right], \end{aligned} \quad (52)$$

where $q' = e^{-\frac{2\pi i}{\tau}}$. Then by Poisson summation one has [14]

$$\hat{\Gamma}_{l,m}^{\epsilon_1,\epsilon_2}\left(\frac{u_1}{\tau}, \frac{u_2}{\tau} \middle| \frac{w}{\tau}, \frac{-1}{\tau}\right) = \sum_{n' \in \mathbb{Z}} f(n') = \sum_{n \in \mathbb{Z}} \tilde{f}(n), \quad (53)$$

where

$$\tilde{f}(y) = \int_{-\infty}^{\infty} dx f(x) e^{-2\pi i x y} \quad (54)$$

is the Fourier transform of f . We now proceed to find $\tilde{f}(y)$ for integer argument.

By changing variable, $x \rightarrow x + \frac{l+1}{2(k+2)} - \frac{u_1}{2(k+2)} + \frac{\tau}{2(k+2)}n$, one obtains

$$\begin{aligned} \tilde{f}(n) &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x} \\ &= e^{2\pi i \frac{l+1}{2(k+2)}n} q'^{-(k+2)(\frac{u_1}{2(k+2)} - \frac{\tau}{2(k+2)}n)^2} \int_{-\infty-i\frac{n\tau_2+\text{Im } u_1}{2(k+2)}}^{\infty-i\frac{n\tau_2+\text{Im } u_1}{2(k+2)}} dx q'^{(k+2)x^2} \\ &\times \left[\left(1 + q'^{(k+2)x+W} e^{i\pi(n+\epsilon_2)} \right)^{-1} - \left(1 + q'^{-(k+2)x+W} e^{i\pi(n+\epsilon_2)} \right)^{-1} \right], \end{aligned} \quad (55)$$

where we introduced the abbreviation $W = \frac{1}{2} \left(-m + \epsilon_1 + \frac{k+2}{k} u_2 - w \right)$.

Now we notice that the integrand is odd under $x \rightarrow -x$. This implies that we can add the real line (with either orientation) to the integration domain without changing the value of the integral. Due to the presence of the gaussian factor we may further close off the combined contour at $\pm\infty$ thus obtaining a closed integration contour [7]. Notice that for $\frac{n\tau_2 + \text{Im } u_1}{2(k+2)} > 0$ (< 0), the integration contour has negative (positive) orientation. Clearly, the integrand has only simple poles and these are situated at

$$x = \pm X_p = \pm \frac{\tau}{2(k+2)}(n + \epsilon_2 + 2p + 1) \mp \frac{W}{k+2} \quad , \quad (56)$$

for $p \in \mathbb{Z}$. The residues at $x = \pm X_p$ are given by

$$\frac{\tau}{2\pi i(k+2)} q^{(k+2)X_p^2} \quad (57)$$

Thus for each n in the summation of Eq. (53) we have to sum p over the four regions

$$\begin{aligned} I : n + \frac{\text{Im } u_1}{\tau_2} &< n + \epsilon_2 + 2p + 1 - \frac{\text{Im } W}{\tau_2} < 0 \\ II : n + \frac{\text{Im } u_1}{\tau_2} &< -(n + \epsilon_2 + 2p + 1) + \frac{\text{Im } W}{\tau_2} < 0 \\ III : n + \frac{\text{Im } u_1}{\tau_2} &> n + \epsilon_2 + 2p + 1 - \frac{\text{Im } W}{\tau_2} > 0 \\ IV : n + \frac{\text{Im } u_1}{\tau_2} &> -(n + \epsilon_2 + 2p + 1) + \frac{\text{Im } W}{\tau_2} > 0 \end{aligned} \quad (58)$$

and the contributions from regions I&II (III&IV) come with positive (negative) sign.

