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**ON LAGRANGIAN TWO DIMENSIONAL
CONFORMAL MODELS.**

by

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Introduction

For a few years now, there has been a revival of interest in the study of special classes of low dimensional field theories and relationships between them. It is fair to say that quantization schemes of a geometric type, typically, quantization over moduli spaces^[1], have been widely preferred to more conventional schemes strongly grounded on some locality principle. Among those one should distinguish the axiomatic schemes where positivity plays an important role but which, so far are unable to accommodate non trivial geometry^[2]. An intermediate road relies on renormalized perturbation theory where the locality principle allows to master the "infinities" which appear as a consequence of locality, and eliminate them in a canonical way modulo ambiguities which span a finite dimensional vector space of distributions of a given order with support on the diagonal of the cartesian product of n copies of a smooth manifold. These appear in a precise formulation where "infinities" are replaced by the ambiguities occurring upon extending distributions^[3]. The results of such constructions are formal power series which, however, can often be summed up in the favourable cases offered by low dimensional field theories of a geometrical type. The only excuse for pursuing this direction, in which there remains much to be done, is that it still offers most of the imperfect understanding we have in gauge theories.

In these notes, we shall work within the euclidean framework, which is best suited for geometrical delicacies, any return to Minkowskian configurations being left over at the level of observables (not of fields). The semiclassical view from which this follows relies on the study of simple Lagrangians, a state of affairs which is by no means satisfactory at a conceptual level.

We shall limit ourselves to the simplest cases of two dimensional conformal models namely, free models, and most of the time we shall neglect "trivial complications due to spin".

Before describing the contents of these lectures we should like to apologize to the numerous authors which have contributed to the bulk of the "non-perturbative" treatments of conformal models and their neighbours, of which we have quoted very few in ref. [1] and [2] as representatives of the main trend.

In Section II we shall describe at the classical level the main geometrical properties of two dimensional Lagrangian conformal models and illustrate them on simple examples.

In Section III we shall describe the main features of the short range properties of a perturbative quantization à la Feynman, in other words we will describe the quantization in infinite euclidean space.

In Section IV we will extend some of the results of Section III when infinite euclidean space is replaced by a compact Riemann surface without boundary.

An appendix is devoted to some technical details repeatedly used in the anomaly calculations.

Much of the material presented here is scattered in the published literature and in other sets of lecture notes, but some of the details may help the reader to find his way.

II. Conformal Two Dimensional Lagrangian Models.

"Euclidean Space time" is a smooth two dimensional manifold Σ . Usually a classical action will involve a smooth riemannian metric on Σ and smooth tensor fields or spinor fields if Σ is endowed with a spin structure. It will describe a conformal model if it only depends on a conformal class of metrics. Thus it can be expressed in terms of local complex analytic coordinates Z and their complex conjugates $\bar{Z}$.

Examples : i) the free massless scalar field

$$S_1 = \int_{\Sigma} \frac{1}{2i} \bar{\partial}\phi \wedge \partial\phi \quad (1)$$

$\phi(Z, \bar{Z})$ is smooth;

$$\bar{\partial}\phi = \partial_{\bar{Z}} \phi d\bar{Z} , \quad \partial\phi = \partial_Z \phi dZ \quad (2)$$

ii) the free chiral spin j field

$$S_2 = \int_{\Sigma} b_{1-j} \bar{\partial} c_j \wedge \frac{dZ}{2i} \quad (3)$$

c_j is the coefficient of the j differential $c_j(Z, \bar{Z}) (dZ)^j$; b_{1-j} is the coefficient of the $1-j$ differential $b_{1-j}(Z, \bar{Z}) (dZ)^{1-j}$.

The dependence on the conformal class of metrics

$$ds^2 = \rho_{Z\bar{Z}} dZ d\bar{Z} \quad (4)$$

for which Z is isothermal can be made explicit through the Beltrami parametrization

$$ds^2 = \rho_{z\bar{z}} |dz + \mu d\bar{z}|^2, \quad |\mu| < 1, \quad \rho > 0 \quad (5)$$

$$dZ = \lambda (dz + \mu d\bar{z}), \quad \rho_{z\bar{z}} = \frac{\rho_{z\bar{z}}}{\lambda \bar{\lambda}} \quad (6)$$

λ is the integrating factor ($= \partial_z Z$) fulfilling :

$$d^2 = 0 \Leftrightarrow (\bar{\partial} - \mu \partial)\lambda = \lambda \partial\mu \quad (7)$$

From now on $\partial, \bar{\partial}$ will denote derivatives or differentials according to the context and small $z, \bar{z}$ variables will be omitted.

