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NEW METHODS FOR THE RENORMALISATION OF COMPOSITE OPERATOR GREEN FUNCTIONS

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Abstract

New methods are developed for the renormalisation of composite operator Green functions, based on the idea of non-linear source renormalisation. The anomalous dimension in the renormalisation group equation (RGE) for the generating functional becomes a non-linear function of the sources, and extra terms are therefore induced in the RGEs for the Green functions. These methods are applied to the operator product expansion, and a simple RGE for the Wilson coefficients is presented. Applications are made to Green functions of conserved and anomalous currents.

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1. Introduction

One of the main ways of deducing the physical properties of a quantum field theory is to study the Green functions of suitably chosen operators. In general these operators will be composites of the elementary fields, with quantum numbers chosen so that they couple to the physical states of interest. Clearly, therefore, the renormalisation of composite operators and their Green functions is a central problem in quantum field theory. It is therefore curious that while a sophisticated and essentially complete formalism exists for the renormalisation of single composite operators and their insertions into Green functions in which all the other fields are elementary, the renormalisation of Green functions involving more than one composite operator is usually treated only on an individual, case by case basis. The purpose of this paper is to develop an improved, systematic formalism describing the renormalisation of general Green functions of composite operators.

The essential features of the renormalisation of single composite operators are well-known (see, e.g., refs.[1,2]). To make insertions of a composite operator O_i^B into a renormalised Green function involving only elementary fields finite, a renormalised operator O_i^R is defined by multiplicative renormalisation, $O_i^R = Z_i^j O_j^B$. The renormalisation matrix Z_i^j is independent of the elementary field renormalisations, and is in general not diagonal, so that different operators of the same dimension, Lorentz structure and quantum numbers can mix. There is an extensive literature on composite operator renormalisation and mixing, and especially for gauge-dependent operators the results can be quite subtle.

However, this multiplicative renormalisation is not sufficient to make Green functions involving more than one composite operator finite. New divergences appear in two and higher point Green functions, e.g., $\langle 0|O_i^R(x) O_j^R(y)|0\rangle$, $\langle 0|O_i^R(x) O_j^R(y) O_k^R(z)|0\rangle$, etc., and an appropriate subtraction has to be made to define a finite, renormalised Green function. At present, these subtractions are usually made separately for each individual Green function, there being no well-developed, systematic formalism for the renormalisation of general Green functions, although some elements of such a formalism have been described in, e.g., ref.[2].

An important contribution has been made recently by Jack and Osborn [3]. They exploit the idea, originally introduced in ref.[4] and developed in the context of 2-dim sigma models in refs.[5,6], of allowing the coupling constants to be spacetime dependent. The coupling “constants” then act as sources for the composite operators which they multiply in the Lagrangian, and Green functions for these operators can be defined by functionally differentiating the generating functional with respect to the couplings. However, as shown in ref.[3], since the couplings are x -dependent, new counterterms must be added to the Lagrangian involving derivatives of the couplings. These new counterterms contribute to the Green functions of the appropriate composite operators, and correspond to the subtractions referred to above. With this technique, the counterterms required to renormalise all the Green functions appear once and for all in the extended action. Renormalisation group equations (RGEs) for the Green functions are also easily written, involving beta functions for the new counterterms.

While this new technique represents a significant advance, it still has some drawbacks, particularly of course that it only deals with composite operators already occurring in the

Lagrangian. In this paper, we show how the key ideas of ref.[3] can be extended to deal with arbitrary composite operators and developed into a systematic theory of composite operator renormalisation.

The main concept developed here is that of non-linear source renormalisation (NLSR). Arbitrary composite operators are introduced into the action coupled to sources in the usual way. The standard linear, multiplicative renormalisation of the bare source gives the required renormalisations for single composite operators. We propose, however, that the bare source should be regarded as a non-linear function of the renormalised source, with the new divergent terms arising in an expansion in powers of the renormalised source providing the counterterms necessary for renormalising the composite operator Green functions. In this formalism, all the counterterms are contained in a concise form in the generating functional, and there is no further need to introduce ad hoc subtractions in individual Green functions.

This conceptual simplicity extends to the renormalisation group equation. Here, the anomalous dimension of the source in the standard formalism becomes a function of the renormalised sources and their derivatives. The RGEs for renormalised Green functions of composite operators are then obtained from the RGE for the generating functional by repeated functional differentiation with respect to the renormalised source.

This new formalism also allows a particularly clear discussion of how the symmetries of the field theory restrict the extra counterterms required for renormalisation of the Green functions. The symmetries are implemented in the quantum theory by demanding that the generating functional, as a functional of the renormalised sources, satisfies a Ward identity. Requiring that the full generating functional, with the non-linear source renormalisation included, satisfies the Ward identity ensures that all the symmetry constraints on the Green function renormalisation counterterms are respected. The same methods are equally applicable to symmetries which at the quantum level are described by anomalous Ward identities.

The renormalisation of composite operator Green functions and the NLSR formalism are described in detail in sections 2 and 3. In section 4, we discuss how this new formalism affects the standard properties of the effective action (the generating functional for 1PI vertices) defined as a Legendre transform, and show how the usual RGEs for the 1PI vertices are modified. In section 5, we show how the NLSR method can be applied to the operator product expansion (OPE), and derive RGEs for the Wilson coefficients in terms of the anomalous dimension function and its derivatives. Finally, in section 6, we discuss as an example the renormalisation of Green functions of currents, both for the case of a conserved current and for the axial current, which satisfies an anomalous Ward identity. The RGEs for the Green functions and the Wilson coefficients in the OPE are derived.