We now define new summation variables, n' and s , in the four regions:

$$\left. \begin{aligned} I : n &= -(k+2)(2n' + s) - \tilde{l} - 1 \quad ; \quad n + 2p + 1 = -(k+2)s - \tilde{m} \\ II : n &= -(k+2)(2n' - s) - \tilde{l} - 1 \quad ; \quad n + 2p + 1 = -(k+2)s - \tilde{m} \\ III : n &= -(k+2)(2n' + s) - \tilde{l} - 1 \quad ; \quad n + 2p + 1 = -(k+2)s - \tilde{m} \\ IV : n &= -(k+2)(2n' - s) - \tilde{l} - 1 \quad ; \quad n + 2p + 1 = -(k+2)s - \tilde{m} \end{aligned} \right\} \begin{aligned} 0 &\leq \tilde{l} \leq 2k+3 \\ 0 &\leq \tilde{m} \leq k+1 \\ \tilde{l} - \tilde{m} &\in 2\mathbb{Z} \end{aligned} \quad (59)$$

In terms of these variables the four regions are given by

$$\begin{aligned} I : n' &> \frac{-\tilde{l} - 1 + \tilde{m} - \epsilon_2}{2(k+2)} + \frac{\text{Im}(u_1 + W)}{2\tau_2(k+2)} \quad ; \quad s > \frac{-\tilde{m} + \epsilon_2}{k+2} - \frac{\text{Im } W}{(k+2)\tau_2} \\ II : n' &> \frac{-\tilde{l} - 1 - \tilde{m} + \epsilon_2}{2(k+2)} + \frac{\text{Im}(u_1 - W)}{2\tau_2(k+2)} \quad ; \quad s < \frac{-\tilde{m} + \epsilon_2}{k+2} - \frac{\text{Im } W}{(k+2)\tau_2} \\ III : n' &< \frac{-\tilde{l} - 1 + \tilde{m} - \epsilon_2}{2(k+2)} + \frac{\text{Im}(u_1 + W)}{2\tau_2(k+2)} \quad ; \quad s < \frac{-\tilde{m} + \epsilon_2}{k+2} - \frac{\text{Im } W}{(k+2)\tau_2} \\ IV : n' &< \frac{-\tilde{l} - 1 - \tilde{m} + \epsilon_2}{2(k+2)} + \frac{\text{Im}(u_1 - W)}{2\tau_2(k+2)} \quad ; \quad s > \frac{-\tilde{m} + \epsilon_2}{k+2} - \frac{\text{Im } W}{(k+2)\tau_2} \end{aligned} \quad (60)$$

These regions look very much like the four summation regions used in Eq. (10), but the restrictions on s are different. However, since we may take u_2 arbitrarily small and since we are only interested in values of w such that $0 \leq \text{Im } w \leq \tau_2$, we see that we may assume

$0 < -\frac{\text{Im } W}{(k+2)\tau_2} < \frac{1}{2(k+2)}$. Thus we always get one of the four “nice” summation ranges: $s > 0$, $s \geq 0$, $s < 0$, or $s \leq 0$. Which one we get depends on the value of $\tilde{m} + \epsilon_2$.

The summands are (using $l - m \in 2\mathbb{Z}$)