Thus we find

$$S_1 = \int \frac{d\bar{z} \wedge dz}{2i} (\bar{\partial} - \mu \partial) \varphi \frac{1}{1 - \mu \bar{\mu}} (\partial - \bar{\mu} \bar{\partial}) \varphi \quad (8)$$

$$S_2 = \int \frac{d\bar{z} \wedge dz}{2i} \beta_{1-j} (\bar{\partial} - \mu \partial - j \partial \mu) \gamma_j \quad (9)$$

with
$$c_j = \frac{\gamma_j}{\lambda^j}, \quad b_{1-j} = \frac{\beta_{1-j}}{\lambda^{1-j}} \quad (10)$$

By construction $S_{1(2)}$ is invariant under the action of $\text{Diff. } \Sigma$, whose Lie algebra is the opposite of the Lie algebra of vector fields^[4] : $(c, \partial) = c\partial + \bar{c}\bar{\partial}$. At the infinitesimal level this invariance is given by

$$s S = 0 \quad (11)$$

with

$$\begin{aligned}
s\mu &= \bar{\partial} C + C\partial\mu - \mu\partial C \\
SC &= C\partial C \\
s\gamma &= (c.\partial)\gamma + j(\partial c + \mu\partial\bar{c})\gamma \\
s\beta &= (c.\partial)\beta + (1-j)(\partial c + \mu\partial\bar{c})\beta \quad \text{and c.c.}
\end{aligned} \tag{12}$$

where

$$C = c + \mu\bar{c} \tag{13}$$

s is the standard differential for the complex of cochains of Lie Diff. Σ with values functional of μ, β, γ , etc... The structure equation for Lie Diff. Σ reads

$$sc = (c.\partial)c \tag{14}$$

Of course, one has

$$s^2 = 0 \tag{15}$$

The action of diffeomorphisms on μ, β, γ can be computed in a standard way, but, remarkably enough they are local at the infinitesimal level (the only non local object in this construction is the integrating factor λ which disappears from the final formulae).

The "chiral splitting" of $(C, \mu), (\bar{C}, \bar{\mu})$ reflects the complex analytic structure of Diff. Σ associated with the complex analytic structure on Σ defined by the Beltrami differential μ ^[5]. The partial loss of this chiral splitting (with respect to $C, \bar{C}$) in the variations of β, γ is not well understood at present. The free field actions S_2 have another invariance σ , with^[4]

$$\begin{aligned}
\sigma\mu &= s\mu \quad \sigma C = sC \\
\sigma\gamma &= C\partial\gamma + j\gamma\partial C \\
\sigma\beta &= C\partial\beta + (1-j)\beta\partial C
\end{aligned} \tag{16}$$

but this is very special to the free field situation.

Note also that chiral splitting is obscured in the scalar case even if one would write it in first order form :

$$S'_1 = \int \frac{d\bar{z} \wedge dz}{2i} \pi (\bar{\partial} - \mu \partial) \varphi + \text{c.c.} - \int \frac{d\bar{z} \wedge dz}{2i} \bar{\pi} \pi (1 - \mu \bar{\mu}) \quad (17)$$

obtained as usual by transformation from the $(Z, \bar{Z})$ to the $(z, \bar{z})$ variables. We shall however see that chiral splitting will emerge at the quantized level when it should.

III. The short range properties of the perturbative expansion and its resummation^[4].

In this section we take $\Sigma = \mathbb{C}$ which is to say we quantize our models in infinite euclidean space. We have the situation of free quantized fields in an external classical field $(\mu, \bar{\mu})$. The goal is to define the quantum extension Γ of the classical action :

$$\Gamma = S_{cl} + \sum_{n=1}^{\infty} \hbar^n \Gamma^{(n)} \quad (18)$$

The arguments of these functionals are those of S_{cl} . Usually the summation over n builds up a formal power series in $\hbar$, but here, this series is truncated at $n=1$. The vacuum functional is obtained by putting the quantized fields to zero. It is the generating functional for the correlation functions of the two components

$$\Theta = \frac{\delta S}{\delta \mu} \Big|_{\mu=\bar{\mu}=0}, \quad \bar{\Theta} = \frac{\delta S}{\delta \bar{\mu}} \Big|_{\mu=\bar{\mu}=0} \quad (19)$$

of the stress energy tensor :

$$\Gamma(\mu, \bar{\mu}) = \sum_{n, \bar{n}} \frac{1}{n! \bar{n}!} \int d^2 z_1 \dots d^2 z_n d^2 z_{\bar{1}} \dots d^2 z_{\bar{n}} \langle \Theta(1) \dots \bar{\Theta}(\bar{n}) \rangle^c \mu(1) \dots \bar{\mu}(\bar{n}) \quad (20)$$

where

$$d^2z = \frac{dz \wedge d\bar{z}}{2i}, \quad \Theta(j) = \Theta(z_j, \bar{z}_j) \quad (21)$$

$$(\text{resp. } \mu(j) = \mu(z_j, \bar{z}_j)).$$

The connected correlation functions $\langle \Theta(1) \dots \bar{\Theta}(\bar{n}) \rangle^c$ have to be defined as distribution kernels since they have to be tested with the smooth function $\mu(1) \dots \bar{\mu}(\bar{n})$.