2. Renormalisation of Composite Operator Green Functions

To study the Green functions of composite operators O_i in the functional formalism, we introduce corresponding sources S^i into the generating functional $W[S]$. The Green functions are then given by functional derivatives of W with respect to the sources. In this paper, we use the same convention for composite operators which already appear in the Lagrangian multiplied by coupling constants g^i . Unlike ref.[3], we leave the couplings g^i spacetime independent, and introduce a separate x -dependent source S^i , whose physical value is zero. In a very condensed notation, where ϕ represents all the elementary fields and O_i the composite operators (the index i is just a label distinguishing the operators, which need not be Lorentz scalars), the generating functional is therefore

$$W = \log \int \mathcal{D}\phi \exp \int d^n x (\mathcal{L}(\phi) + \mathcal{S}_B \phi + S_B^i O_i + \dots), \quad (2.1)$$

where we have, for definiteness, assumed dimensional regularisation, although any mass independent regularisation scheme would be appropriate. The dots in eq.(2.1) indicate the new counterterms to be discussed below.

The renormalisation of Green functions with a single composite operator and any number of elementary fields is standard. In addition to the usual source renormalisation for the elementary fields, the sources for the composite operators are multiplicatively renormalised, i.e.,

$$S_B^i = \mu^{-\epsilon} Z_j^i S_R^j, \quad (2.2)$$

where $Z(g_R)_j^i$ is a divergent function of the couplings, and $\epsilon = 4 - n$. In general, this renormalisation matrix is not diagonal, and the composite operators can mix under renormalisation.

However, new divergences arise when Green functions with more than one composite operator are considered. To renormalise these in a systematic way, we follow and extend the methods of ref.[3], which were restricted to operators occurring already in the Lagrangian. For simplicity, we consider theories in flat spacetime with no mass parameters, since the extension should involve no new points of principle, although the number of possible counterterm structures is increased.

Translated into the language of sources, the basic idea of ref.[3] is that the regularised action should be extended to include all possible terms (gauge invariant, Lorentz scalars of dimension n) formed from a single composite operator and an arbitrary number of sources and derivatives. It is crucial here that the unit operator $O_0 \equiv \mathbf{1}$ is included in the basis of composite operators O_i , since the extended action must include terms involving sources only.

The need for such terms has been recognised in ref.[2]. To see how these new counterterms are related to the divergences in composite operator Green functions, consider first the simplest case of insertions of two composites. For this, we need the counterterm quadratic in the sources S_R^i . The most general such term can be written in the form

$$f_{B(p,q)}^i{}_{jk} \partial^p S_B^j \partial^q S_B^k O_i, \quad (2.3)$$

where p and q denote the number of derivatives acting on the source.* The notation here is highly condensed, and in particular the Lorentz structure is contained implicitly in the labels i, j, k and p, q .

The term in eq.(2.3) where O_i denotes the unit operator provides the counterterm necessary to renormalise the Green function $\langle 0|O_j(x) O_k(y)|0\rangle$ in the case where none of the operators has a non-zero VEV. Here, the 2-point function is renormalised simply by the subtraction of a contact term of the form $Z_j^m Z_k^n f_{B(p,q)}^0{}_{mn} \partial^p \partial^q \delta(x, y)$. As is well-known, the divergent coefficient $f_{B(p,q)}^0{}_{jk}$ can be non-zero even in free field theory.

The counterterms in eq.(2.3) with $O_i \neq 1$ are required to remove the divergences from the Green functions $\langle 0|O_j(x) O_k(y)|0\rangle$ in the case where some of the composite operators can acquire a VEV, or from Green functions which involve also elementary fields, viz. $\langle 0|O_j(x) O_k(y) \phi(z_1) \dots \phi(z_r)|0\rangle$. To see this in more detail, consider the explicit expressions for these renormalised Green functions. Differentiating the generating functional, we find

$$\begin{aligned} \frac{\delta^{r+2} W}{\delta S_R^j(x) \delta S_R^k(y) \delta S_R(z_1) \dots \delta S_R(z_r)} &\equiv \langle 0|O_j(x) O_k(y) \phi(z_1) \dots \phi(z_r)|0\rangle_R \\ &= (Z)^r Z_j^m Z_k^n \left(\langle 0|O_m(x) O_n(y) \phi(z_1) \dots \phi(z_r)|0\rangle \right. \\ &\quad \left. + f_{B(p,q)}^i{}_{mn} \partial_x^p \partial_y^q (\langle 0|O_i(x) \phi(z_1) \dots \phi(z_r)|0\rangle \delta(x, y)) \right). \end{aligned} \quad (2.4)$$

This can serve as a definition of the normal product of two composites. Writing $O_i^R = Z_i^j O_j^B$, etc., we can define

$$[O_j(x) O_k(y)]_R = O_j^R(x) O_k^R(y) + (Z^{-1})_l^i Z_j^m Z_k^n f_{B(p,q)}^l{}_{mn} \partial_x^p \partial_y^q (O_i^R(x) \delta(x, y)). \quad (2.5)$$

However, this expression still only gives finite results when inserted into Green functions with elementary fields only. For higher point Green functions of composites, still more divergences occur.