$$\begin{aligned}
I\&III : & \frac{\tau}{k+2} q'^{\frac{-u_1^2}{4(k+2)}} q'^{\frac{1}{4(k+2)}} (m - \epsilon_1 - \frac{k+2}{k} u_2 + w)^2 e^{i\pi \frac{(\tilde{m} - \epsilon_2)(m - \epsilon_1)}{k+2} - i\pi \frac{(\tilde{l}+1)(l+1)}{k+2}} \\
& e^{2\pi i (\tilde{m} - \epsilon_2) (\frac{w}{2(k+2)} - \frac{u_2}{2k})} (-1)^s q^{-\frac{(\tilde{m} - \epsilon_2)^2}{4(k+2)}} q^{(k+2) \left(n' + \frac{\tilde{l}+1}{2(k+2)} \right)^2} e^{2\pi i u_1 \left(n' + \frac{\tilde{l}+1}{2(k+2)} \right)} \\
& q^{\left((k+2)n' + \frac{\tilde{l}+1 - \tilde{m} + \epsilon_2}{2} \right) s} e^{2\pi i \left(\frac{1}{2} u_1 - \frac{k+2}{2k} u_2 + \frac{1}{2} w - \frac{1}{2} \epsilon_1 \right) s} \\
II\&IV : & \frac{\tau}{k+2} q'^{\frac{-u_1^2}{4(k+2)}} q'^{\frac{1}{4(k+2)}} (m - \epsilon_1 - \frac{k+2}{k} u_2 + w)^2 e^{i\pi \frac{(\tilde{m} - \epsilon_2)(m - \epsilon_1)}{k+2} - i\pi \frac{(\tilde{l}+1)(l+1)}{k+2}} \\
& e^{2\pi i (\tilde{m} - \epsilon_2) (\frac{w}{2(k+2)} - \frac{u_2}{2k})} (-1)^s q^{-\frac{(\tilde{m} - \epsilon_2)^2}{4(k+2)}} q^{(k+2) \left(n' + \frac{\tilde{l}+1}{2(k+2)} \right)^2} e^{2\pi i u_1 \left(n' + \frac{\tilde{l}+1}{2(k+2)} \right)} \\
& q^{\left(-(k+2)n' + \frac{-\tilde{l} - 1 - \tilde{m} + \epsilon_2}{2} \right) s} e^{2\pi i \left(-\frac{1}{2} u_1 - \frac{k+2}{2k} u_2 + \frac{1}{2} w - \frac{1}{2} \epsilon_1 \right) s}. \tag{61}
\end{aligned}$$

Thus the four sums are all absolutely convergent and we may do the summation over s explicitly.

Now, remembering the overall minus signs on the contributions from regions III and IV, we obtain Eq. (51).

C The D -coefficients

In the diagonal (i.e. left-right symmetric) models the D -coefficients of the $\widehat{SU}(2)_k$ model are defined by

$$C \begin{pmatrix} l_a & m_a \\ l_b & m_b \\ l_c & m_c \end{pmatrix} = D_{l_b, m_b; l_a, m_a}^{l_c, m_c} \left| \mathcal{N}_{l_b, m_b; l_a, m_a}^{l_c, m_c} \right|^2, \tag{62}$$

where the left hand side are the structure constants and the $\mathcal{N}$'s are the normalizations of the screened vertex operators, i.e.

$$\mathcal{N}_{l_b, m_b; l_a, m_a}^{l_c, m_c} = \langle l_c, m_c | \Psi^{\left(\frac{1}{2}(l_a + l_b - l_c)\right)}(1) | l_a, m_a \rangle. \tag{63}$$

As in the case of the $N = 0$ minimal models [10] these normalizations may be computed explicitly using the methods described in the appendices of [15]. However, due to the fact that in the present case the screening operators are not pure exponentials the computation becomes in general rather tedious. We shall therefore use the fact that the normalizations are proportional to Wigner $3j$ -symbols. This is easily seen by inserting in Eq. (63) the operators $J^\pm(z)$ and integrating z along a contour encircling the point $z = 1$. In this way one obtains an identity between the result obtained by doing the operator product expansion at $z = 1$ and the result obtained by first deforming the contour to a sum of two contours encircling $z = 0, \infty$ and then doing the operator product expansion at those points. The ensuing recursion relation reads

$$\begin{aligned}
\sqrt{(l_c \pm m_c)(l_c \mp m_c + 2)} \mathcal{N}_{l_b, m_b; l_a, m_a}^{l_c, m_c \mp 2} &= \sqrt{(l_a \mp m_a)(l_a \pm m_a + 2)} \mathcal{N}_{l_b, m_b; l_a, m_a \pm 2}^{l_c, m_c} \\
&+ \sqrt{(l_b \mp m_b)(l_b \pm m_b + 2)} \mathcal{N}_{l_b, m_b \pm 2; l_a, m_a}^{l_c, m_c}, \tag{64}
\end{aligned}$$