An easy and well known calculation shows for instance that

$$\langle \Theta(1) \Theta(2) \rangle = \frac{\kappa}{(z_1 - z_2)^4} \quad \text{for } 1 \neq 2 \quad (22)$$

where the coefficient κ is proportional to the so-called central charge of the model. We also know that Θ has canonical dimension 2 so that all correlation functions "diverge quadratically". This is to say they are recursively computable outside the diagonal $D_{n\bar{n}} = \{1 = \dots = \bar{n}\}$ and have to be extended to all of $\mathbb{C}^{n+\bar{n}}$. In the case of the two point function it is trivial to find all extensions :

$$\langle \Theta(1) \Theta(2) \rangle = \frac{-\kappa}{6} \partial_{z_1}^3 \frac{1}{(z_1 - z_2)} + P_2(\partial) \delta^2(1,2) \quad (23)$$

where P_2 is a second degree polynomial and δ^2 the Dirac measure. For higher point functions the extension is more difficult.

The ambiguity of the renormalization can however be completely fixed by imposing the Ward identity which follows from Diffeomorphism invariance : if there were no anomaly one would expect from Eq.(12)

$$(\bar{\partial} - \mu \partial - 2\partial\mu) \frac{\delta\Gamma}{\delta\mu} = 0 \quad \text{and c.c.} \quad (24)$$

But there is indeed an anomaly produced by the renormalization of the two point function, and the theorem^[4] is that there is a unique renormalization fulfilling

$$(\bar{\partial} - \mu\partial - 2\partial\mu) \frac{\delta\Gamma}{\delta\mu} = \kappa \partial^3 \mu \quad \text{and c.c.} \quad (25)$$

which furthermore is chirally split :

$$\Gamma(\mu, \bar{\mu}) = \Gamma(\mu, 0) + \Gamma(0, \bar{\mu}) \quad (26)$$

Furthermore, one can formulate a conjecture supported by checks up to $O(\mu^4)$ ^[4], namely, $\Gamma(\mu, 0) \stackrel{\text{Def}}{=} \Gamma(\mu)$ is summed up by the Polyakov formula^[6]

$$\Gamma_{\text{Polyakov}}(\mu) = \frac{\kappa}{2} \int d^2z \mu \partial^2 \ln \partial Z \quad (27)$$

where Z is the solution of the Beltrami equation

$$(\bar{\partial} - \mu\partial)Z = 0 \quad (28)$$

which goes to z for $|z|$ large, as long as μ has compact support. The corresponding combinatorics is similar to but different from that of gauge theories^[7].

It is interesting to note the relationship^[6]

$$\frac{\delta\Gamma}{\delta\mu} = 2\kappa S(Z; z) \quad (29)$$

where $S(Z; z)$ is the Schwartzian derivative

$$S(Z; z) = \frac{\partial^3 Z}{\partial Z^3} - \frac{3}{2} \left(\frac{\partial^2 Z}{\partial Z^2} \right)^2 \quad (30)$$

One can show^[8] that the Polyakov action can be obtained from the Wess Zumino action corresponding to the Diffeomorphism anomaly $\partial^3 \mu$,

$$\Gamma_{wz}(\varphi, \mu) = \kappa \int_{\mathbb{C} \times [0,1]} \Omega_t \partial^3 \mu^{\varphi_t} d^2 z, \quad (31)$$

where φ_t is a family of diffeomorphisms $0 \leq t \leq 1$, $\varphi_0 = \text{id}$. $\varphi_1 = \varphi$, μ^{φ_t} the transform of μ by φ_t and Ω_t the restriction to the curve φ_t in $\text{Diff } \mathbb{C}$ of the Maurer Cartan form Ω of $\text{Diff } \mathbb{C}$, by imposing the constraint

$$\partial^3 \mu^\varphi = 0 \quad (32)$$

which yields a solution of the anomalous Ward identity Eq.(25). There is an algebraic miracle through which the integral over t can be explicitly performed, unlike in the non abelian gauge case; however the calculation involves dropping boundary terms, which is not allowed on a general Riemann surface.

We should like to comment on the renormalization scheme adopted here : it involves no metric; as a result the Feynman propagators are inverses of an operator from one space to another space (e.g. from functions to 1-1 forms) which, on a compact Σ poses problems in connection with the occurrence of non vanishing kernels (zero-modes).

The extension to correlation functions of other conformal operators has not been looked at in detail. For instance, in the scalar model, one conjectures that there is a renormalization such that*

$$\Gamma^L(\mu, \pi; \bar{\mu}, \bar{\pi}) = \Gamma^L(\mu, \pi; 0, 0) + \Gamma^L(0, 0; \bar{\mu}, \bar{\pi}) \quad (33)$$

as a consequence of the (non anomalous) Diffeomorphism Ward identity (cf Eq.(12)).

IV. Free conformal fields on a compact Riemann surface without boundary.

It would be nice to extend the scheme presented in Section III in the case when Σ is a compact Riemann surface without boundary. There are however difficulties related, in particular with the occurrence of zero modes.