For such higher point Green functions, we need the counterterms of higher order in the sources, viz.

$$f_{B(p,q,r)}^i{}_{jkl} \partial^p S_B^j \partial^q S_B^k \partial^r S_B^l O_i + \dots \quad (2.6)$$

Whether this series terminates or not depends on the dimensionality of the sources, and reflects, through power counting, how divergences occur in the Green functions. For example, for dimension 2 operators, the dimension of S_R^i is 2 and so the series terminates at $O(S_R^2)$ with a term involving no derivatives, viz. $f^0{}_{jk} S_B^j S_B^k$. By power counting, it is easy to check that a Green function with more than two dimension 2 composite operators has no new primitive divergence. In contrast, for dimension 4 operators, $\dim(S_R^i) = 0$, and the counterterms can contain an arbitrary number of sources but at most four derivatives, and this only in the case where the operator O_i in eqs. (2.3) or (2.6) is the unit

* For a detailed analysis of the possible counterterms for general renormalisable field theories, including both mass and curvature dependence, see refs.[3] and [13].

operator. This reflects the power counting result that Green functions involving arbitrary numbers of dimension 4 operators can have new primitive divergences. In the absence of VEVs, such divergences are contact terms of order 4 in the momenta. For dimension 6 operators, for which $\dim(S_R^i) = -2$, even this restriction disappears and counterterms with arbitrary numbers of sources compensated by powers of derivatives occur, cancelling the divergences in Green functions proportional to higher powers of momenta. Of course, counterterms involving sources of different dimensions also arise, corresponding to mixed Green functions.

The $O(S_R^3)$ counterterm in eq.(2.6) represents the new primitive divergence occurring in the Green functions with three composite operators. In addition to the primitive divergence, these Green functions also have sub-divergences. These are cancelled by contributions from the counterterms with fewer sources. For example,

$$\begin{aligned} \frac{\delta^3 W}{\delta S_R^j(x) \delta S_R^k(y) \delta S_R^l(z)} &\equiv \langle 0 | O_j(x) O_k(y) O_l(z) | 0 \rangle_R \\ &= Z_j^m Z_k^n Z_l^p \left(\langle 0 | O_m(x) O_n(y) O_p(z) | 0 \rangle \right. \\ &\quad + f_{B(p,q)}^i{}_{mn} \partial_x^p \partial_y^q (\langle 0 | O_i(x) O_p(z) | 0 \rangle \delta(x, y)) + 2 \text{ perms} \\ &\quad \left. + f_{B(p,q,r)}^i{}_{mnp} \partial_x^p \partial_y^q \partial_z^r (\langle 0 | O_i(x) | 0 \rangle \delta(x, y) \delta(x, z)) \right). \end{aligned} \quad (2.7)$$

The primitive divergence is cancelled by the term involving $f_{B(p,q,r)}^i{}_{jkl}$, the various sub-divergences being cancelled by the $f_{B(p,q)}^i{}_{jk}$.

It follows that to renormalise an n -point Green function, it is not sufficient simply to consider the counterterm of $O(S_R^n)$. All the possible counterterms involving lower powers of S_R must also be included.

We have assumed above that the sources for the elementary fields only occur linearly in the counterterm action, i.e. there are no further terms involving these sources beyond the original multiplicative renormalisation. This is equivalent to the statement that, modulo multiplicative renormalisations, the counterterms required to renormalise Green functions with a given set of composite operators and any number of elementary fields is the same. It is easy to see why this is so. Adding extra elementary fields to the Green function only introduces the usual dressed propagators and vertices controlled by the conventional field and coupling constant renormalisations. No new loops of the type characteristic of composite operator insertions are induced.

Further restrictions on the possible counterterms, corresponding to further identifications amongst the divergences arising from different Feynman diagrams, can be made by imposing the symmetries of the theory, i.e. by requiring that the full generating functional including the new counterterms obeys the relevant Ward identities. However, we prefer to postpone this discussion until section 3 when we have introduced the streamlined formalism of non-linear source renormalisation.

We now discuss briefly the RGE for composite operator Green functions within the above framework. An alternative approach based on the NLSR formalism is considered in the next section. Because of the new counterterms, we can view the theory as depending on an increased number of coupling constants. The coefficients f_B introduced above are

regarded as bare couplings, related to renormalised couplings f_R in such a way that the generating functional and its derivatives are finite functions of the g_R and f_R . (Eventually, we set $f_R = 0$ to recover physical results.) We therefore write

$$f_B^i{}_{jk} = \mu^\epsilon Z_l^i(Z^{-1})_j{}^m(Z^{-1})_k{}^n(f_R^l{}_{mn} + L^l{}_{mn}(g_R)), \quad (2.8)$$

where $L^i{}_{jk}(g_R)$ is a power series of poles in ϵ whose residues are functions of the couplings g_R only. Similar equations hold for all the new couplings $f_B^i{}_{jkl}, \dots$, etc. (Of course, each Lorentz structure (p, q) gives an independent coupling, but we suppress this notation here for simplicity.)