showing that up to an overall (i.e. m -independent) normalization the $\mathcal{N}$'s are just Wigner $3j$ -symbols:

$$\mathcal{N}_{l_b, m_b; l_a, m_a}^{l_c, m_c} = K(l_a, l_b, l_c) \times \begin{pmatrix} \frac{l_a}{2} & \frac{l_b}{2} & \frac{l_c}{2} \\ \frac{m_a}{2} & \frac{m_b}{2} & \frac{m_c}{2} \end{pmatrix}. \quad (65)$$

To determine the overall normalization K we compute $\mathcal{N}_{l_b, m_b; l_a, m_a}^{l_c, m_c}$, since in this case the “unscreened” vertex operator decouples from the screening operators. From [2] we have⁷⁾

$$\begin{aligned} \mathcal{N}_{l_b, m_b; l_a, m_a}^{l_c, m_c} &= \begin{pmatrix} l_c \\ \frac{l_c - m_c}{2} \end{pmatrix}^{-1/2} \begin{pmatrix} l_a \\ \frac{l_a - m_a}{2} \end{pmatrix}^{1/2} e^{\frac{2\pi i}{k+2} \frac{(l_a + l_b - l_c)(l_b - 2 - 3l_a - l_c)}{8}} \\ &\cdot \prod_{j=1}^{\frac{1}{2}(l_a + l_b - l_c)} \left[\left(\frac{m_a - l_a}{2} + j - 1 \right) \frac{(1 - e^{\frac{2\pi i}{k+2} j})(1 - e^{\frac{2\pi i}{k+2} (l_a + 1 - j)})}{1 - e^{\frac{2\pi i}{k+2}}} \right. \\ &\cdot \left. \frac{\Gamma\left(\frac{j}{k+2}\right) \Gamma\left(\frac{j-1-l_a}{k+2}\right) \Gamma\left(\frac{j-1-l_b}{k+2} + 1\right)}{\Gamma\left(\frac{1}{k+2}\right) \Gamma\left(\frac{j-2-\frac{1}{2}(l_a + l_b + l_c)}{k+2} + 1\right)} \right]. \end{aligned} \quad (66)$$

One has for the $3j$ -symbol

$$\begin{pmatrix} \frac{l_a}{2} & \frac{l_b}{2} & \frac{l_c}{2} \\ -\frac{l_a}{2} & \frac{l_b}{2} & \frac{l_a - l_b}{2} \end{pmatrix} = \left[\frac{l_a! l_b!}{\left(\frac{l_a + l_b + l_c}{2} + 1\right)! \left(\frac{l_a + l_b - l_c}{2}\right)!} \right]^{1/2} \quad (67)$$

and thus

$$\begin{aligned} K(l_a, l_b, l_c) &= (-1)^{\frac{l_a + l_b - l_c}{2}} \left[\frac{l_a! \left(\frac{l_c + l_b - l_a}{2}\right)! \left(\frac{l_c + l_b + l_a}{2} + 1\right)! \left(\frac{l_a + l_b - l_c}{2}\right)!}{l_b! l_c! \left(\frac{l_c - l_b + l_a}{2}\right)!} \right]^{1/2} \\ &\cdot e^{\frac{2\pi i}{k+2} \frac{(l_a + l_b - l_c)(l_b - 2 - 3l_a - l_c)}{8}} \prod_{j=1}^{\frac{1}{2}(l_a + l_b - l_c)} \left[\frac{(1 - e^{\frac{2\pi i}{k+2} j})(1 - e^{\frac{2\pi i}{k+2} (l_a + 1 - j)})}{1 - e^{\frac{2\pi i}{k+2}}} \right. \\ &\cdot \left. \frac{\Gamma\left(\frac{j}{k+2}\right) \Gamma\left(\frac{j-1-l_a}{k+2}\right) \Gamma\left(\frac{j-1-l_b}{k+2} + 1\right)}{\Gamma\left(\frac{1}{k+2}\right) \Gamma\left(\frac{j-2-\frac{1}{2}(l_a + l_b + l_c)}{k+2} + 1\right)} \right]. \end{aligned} \quad (68)$$