The anomaly $\mathcal{A}(\mathbb{C}, \mu)$ with local expression

* The upper script L is an abbreviation for "Linked" and refers to the part which does vanish at $\pi = \bar{\pi} = 0$.

$$\mathcal{A}^{\text{loc}}(C, \mu) = \kappa \int d^2 z C \partial^3 \mu \quad (34)$$

can easily be written down on a compact Σ :

$$\mathcal{A}(C, \mu) = \frac{\kappa}{2} \int_{\Sigma} d^2 z [C \partial^3 \mu - \mu \partial^3 C + 2R(C \partial \mu - \mu \partial C)] \quad (35)$$

Here, R is a projective connection^[9], i.e. locally, a holomorphic quadratic differential whose glueing between two holomorphic charts picks up a Schwarzian derivative :

$$R^{\alpha} (z^{\alpha})(dz^{\alpha})^2 = R^{\beta} (z^{\beta})(dz^{\beta})^2 - S(z^{\alpha}, z^{\beta})(dz^{\beta})^2. \quad (36)$$

R is undetermined up to a quadratic holomorphic differential, which reflects the ambiguity on $\langle \Theta \rangle$ (a holomorphic quadratic differential, by virtue of the Ward identity). In a projective (z) atlas^[9] one may normalize $\langle \Theta \rangle$ in such a way that $R=0$.

Thus, one may extend the Ward identity (Eq.25) to constrain the renormalization. However all efforts have so far failed to extend the Polyakov formula (Eq.27). This is no surprise since one knows from general knowledge, and one can check in perturbation theory^[8] that the partition function depends on the details of the model, besides the central charge κ . So, one should rather look for a class of Polyakov formulae, to fit various models. The Ward identity can be solved (cf. Eq. 29) but in a non unique way :

$$\frac{\delta \Gamma}{\delta \mu} = 2\kappa S(Z, z) + Q(Z) (\partial Z)^2 \quad (37)$$

where, now, in order to make sense not only z (since we have taken $R=0$) but also Z , should belong to a projective atlas. $Q(Z)$ is the coefficient of a holomorphic quadratic differential, which may explicitly (i.e. not via Z) depend on $\mu, \bar{\mu}$ and the model. So far, the integrability conditions which follow from Eq.37 have not been used successfully. Eq.(37) has been shown to hold in particular cases with Σ a torus^[10].

Given this unfortunate state of affairs one may go back to a renormalization scheme involving a metric^[10] and extract $\Gamma(\mu)$ from there.

Let us first point out that if one has constructed a $\Gamma(\rho, \mu, \bar{\mu})$ corresponding to a metric $ds^2 = \rho_{z\bar{z}} |dz + \mu d\bar{z}|^2$ which is Diffeomorphism invariant and has the Weyl anomaly^[11] :

$$\begin{aligned} \int d^2z \left[C (\bar{\partial} - \mu \partial - 2\partial\mu) \frac{\delta \Gamma}{\delta \mu} + \text{c.c.} \right. \\ \left. - (c \cdot \partial + (\partial c + \mu \partial \bar{c}) + (\bar{\partial} \bar{c} + \bar{\mu} \partial c)) \rho_{z\bar{z}} \frac{\delta \Gamma}{\delta \rho_{z\bar{z}}} \right] = 0 \end{aligned} \quad (38)$$

$$\int d^2z \Omega \frac{\delta \Gamma}{\delta \rho} = -\kappa \int \frac{d\bar{Z} \wedge dZ}{2i} \Omega \partial_Z \partial_{\bar{Z}} \ln \rho_{ZZ}$$

where

$$\rho_{ZZ} = \frac{\rho_{z\bar{z}}}{\lambda \bar{\lambda}}, \quad (39)$$

with Z and λ defined by Eq. (6), then, one may construct a $\Gamma(\mu, \bar{\mu})$ fulfilling the Ward identity Eq.(25) as follows^[11] :

One first subtracts from $\Gamma(\rho, \mu, \bar{\mu})$ a local Liouville counterterm

$$\begin{aligned} \Gamma_{\text{Liouville}}(\rho, \mu, \bar{\mu}; \mathring{\rho}) = \kappa \int \frac{d\bar{Z} \wedge dZ}{2i} \left[\frac{1}{2} (\ln \rho_{ZZ} - \ln \mathring{\rho}_{ZZ}) \right. \\ \left. \times \partial_Z \partial_{\bar{Z}} (\ln \rho_{ZZ} - \ln \mathring{\rho}_{ZZ}) - (\ln \rho_{ZZ} - \ln \mathring{\rho}_{ZZ}) \partial_Z \partial_{\bar{Z}} \ln \rho_{ZZ} \right] \end{aligned} \quad (40)$$

with

$$\mathring{\rho}_{ZZ} = \frac{\mathring{\rho}_{z\bar{z}}}{\lambda \bar{\lambda}}, \quad (41)$$

$\mathring{\rho}_{z\bar{z}}$ being a background on which neither Diff. nor Weyl operates. This has the effect to replace ρ by $\mathring{\rho}$.