We now define beta functions for each of these new couplings in the standard way, viz.

$$\begin{aligned} \mu \frac{d}{d\mu} f_R^i{}_{jk} &= \hat{B}^i{}_{jk} = -\epsilon f_R^i{}_{jk} + B^i{}_{jk} \\ &= -\epsilon f_R^i{}_{jk} + \beta^i{}_{jk} - \gamma^{O_i}{}^l f_R^l{}_{jk} + \gamma^{O_j}{}^m f_R^i{}_{mk} + \gamma^{O_k}{}^i f_R^i{}_{jm}, \end{aligned} \quad (2.9)$$

etc. (see appendix A). The RGE for the generating functional is therefore

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g_R} + B^i{}_{jk} \frac{\partial}{\partial f_R^i{}_{jk}} + B^i{}_{jkl} \frac{\partial}{\partial f_R^i{}_{jkl}} + \dots - \gamma^\phi \mathcal{S}_R \frac{\delta}{\delta \mathcal{S}_R} - \gamma^{O_j}{}^i \mathcal{S}_R^j \frac{\delta}{\delta \mathcal{S}_R^i} \right) W = 0, \quad (2.10)$$

where $\gamma^{O_i}{}^j$ is the conventional anomalous dimension matrix corresponding to the multiplicative renormalisation of the composite operator O_i , and γ^ϕ is the anomalous dimension for the elementary fields ϕ . RGEs for the Green functions themselves are found as usual by differentiating eq.(2.10). An integration over x is assumed in this notation, i.e. $\mathcal{S}_R^j \delta / \delta \mathcal{S}_R^i \equiv \int d^n x \mathcal{S}_R^j \delta / \delta \mathcal{S}_R^i$.

3. Non-linear Source Renormalisation

While the discussion in section 2 is complete and sufficient to discuss the renormalisation of composite operator Green functions, it is in some respects cumbersome, particularly in the treatment of the RGE with many new couplings. In this section, we introduce a more concise and elegant formalism.

The essential idea is to group together all the counterterms involving a particular composite operator O_i , and to regard them all as arising from a non-linear renormalisation of the corresponding source. The generating functional is simply

$$W = \log \int \mathcal{D}\phi \exp \int d^n x (\mathcal{L}(\phi) + \mathcal{S}_B \phi + \mathcal{S}_B^i O_i), \quad (3.1)$$

where now $\mathcal{S}_B^i = \mathcal{S}_B^i(\mathcal{S}_R)$ is a completely general function of $\mathcal{S}_R^i$ and derivatives. (Of course, this is a redefinition of $\mathcal{S}_B^i$ compared to section 2.) Notice again that it is essential

in this formalism to include a source S_B^0 for the unit operator $O_0 \equiv 1$ on the same footing as all the others. In the notation of section 2, we therefore have the expansion

$$S_B^i = \mu^{-\epsilon} Z_l^i (S_R^l + L_{(p,q)}^l{}_{jk} \partial^p S_R^j \partial^q S_R^k + \dots). \quad (3.2)$$

Clearly this reproduces all the results of section 2. The renormalised Green functions are still given simply by functional derivatives of W with respect to the renormalised sources S_R, S_R^i . To write the renormalisation group equation in this NLSR formalism we need the analogue of the anomalous dimension matrix for conventional source renormalisation. Here, the anomalous dimension becomes a function of the sources. We define

$$\mu \frac{dS_R^i}{d\mu} = \hat{\gamma}^i(S_R) = \epsilon S_R^i + \gamma^i(S_R). \quad (3.3)$$

The RGE for the generating functional is now

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g_R} + \gamma S_R \frac{\delta}{\delta S_R} + \gamma^i(S_R) \frac{\delta}{\delta S_R^i} \right) W = 0. \quad (3.4)$$

(Here, and below, an integration over x is assumed together with the summation on repeated indices.) The equivalence of this form of the RGE to eq.(2.10) is demonstrated in appendix A.

This shows clearly the relation of the NLSR method with the standard textbook discussion. If $\gamma^i(S_R)$ is taken to be a linear function of S_R^i , we only include the counterterms necessary to renormalise single composite operator insertions. To renormalise the theory fully and correctly, the non-linear terms in $\gamma^i(S_R)$ must be included.

The RGEs for the composite operator Green functions themselves are given by repeated functional differentiation of eq.(3.2). We find, of course, new contributions from differentiating the non-linear terms in the anomalous dimension function. For example, letting $\mathcal{D} \equiv \mu \partial / \partial \mu + \beta \partial / \partial g_R$, and including only the composite operator sources, we find (temporarily writing all the spacetime arguments explicitly)

$$\mathcal{D} \frac{\delta W}{\delta S_R^j(x)} + \int dz \frac{\delta \gamma^i(S_R(z))}{\delta S_R^j(x)} \frac{\delta W}{\delta S_R^i(z)} + \int dz \gamma^i(S_R(z)) \frac{\delta^2 W}{\delta S_R^i(z) \delta S_R^j(x)} = 0, \quad (3.5)$$

i.e.

$$\mathcal{D} \frac{\delta W}{\delta S_R^j} + \gamma^i{}_{,j} \frac{\delta W}{\delta S_R^i} + \gamma^i \frac{\delta^2 W}{\delta S_R^i \delta S_R^j} = 0. \quad (3.6)$$