Now that we know the general expression for the normalizations we can compute the D -coefficients directly from Eq. (62) by plugging in the structure constants [3]

$$C \begin{pmatrix} l_a & m_a \\ l_b & m_b \\ l_c & m_c \end{pmatrix} = \begin{pmatrix} \frac{l_a}{2} & \frac{l_b}{2} & \frac{l_c}{2} \\ \frac{m_a}{2} & \frac{m_b}{2} & \frac{m_c}{2} \end{pmatrix}^2 \rho\left(\frac{l_a}{2}, \frac{l_b}{2}, \frac{l_c}{2}\right), \quad (69)$$

with

$$\begin{aligned} \frac{\rho^2}{(l_a + 1)(l_b + 1)(l_c + 1)} &= \frac{\Gamma\left(\frac{k+3}{k+2}\right)}{\Gamma\left(\frac{k+1}{k+2}\right)} \prod_{j=abc} \frac{\Gamma\left(1 - \frac{l_j + 1}{k+2}\right)}{\Gamma\left(1 + \frac{l_j + 1}{k+2}\right)} \\ &\cdot \tilde{P}^2\left(\frac{l_a + l_b + l_c}{2} + 1\right) \prod_{j=abc} \frac{\tilde{P}^2\left(\frac{l_a + l_b + l_c}{2} - l_j\right)}{\tilde{P}^2(l_j)}, \end{aligned} \quad (70)$$

⁷⁾ We take this opportunity to correct a misprint in Eq. (25) of [2].

where

$$\tilde{P}(l) = \prod_{j=1}^l \frac{\Gamma\left(1 + \frac{j}{k+2}\right)}{\Gamma\left(1 - \frac{j}{k+2}\right)}. \quad (71)$$

We shall not need the general D -coefficients but only $D_{l',0;l,m}^{l,m}$. After some simplifications involving elementary properties of the Γ function one has up to an overall l -independent factor

$$\rho\left(\frac{l}{2}, \frac{l'}{2}, \frac{l}{2}\right) \propto \frac{(l + \frac{1}{2}l' + 1)!}{(l - \frac{1}{2}l')!} \prod_{j=1}^{\frac{1}{2}l'} \frac{\sin\left(\pi \frac{l+1-j}{k+2}\right)}{\sin\left(\pi \frac{l+2+\frac{1}{2}l'-j}{k+2}\right)} \frac{\Gamma^2\left(\frac{j-1-l}{k+2}\right)}{\Gamma^2\left(\frac{j-2-l-\frac{1}{2}l'}{k+2} + 1\right)}. \quad (72)$$

By combining what we have found sofar in this appendix one may obtain

$$\begin{aligned} D_{l',0;l,m}^{l,m} &= C_{l'} \prod_{j=1}^{\frac{1}{2}l'} \left[\sin\left(\pi \frac{l+1-j}{k+2}\right) \sin\left(\pi \frac{l+2+\frac{1}{2}l'-j}{k+2}\right) \right]^{-1} \\ &= C_{l'} \prod_{\substack{j=l+1-\frac{1}{2}l' \\ j \neq l+1}}^{l+1+\frac{1}{2}l'} \left[\sin \frac{\pi j}{k+2} \right]^{-1}, \end{aligned} \quad (73)$$

where $C_{l'}$ is an l -independent factor. We notice that these D -coefficients are very similar to the ones of the $N = 0$ diagonal minimal models [11].

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