There results a Diff. anomaly which can be transformed into $\mathcal{A}(C, \mu)$ (Eq.35) by adding a further counterterm :

$$\begin{aligned} \Delta\Gamma(\mu, \bar{\mu}; \dot{\rho}) = & -\kappa \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \left\{ \frac{1}{1-\mu\bar{\mu}} [(\partial + \overset{\circ}{\Gamma})\mu (\bar{\partial} + \overset{\circ}{\bar{\Gamma}})\bar{\mu} \right. \\ & \left. - \frac{1}{2} \bar{\mu} ((\partial + \overset{\circ}{\Gamma})\mu)^2 - \frac{1}{2} \mu ((\bar{\partial} + \overset{\circ}{\bar{\Gamma}})\bar{\mu})^2] - \mu (R - \overset{\circ}{R}) + \text{c.c.} \right\} \end{aligned} \quad (42)$$

with

$$\overset{\circ}{\Gamma} = \partial \ln \dot{\rho} \quad , \quad \overset{\circ}{\bar{\Gamma}} = \bar{\partial} \ln \dot{\rho} \quad , \quad \overset{\circ}{R} = \partial \overset{\circ}{\Gamma} - \frac{1}{2} (\overset{\circ}{\Gamma})^2. \quad (43)$$

The first term of $\Delta\Gamma$ between brackets is a covariantized form of a local expression known in the literature^[12], with the property that it glues between holomorphic charts on an arbitrary Σ . To sum up,

$$\Gamma'(\mu, \bar{\mu}; \dot{\rho}) = \Gamma(\rho, \mu, \bar{\mu}) - \Gamma_{\text{Liouville}}(\rho, \mu, \bar{\mu}; \dot{\rho}) + \Delta\Gamma(\mu, \bar{\mu}; \dot{\rho}) \quad (44)$$

has no Weyl anomaly and a chirally split Diff. anomaly, independent of $\dot{\rho}$. (Recall none of these transformations acts on $\dot{\rho}$). One may now investigate under which condition Γ' is independent of $\dot{\rho}$. One finds

$$\frac{\delta \Gamma'}{\delta \dot{\rho}} = -\kappa \partial \bar{\partial} \ln \dot{\rho}. \quad (45)$$

The $\dot{\rho}$ independence of Γ' is achieved provided one takes

$$\Gamma(\rho, \mu, \bar{\mu}; \dot{\rho}) = \Gamma(\rho, \mu, \bar{\mu}) - \Gamma(\dot{\rho}, 0, 0) \quad (46)$$

instead of $\Gamma(\rho, \mu, \bar{\mu})$ since $-\kappa \partial \bar{\partial} \ln \hat{\rho}$ is just the Weyl anomaly we started from. Thus, there remains to find such a $\Gamma(\rho, \mu, \bar{\mu}; \hat{\rho})$. The main ingredient for this construction is the ζ renormalized* determinant formula of Ray and Singer^[13] :

$$\log \det'_{\zeta} (-\Delta) - \log \det'_{\zeta} (-\Delta_0) = -\frac{d}{ds} \Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} dt \operatorname{Tr} (e^{\Delta t} - e^{\Delta_0 t}) \quad (47)$$

where

$$\Delta \equiv \Delta(\rho, \mu, \bar{\mu}) ; \Delta_0 \equiv \Delta(\hat{\rho}, 0, 0), \quad (48)$$

corresponding to the metrics

$$ds^2 = \rho |dz + \mu d\bar{z}|^2, \quad d\hat{s}^2 = \hat{\rho} dz d\bar{z} \quad (49)$$

The prime symbol refers to the proper subtraction of the zero mode contribution, which cancels out in the above "relative determinant" : $\operatorname{Tr} e^{\Delta t} \rightarrow \operatorname{Tr} e^{\Delta t} - \text{number of zero modes of } \Delta$, and similarly for $\operatorname{Tr} e^{\Delta_0 t}$.

In the following, we shall only look at the scalar Laplacian which is almost the general case. The study of Laplacians of spin j ^{[14], [15]} is deferred to a later publication^[16].

In formula Eq.(47), the zero mode contribution^[13] has cancelled out. This is a non local effect. In order to check locality in the perturbative sense, one may introduce a mass as an infrared regulator which eliminates the zero mode problem :

$$\begin{aligned} & \log \det_{\zeta} (-\Delta + C) - \log \det_{\zeta} (-\Delta_0 + C) = \\ & -\frac{d}{ds} \Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} dt \operatorname{Tr} (e^{-(\Delta+C)t} - e^{-(\Delta_0+C)t}) \end{aligned} \quad (50)$$

* In the mathematics literature, one talks about regularized determinants. In the physical literature, regularization refers to the use of a cut off procedure e.g. proptime regularization^[14] whose removal after subtraction of the singular parts and passage to the limit is referred to as renormalization since the fifties. We adhere to the physical terminology. ζ renormalization is equivalent to proptime regularization followed by subtraction of the polar parts in the proper time cut off.