Differentiating again,

$$\mathcal{D} \frac{\delta^2 W}{\delta S_R^j \delta S_R^k} + \gamma^i \frac{\delta^3 W}{\delta S_R^i \delta S_R^j \delta S_R^k} + \gamma^i{}_{,k} \frac{\delta^2 W}{\delta S_R^j \delta S_R^i} + \gamma^i{}_{,j} \frac{\delta^2 W}{\delta S_R^i \delta S_R^k} + \gamma^i{}_{,j,k} \frac{\delta W}{\delta S_R^i} = 0. \quad (3.7)$$

The Green functions are given by the functional derivatives with the sources set to zero. The second term above, proportional to $\gamma^i(S_R)$, therefore vanishes. The third and fourth

terms give the standard anomalous dimension contributions. The final term picks up the new counterterms of $O(S_R^2)$ required to renormalise the divergence appearing in the 2-point function. Denoting $\gamma^i_{,j}|_{S_R^i=0} = \gamma^i_j$, $(\delta W/\delta S_R^i)|_{S_R^i=0} = W_i$, etc., we therefore have the following RGEs for Green functions:

$$\mathcal{D}W_{jk} + \gamma^i_j W_{ik} + \gamma^i_k W_{ji} + \gamma^i_{jk} W_i = 0, \quad (3.8)$$

$$\mathcal{D}W_{jkl} + \gamma^i_j W_{ikl} + \gamma^i_k W_{jil} + \gamma^i_l W_{jki} + \gamma^i_{jk} W_{il} + \gamma^i_{jl} W_{ik} + \gamma^i_{kl} W_{ij} + \gamma^i_{jkl} W_i = 0, \quad (3.9)$$

and so on.

These equations exemplify the conceptual simplicity of the NLSR formalism. The RGEs for Green functions of arbitrary numbers of composite operators are described in a simple, unified way based on a single renormalisation prescription in the generating functional.

A special case of these RGEs occurs when the operators O_i already appear in the Lagrangian. This situation has been extensively studied in ref.[3]. Here, the multiplicative renormalisation of the operators O_i is determined by the usual coupling constant renormalisation of the g_B^i . As a result, the anomalous dimension matrix is related to the beta functions. In appendix B, we verify explicitly in our formalism that for such operators, $\gamma^i_j = \partial_j \beta^i$ (where $\partial_j \equiv \partial/\partial g_R^j$).

We now discuss how the possible counterterms in $S_B^i(S_R)$ are restricted by the symmetries of the field theory. The NLSR formalism is well-suited to implement these constraints in a simple way.

Symmetries in quantum field theory are expressed as Ward identities. Introducing a source S_R^μ corresponding to a conserved current j_μ , the (non-anomalous) Ward identity for W has the generic form

$$\left(\partial^\mu \frac{\delta}{\delta S_R^\mu} + (\Delta S_R) \frac{\delta}{\delta S_R} + (\Delta S_R)^i \frac{\delta}{\delta S_R^i} \right) W = 0, \quad (3.10)$$

where $(\Delta S_R)^i$ are the variations of the sources (including the source for the current itself) under the infinitesimal symmetry transformation for which j_μ is the Noether current. (Notice there is no spacetime integration implied by the sum over i in eq.(3.10).)

The Ward identity holds for the renormalised sources. It is demanded that the full generating functional with the counterterms in the expansion of $S_B^i(S_R)$ satisfies the Ward identity.

The determination of allowed counterterms in W is therefore in practice very similar to the construction of an effective Lagrangian subject to the constraints of a functional Ward identity. Several relevant techniques can be found in ref.[7]. How this procedure works is best demonstrated in terms of particular examples. In section 6, we compute the non-linear counterterms relevant for Green functions of the electromagnetic current in QCD, and for the operators occurring in the axial anomaly.

It should be emphasised that one advantage of these methods is that anomalous symmetries can be implemented just as straightforwardly as true symmetries. In the presence of anomalies, the generating functional is simply required to satisfy the appropriate anomalous Ward identity with renormalised sources.

4. Effective Action in the NLSR Formalism

It is frequently useful in quantum field theory to define an effective action Γ , i.e. the generating functional for proper vertices which are 1PI with respect to some set of fields. This effective action is related to the generating functional W by a Legendre transform with respect to the sources. To keep the standard interpretation of Γ in relation to renormalised Green functions in the NLSR formalism, we define

$$\Gamma[\Phi_i] = W[S_R] - \int dx S_R^i \Phi_i, \quad (4.1)$$

so that

$$\frac{\delta W}{\delta S_R^i} = \Phi_i, \quad , \quad \frac{\delta \Gamma}{\delta \Phi_i} = -S_R^i. \quad (4.2)$$

The RGE for Γ is, however, changed in the NLSR formalism. Differentiating eq.(4.1) with respect to μ , we find as usual

$$\mathcal{D}\Gamma + \mu \frac{dS_R^i}{d\mu} \Phi_i = 0, \quad (4.3)$$

where $\mathcal{D} = (\mu\partial/\partial\mu + \beta\partial/\partial g_R) |_{\Phi}$. Here, however, the derivative of the source gives the anomalous dimension function $\gamma^i(S_R)$. Substituting eq.(4.2), we find the RGE

$$\mathcal{D}\Gamma + \Phi_i \gamma^i \left(-\frac{\delta \Gamma}{\delta \Phi} \right) = 0. \quad (4.4)$$

The RGE is therefore non-linear in $\delta\Gamma/\delta\Phi$.