One may convince oneself that this formula sums up a renormalization of the perturbative series for Δ relative to Δ_0 , and as it stands, has a limit as $C \rightarrow 0$. We now investigate the Weyl anomaly i.e. differentiate with respect to ρ ^[13]. Of course, the mass term breaks Weyl invariance explicitly which results into two additional terms besides the usual Weyl anomaly, namely a cosmological term $C\rho$ and a non local contribution corresponding to a mass insertion into the vacuum functional. As $C \rightarrow 0$ this non local contribution provides the non local term $\Pi(z,z)$, the diagonal of the kernel of the projector on the zero modes of Δ (here, the constants!)*. Provided one has chosen zero modes not to depend on ρ , one finds that this non local contribution is cancelled by subtracting from $\det'_\zeta$ the Gram determinant of a basis of zero modes.

The next property we are after is holomorphic factorization in $\mu, \bar{\mu}$ (not on Teichmüller space but on the infinite dimensional space of complex structures), which provides a local strengthened form of the Belavin Knizhnik^{[14], [15]} holomorphic factorization theorem, for non vanishing central charge κ . The complete calculation, still in progress, is rather lengthy and will be described elsewhere^[16]. The conjectured result is as follows : there exists a functional $\Gamma(\mu, \bar{\mu})$ such that

$$\delta_\mu \delta_{\bar{\mu}} \Gamma(\mu, \bar{\mu}) = 0 \quad (51)$$

$\Gamma(\mu, \bar{\mu})$ is constructed as

$$\begin{aligned} \Gamma(\mu, \bar{\mu}) = & -\frac{1}{2}(\log \det'_\zeta (-\Delta) - \log \det'_\zeta (-\Delta_0)) \\ & - \log \det \| \langle \phi_i | \phi_j \rangle \|_{\mu \bar{\mu}} - \log \det \| \langle \psi_\alpha | \psi_\beta \rangle \|_{\mu \bar{\mu}} \\ & + \log \det \| \langle \phi_i | \phi_j \rangle \|_{0,0} + \log \det \| \langle \psi_\alpha | \psi_\beta \rangle \|_{0,0} \\ & - \text{local counterterm} \end{aligned} \quad (52)$$

where the notation is as follows : $\log \det'_\zeta (-\Delta) - \log \det'_\zeta (-\Delta_0)$ is given by Eq.(47).

Given a factorization

$$\Delta(\rho, \mu, \bar{\mu}) = -\mathcal{T}^\dagger \mathcal{T} \quad (53)$$

* For general spin, to each Laplacian $\Delta = -\mathcal{T}^\dagger \mathcal{T}$ where $\mathcal{T}$ is a representation of the $\bar{\partial}$ operator, there corresponds another Laplacian $\tilde{\Delta} = -\mathcal{T} \mathcal{T}^\dagger$, and one would also find a term $\tilde{\Pi}(z,z)$ corresponding to $\ker \tilde{\Delta}$.

of the Laplacian, where $\dagger$ denotes the adjoint with respect to the natural pair of metrics on the space of functions and type (0,1) forms on Σ inherited from a metric on Σ , e.g. $ds^2 = \rho |dz + \mu d\bar{z}|^2$, $\mathcal{T}$ and $\mathcal{T}^\dagger$ are conjugate to $\bar{\partial}_Z$ and $\bar{\partial}_Z^c$ on the space of functions and type (0,1) differentials. (Here, the upperscript c denotes complex conjugation). Then $\{\phi_i\}$ and $\{\psi_\alpha\}$ span bases of $\ker \mathcal{T}$ and $\ker \mathcal{T}^\dagger$, respectively, which are chosen conjugate to a holomorphic and an antiholomorphic basis of $\ker \bar{\partial}_Z$ and $\ker \bar{\partial}_Z^c$. Holomorphicity, here, is with

respect to μ , on which the defining (elliptic) equations depend holomorphically up to conjugation. "Local counterterm" is $\Gamma_{\text{Liouville}} - \Delta\Gamma$ as previously described in Eq.(42).

The resulting expression does not depend on ρ anymore since the Weyl anomaly has been absorbed. It does not depend on ρ_0 either, as previously noticed. Unfortunately, it is impossible to find a holomorphic splitting of the Laplacian, which makes the calculation involved : at the classical level, in first order form, holomorphic splitting is obvious up to a contact term $\int \bar{\pi} \wedge \pi (1 - \mu \bar{\mu}) dz^2$ and to infrared effects which show up for instance in a regularizing mass term. What has to be checked, then, is that factorization can be recovered by inclusion of correct ultraviolet and infrared counterterms. Some technical details pertinent to this calculation are collected in the appendix. The only ambiguity that is left over is an additive constant (independent from $\rho, \mu, \bar{\mu}, \bar{\rho}$) corresponding to the choice of bases $\phi_i, \psi_\alpha, \bar{\phi}_i, \bar{\psi}_\alpha$, which can be adjusted so that $\Gamma(\mu, \bar{\mu})|_{\mu=\bar{\mu}=0} = 0$. By construction, $\Gamma(\mu, \bar{\mu})$ fulfills the anomalous Ward identity Eq.(25). So, for a system of free fields for which the total central charge $\kappa \equiv \sum_{\text{fields}} \kappa_{\text{field}} = 0$, Γ is defined on Teichmüller space. Otherwise one may write