To see how this works in practice, we calculate explicitly the RGE for $\delta^2\Gamma/\delta\Phi_j\delta\Phi_k \equiv \Gamma^{jk}$. Differentiating eq.(4.4) twice with respect to Φ , we find

$$\mathcal{D}\Gamma^{jk} - \gamma^j_{,l} \Gamma^{lk} - \gamma^k_{,l} \Gamma^{jl} + \Phi_i \gamma^i_{,l,m} \Gamma^{jl} \Gamma^{mk} - \Phi_i \gamma^i_{,l} \Gamma^{ljk} = 0. \quad (4.5)$$

In the notation of section 3, therefore, evaluating with sources set to zero, we have the following RGE for the 2-point proper vertices,

$$\left(\mathcal{D} - \gamma^i_l \Phi_i \frac{\delta}{\delta \Phi_l} \right) \Gamma^{jk} - \gamma^j_l \Gamma^{lk} - \gamma^k_l \Gamma^{jl} + \gamma^i_{lm} \Phi_i \Gamma^{jl} \Gamma^{mk} = 0. \quad (4.6)$$

This is the analogue of the RGE (3.6) for W_{jk} . It is straightforward to check the consistency of these equations using $\Gamma^{jk} = -(W^{-1})^{jk}$, and taking careful account of the distinction between $\mathcal{D}$ at fixed Φ_i as above, and $\mathcal{D}$ at fixed S_R^i as in section 3.

For completeness, we also give the RGE for the 3-point proper vertices,

$$\begin{aligned} & \left(\mathcal{D} - \gamma^i_m \Phi_i \frac{\delta}{\delta \Phi_m} \right) \Gamma^{jkl} - \gamma^j_m \Gamma^{mkl} - \gamma^k_m \Gamma^{jml} - \gamma^l_m \Gamma^{jkm} \\ & + \gamma^j_{mn} \Gamma^{mk} \Gamma^{nl} + \gamma^k_{mn} \Gamma^{mj} \Gamma^{nl} + \gamma^l_{mn} \Gamma^{mj} \Gamma^{nk} \\ & + \Phi_i \gamma^i_{mn} \Gamma^{mj} \Gamma^{nkl} + \Phi_i \gamma^i_{mn} \Gamma^{mk} \Gamma^{njl} + \Phi_i \gamma^i_{mn} \Gamma^{ml} \Gamma^{njk} - \Phi_i \gamma^i_{mnp} \Gamma^{mj} \Gamma^{nk} \Gamma^{pl} \\ & = 0. \end{aligned} \quad (4.7)$$

5. Operator Product Expansion

One immediate application of the NLSR formalism is to the operator product expansion. In this section, we show how the RGEs for the Green functions in this approach may be used to derive a simple expression for the RGE for the Wilson coefficients.

The starting point is the operator product expansion (see, e.g., refs.[1,8]),

$$[O_j(x) O_k(y)]_R \underset{|x-y| \rightarrow 0}{\sim} c^i_{jk}(x-y) O_i^R(y). \quad (5.1)$$

Notice that the LHS is the renormalised normal product, defined so that its insertions into Green functions of elementary fields is finite.

The RGE for these Green functions follows from eq.(3.2), exactly as in the derivation of eq.(3.8). We have,

$$\mathcal{D}W_{jkS\dots S} = -n\gamma W_{jkS\dots S} - \gamma^i_j W_{ikS\dots S} - \gamma^i_k W_{jis\dots S} - \gamma^i_{jk} W_{iS\dots S} = 0. \quad (5.2)$$

Now, applying the OPE to this Green function, we find

$$W_{jkS\dots S} \sim c^i_{jk} W_{iS\dots S}. \quad (5.3)$$

(Notice that the repeated index i involving c^i_{jk} implies only a summation, not the integration implicit in, e.g., eq.(5.2).) Therefore,

$$\mathcal{D}W_{jkS\dots S} \sim \mathcal{D}c^i_{jk} W_{iS\dots S} + c^i_{jk} \mathcal{D}W_{iS\dots S} \quad (5.4)$$

$$= \mathcal{D}c^i_{jk} W_{iS\dots S} - c^i_{jk} \gamma^l_i W_{lS\dots S} - c^i_{jk} n\gamma W_{iS\dots S}, \quad (5.5)$$

using the RGE for $W_{iS\dots S}$. On the other hand, substituting the OPE on the RHS of eq.(5.2), we have

$$\mathcal{D}W_{jkS\dots S} \sim -c^i_{jk} n\gamma W_{iS\dots S} - \gamma^i_j c^l_{ik} W_{lS\dots S} - \gamma^i_k c^l_{ji} W_{lS\dots S} - \gamma^i_{jk} W_{iS\dots S}. \quad (5.6)$$

Comparing eqs.(5.5) and (5.6), we find

$$\left(\mathcal{D}c^i_{jk} - \gamma^i_l c^l_{jk} - \gamma^l_j c^i_{lk} - \gamma^l_k c^i_{jl} + \gamma^i_{jk} \right) W_{iS\dots S} = 0. \quad (5.7)$$

Before factoring off the Green function to obtain the RGE for the Wilson coefficients, we have to take account of the fact that the γ^i_{jk} in eq.(5.7) in general involve derivatives acting on delta functions. This gives rise to some subtleties, which are addressed in appendix C. The final result can be simply written as

$$\mathcal{D}c^i_{jk} = \bar{\gamma}^i_l c^l_{jk} - \bar{\gamma}^l_j c^i_{lk} - \bar{\gamma}^l_k c^i_{jl} - \bar{\gamma}^i_{jk}, \quad (5.8)$$

where,

$$\bar{\gamma}^i_j = \int dz \gamma^i_j(z, x) = -\gamma^O_j{}^i, \quad (5.9)$$

$$\bar{\gamma}^i_{jk} = \int dz \gamma^i_{jk}(z, x, y) = \beta^i_{jk} \partial_x^p \partial_y^q \delta(x, y) + j \leftrightarrow k. \quad (5.10)$$

(For notation, see eq.(A.8).)

The RGE for c^i_{jk} therefore has a very simple form. The first three terms, involving the anomalous dimension matrix, are quite standard. The novel feature is the final, inhomogeneous term, whose origin is the new divergence in the product of two renormalised composite operators, and which appears naturally in the NLSR formalism as a coefficient in the expansion of the anomalous dimension function.

Eq.(5.8) can readily be solved to give the leading short distance behaviour of the Wilson coefficients $c^i_{jk}(x-y)$ (or equivalently the leading large momentum behaviour of the Fourier transform) using the standard techniques for the solution of inhomogeneous RGEs. Rather than write a general solution, however, we will discuss the RGE in the context of specific examples in the following section.

6. Green functions of currents and (anomalous) Ward identities

In this final section, we consider two examples of the NLSR formalism. The first is the simplest possible case of the 2-point Green function of a conserved electromagnetic current. The second application is to the Green functions of the operators j_μ^5 and FF^* occurring in the axial anomaly equation. This is a particularly interesting example because, as well as involving operator mixing, the new counterterms are restricted by an anomalous Ward identity.

(1) RGE for conserved current Green functions

The Ward identity for the conserved electromagnetic current in QCD is simply

$$\partial^\mu \frac{\delta W}{\delta S_R^\mu} + \Delta W = 0, \quad (6.1)$$

where ΔW represents the variation terms for the sources for the elementary fields ϕ , and S_R^μ is the source for the current.

Since the current is conserved, it is not renormalised, so $S_B^\mu = S_R^\mu$ at least to linear order in the sources. The only operators needed to discuss the renormalisation of Green functions of the current are j_μ and $\mathbf{1}$. The corresponding sources S_R^μ and S_R^0 have dimensions 1 and 4 respectively. The only counterterm of second order in the sources is therefore of $O(S_R^{\mu 2})$ with two derivatives. The Ward identity restricts this term to be transverse, i.e. $(\partial^\mu S_R^\nu - \partial^\nu S_R^\mu)^2$. The generating functional is therefore

$$W[S_R, S_R^\mu, S_R^0] = \log \int \mathcal{D}\phi \exp \int d^n x \left(\mathcal{L}(\phi) + S_B \phi + S_B^\mu j_\mu + S_B^0 \mathbf{1} \right), \quad (6.4)$$

where to $O(S_R^2)$,

$$S_B^\mu = \mu^{-\epsilon} S_R^\mu, \quad (6.3)$$

$$S_B^0 = \mu^{-\epsilon} \left(S_R^0 + L^0{}_{11} (\partial^\mu S_R^\nu - \partial^\nu S_R^\mu)^2 \right). \quad (6.4)$$

The anomalous dimensions are (letting the index 1 represent S_R^μ)

$$\hat{\gamma}^1(S_R^\mu, S_R^0) = \mu \frac{dS_R^\mu}{d\mu} = \epsilon S_R^\mu, \quad (6.5)$$

$$\hat{\gamma}^0(S_R^\mu, S_R^0) = \mu \frac{dS_R^0}{d\mu} = \epsilon S_R^0 + \beta^0{}_{11} (\partial^\mu S_R^\nu - \partial^\nu S_R^\mu)^2, \quad (6.6)$$

where (c.f. eq.(A.7)),

$$\beta^0{}_{11} = -\epsilon L^0{}_{11} - \mu \frac{dL^0{}_{11}}{d\mu}. \quad (6.7)$$

The RGE, to this order, is therefore

$$\mathcal{D}W + \beta^0{}_{11} \int dx (\partial^\mu S_R^\nu - \partial^\nu S_R^\mu)^2 \frac{\delta W}{\delta S_R^0(z)} = 0. \quad (6.8)$$

The RGE for the current-current Green function follows by differentiating eq.(6.8). We find,

$$\mathcal{D} \frac{\delta^2 W}{\delta S_R^\mu(x) \delta S_R^\nu(y)} = 2\beta^0{}_{11} \frac{\delta W}{\delta S_R^0(x)} (\partial^2 g_{\mu\nu} - \partial_\mu \partial_\nu) \delta(x, y). \quad (6.9)$$

Recalling that $\delta W / \delta S_R^0 = \langle 1 \rangle = 1$, and writing the momentum space Green function as $\langle 0 | j_\mu j_\nu | 0 \rangle_R = \Pi(k^2) (k^2 g_{\mu\nu} - k_\mu k_\nu)$, we therefore have

$$\mathcal{D}\Pi(k^2) = -2\beta^0{}_{11}. \quad (6.10)$$

In this form, we recover the standard result for the RGE of the current-current Green function (see, e.g., ref.[8], eq.(8.7), where $D(g)$ is to be identified with $-2\beta^0{}_{11}$).