$$\Gamma = \Gamma_2 + \Gamma' \quad (54)$$

where Γ_2 is the second order truncation of the Taylor expansion, which exhausts the anomalous contribution. Γ' is then defined on Teichmüller space.

For lack of further knowledge (see however [17]), we shall stop here our review of the consequences of the locality principle in the realm of Lagrangian conformal models.

Conclusion

In these notes we have addressed the question whether "conventional" field theory, typically à la Feynman is able to account for the special features exhibited by a restricted class of conformal models, namely free fields. The price to pay is that the locality principle demands the use of infinite dimensional spaces, whereas preference is commonly given to geometric quantization over finite dimensional moduli spaces. The advantage of using the locality principle is however to stay within a long tested scheme within which the ultraviolet problems receive a canonical solution. When geometry is properly treated, the global effects are included as well by the summation formulae which are due to specific properties of these models. Whereas it is unlikely that new phenomena will be discovered this way, it is probably of some intellectual interest to fill in these gaps, also in the case of the popular three dimensional theories for which similar experiments have been successfully performed^[18].

References

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Appendix

The scalar Laplacian for the metric

$$ds^2 = \rho |dz + \mu d\bar{z}|^2 \quad (\text{A1})$$

reads :

$$\Delta(\rho, \mu, \bar{\mu}) = \frac{1}{\rho(1-\mu\bar{\mu})} (\partial - \bar{\partial}\bar{\mu}) \frac{1}{1-\mu\bar{\mu}} (\bar{\partial} - \mu\partial). \quad (\text{A2})$$

We introduce the Hilbert space $L^2(\Omega^{0,0}, \rho(1-\mu\bar{\mu})d^2z)$ of square integrable functions with the norm :

$$\|\phi\|^2 = \int |\phi|^2 \rho(1-\mu\bar{\mu}) d^2z \quad (\text{A3})$$

and the Hilbert space $L^2(\Omega^{0,1}, (1-\mu\bar{\mu}) d^2z)$ of differential forms of type (0,1) with the norm

$$\|w_{\bar{z}} d\bar{z}\|^2 = \int w_{\bar{z}} d\bar{z} \wedge \bar{w}_z dz \frac{1}{2i} (1-\mu\bar{\mu}) \quad (\text{A4})$$

One may factorize $\Delta(\rho, \mu, \bar{\mu})$ as

$$\Delta(\rho, \mu, \bar{\mu}) \equiv \Delta = -\mathcal{T}^\dagger \mathcal{T}$$

with $\mathcal{T} : L^2(\Omega^{0,0}, \rho(1-\mu\bar{\mu}) d^2z) \rightarrow L^2(\Omega^{0,1}, (1-\mu\bar{\mu}) d^2z)$
and $\mathcal{T}^\dagger : L^2(\Omega^{0,1}, (1-\mu\bar{\mu}) d^2z) \rightarrow L^2(\Omega^{0,0}, \rho(1-\mu\bar{\mu}) d^2z)$.

All these constructions are the images in the $z\bar{z}$ variables of canonical constructions in the $Z\bar{Z}$ variables, and easily generalize to spin j where $\Omega^{0,0}$ (resp. $\Omega^{0,1}$) is replaced by $\Omega^{j,0}$ resp. $\Omega^{j,1}$.

Here we have :

$$\mathcal{T} = \frac{1}{1-\mu\bar{\mu}} (\bar{\partial} - \mu\partial) \quad (A6)$$

$$\mathcal{T}^\dagger = \frac{-1}{\rho(1-\mu\bar{\mu})} (\partial - \bar{\partial}\bar{\mu})$$

The Laplacian $\tilde{\Delta}$ on $\Omega^{0,1}$ also appears in the calculation :

$$\tilde{\Delta} = -\mathcal{T}\mathcal{T}^\dagger = \frac{1}{1-\mu\bar{\mu}} (\bar{\partial} - \mu\partial) \frac{1}{\rho(1-\mu\bar{\mu})} (\partial - \bar{\partial}\bar{\mu}) \quad (A7)$$

We call Π the projector on $\ker\Delta = \ker\mathcal{T}$ for which we choose a constant basis

$$\varphi^{0,0} \quad (A8)$$

(holomorphic in μ) :

$$\Pi = 1 - \mathcal{T}^\dagger \frac{1}{\mathcal{T}\mathcal{T}^\dagger} \mathcal{T} \quad (A9)$$

and $\tilde{\Pi}$ the projector on $\ker\tilde{\Delta} = \ker\mathcal{T}^\dagger$ for which we choose a basis

$$\{\bar{\omega}^i\} \equiv \{\bar{\psi}^i\} \quad (A10)$$

where the ψ^i 's are antiholomorphic in μ :

$$\tilde{\Pi} = 1 - \mathcal{T} \frac{1}{\mathcal{T}^\dagger\mathcal{T}} \mathcal{T}^\dagger \quad (A11)$$

Denoting δ_μ by δ , $\delta_{\bar{\mu}}$ by $\bar{\delta}$ and $m = \ln(1-\mu\bar{\mu})$, one has

$$\bar{\delta} \mathcal{T} = -\bar{\delta} m \mathcal{T}$$

(A12)

$$\delta \mathcal{T}^\dagger = -\delta m \mathcal{T}^\dagger$$

whereas $\delta \mathcal{T}$, $\bar{\delta} \mathcal{T}^\dagger$ cannot be reduced to equivalent Weyl transformations. Among the useful identities, for any function F note^[14]

$$\mathcal{T} F (\mathcal{T}^\dagger \mathcal{T}) = F (\mathcal{T} \mathcal{T}^\dagger) \mathcal{T}$$

(A13)

$$\mathcal{T}^\dagger F (\mathcal{T} \mathcal{T}^\dagger) = F (\mathcal{T}^\dagger \mathcal{T}) \mathcal{T}^\dagger$$

The main steps of the calculation go as follows :

i) The double variation of $\log \det |\varphi^{00}|^2$ (resp. $\log \det \|\langle \bar{\psi}^i | \bar{\psi}^j \rangle\|$), taking into account the holomorphicity in μ , (resp. $\bar{\mu}$) of the chosen basis, decomposes into the effect of a Weyl transformation of parameter m, and an expression which only involves the components of $\delta \varphi^{00}$, $\bar{\delta} \varphi^{00}$, (resp. $\delta \bar{\psi}^i$, $\bar{\delta} \bar{\psi}^i$) orthogonal to $\ker \mathcal{T}$, (resp. $\ker \mathcal{T}^\dagger$)^[14] :

$$(\bar{\delta}) \mathcal{T} \varphi^{00} + \mathcal{T} (\bar{\delta}) \varphi^{00} = 0$$

(A14*)

$$(\bar{\delta}) \mathcal{T}^\dagger \bar{\psi}^i + \mathcal{T}^\dagger (\bar{\delta}) \bar{\psi}^i = 0$$

$$(\bar{\delta}) \varphi^{00} \big|_{\perp} = \frac{-1}{\mathcal{T}^\dagger \mathcal{T}} \mathcal{T}^\dagger (\bar{\delta}) \mathcal{T} \varphi^{00}$$

(A15)

$$(\bar{\delta}) \bar{\psi}^i \big|_{\perp} = \frac{-1}{\mathcal{T} \mathcal{T}^\dagger} \mathcal{T} (\bar{\delta}) \mathcal{T}^\dagger \bar{\psi}^i .$$

* In the case of spin 0, the equations involving φ^{00} are of course empty, but in the general spin case $\ker \mathcal{T}$ and $\ker \mathcal{T}^\dagger$ occur symmetrically.

The double variations of $\log \det |\phi^{00}|^2$ (resp. $\log \det \|\langle \bar{\psi}_j, \bar{\psi}_i \rangle\|$) can then be transformed into traces involving Π (resp. $\tilde{\Pi}$) (Eq. A.9, A.11).

ii) A formal calculation of $\delta\bar{\delta} \log \det \frac{1}{\mathcal{T}^\dagger \mathcal{T} + C}$ shows that, in the limit $C \rightarrow 0$ where

$$\frac{C}{\mathcal{T}^\dagger \mathcal{T} + C} \rightarrow \Pi, \quad \frac{C}{\mathcal{T} \mathcal{T}^\dagger + C} \rightarrow \tilde{\Pi} \quad (\text{A16})$$

one gets exact cancellation from the double variations of the zero modes determinants.

This shows that after ultraviolet renormalization $\delta\bar{\delta} \log \det_{\text{renorm.}} \frac{1}{\mathcal{T}^\dagger \mathcal{T} + C}$ is purely local, because only one loop diagrams are involved, via traces.

The ζ renormalization, which is equivalent to the proper time cut off regularization followed by the subtraction of poles in the propertime cut off is currently used^[16] to check that the local counterterms produced by renormalization are indeed of the form $\delta\bar{\delta}$ (local counterterm $\Gamma_{\text{Liouville}} - \Delta\Gamma$, Eq. 42).

More abstractly, the result should follow from an application of the index theorem for families as indicated by J.B. Bost in his Bourbaki seminar (Eqs 2.5.2, 2.6.2) [1] and J.B. Bost, Th. Jolicoeur P.L. **B174**, 273 (1986).