The solution of the inhomogeneous RGE allows the determination of the large momentum behaviour of $\Pi(k^2)$, and therefore the total cross section $\sigma(e^+e^- \rightarrow \text{hadrons})$ in QCD. These are controlled by the new RG function $\beta^0{}_{11}$.

We can also restate this result in terms of the OPE formalism of section 5. In momentum space, the OPE gives

$$\langle 0 | j_\mu j_\nu | 0 \rangle_R \underset{k \rightarrow \infty}{\sim} c^0{}_{11}(k^2) (k^2 g_{\mu\nu} - k_\mu k_\nu) \mathbf{1} + \dots \quad (6.11)$$

From the general RGE (5.8) for Wilson coefficients, we find here

$$\mathcal{D}c^0{}_{11} = -2\beta^0{}_{11}. \quad (6.12)$$

(2) RGEs for the axial anomaly operators

The axial-vector current in massless QCD satisfies the anomalous Ward identity

$$\partial^\mu \frac{\delta W}{\delta S_R^{5\mu}} - \frac{\delta W}{\delta \theta_R} + \Delta W = 0, \quad (6.13)$$

where $S_R^{5\mu}$ is the source for $j_{\mu R}^5$ and θ_R is the source for $Q_R \equiv \frac{g_R^2}{32\pi^2} FF^*_R$. These operators mix under renormalisation[9],

$$j_{\mu B}^5 = Z^{-1} j_{\mu R}^5, \quad Q_B = Q_R - Z^{-1}(1 - Z)\partial^\mu j_{\mu R}^5, \quad (6.14)$$

implying a corresponding mixing of the renormalised sources.

The higher order counterterms for the sources $S_R^{5\mu}$, θ_R must satisfy eq.(6.13). This is equivalent to demanding invariance of $W[S_R^{5\mu}, \theta_R]$ under the local symmetry $S_R^{5\mu} \rightarrow S_R^{5\mu} + \partial^\mu \alpha$, $\theta_R \rightarrow \theta_R + \alpha$. It is therefore clear that θ_R can occur only with a derivative. The only dimension 1 invariant is $(S_R^{5\mu} - \partial^\mu \theta_R)$, which we denote for brevity by A_R^μ . It is then easy to see that the most general counterterm of second order in the sources is a linear combination of $(\partial^\mu A_R^\nu - \partial^\nu A_R^\mu)^2$ and $(\partial_\mu A_R^\mu)^2$.

The generating functional is therefore*

$$W[S_R, S_R^{5\mu}, \theta_R, S_R^0] = \log \int \mathcal{D}\phi \exp \int d^n x \left(\mathcal{L}(\phi) + S_B \phi + S_B^{5\mu} j_\mu^5 + \theta_B Q + S_B^0 \mathbf{1} \right), \quad (6.15)$$

where, to this order,

$$S_B^{5\mu} = \mu^{-\epsilon} (Z S_R^{5\mu} + (1 - Z)\partial^\mu \theta_R), \quad (6.16)$$

$$\theta_B = \mu^{-\epsilon} \theta_R, \quad (6.17)$$

$$S_B^0 = \mu^{-\epsilon} \left(S_R^0 + L^{(T)} (\partial^\mu A_R^\nu - \partial^\nu A_R^\mu)^2 + L^{(L)} (\partial_\mu A_R^\mu)^2 \right). \quad (6.18)$$

The anomalous dimensions are therefore

$$\hat{\gamma}^{5\mu} = \epsilon S_R^{5\mu} - \gamma S_R^{5\mu} + \gamma \partial^\mu \theta_R, \quad (6.19)$$

$$\hat{\gamma}^\theta = \epsilon \theta_R, \quad (6.20)$$

$$\hat{\gamma}^0 = \epsilon S_R^0 + \beta^{(T)} (\partial^\mu A_R^\nu - \partial^\nu A_R^\mu)^2 + \beta^{(L)} (\partial_\mu A_R^\mu)^2, \quad (6.21)$$

* Following ref.[9], we continue to use dimensional regularisation here. However, due to the well-known difficulties in defining γ^5 in dimensional regularisation, this method may not be well-defined to all orders. In fact, for our subsequent results, all we need is the RGE (6.25). This follows if we assume only that a renormalisation scheme exists in which the Ward identity (6.13) holds in the same form for both bare and renormalised quantities (non-renormalisation of the anomaly).

where,

$$\gamma = Z^{-1} \mu \frac{dZ}{d\mu}, \quad (6.22)$$

$$\beta^{(T)} = -\epsilon L^{(T)} + \mu \frac{dL^{(T)}}{d\mu}, \quad (6.23)$$

$$\beta^{(L)} = -\epsilon L^{(L)} + \mu \frac{dL^{(L)}}{d\mu}. \quad (6.24)$$

The RGE for the generating functional, to second order in the sources, is therefore

$$\mathcal{D}W - \gamma A_R^\mu \frac{\delta W}{\delta S_R^{5\mu}} + \left(\beta^{(T)} (\partial^\mu A_R^\nu - \partial^\nu A_R^\mu)^2 + \beta^{(L)} (\partial_\mu A_R^\mu)^2 \right) \frac{\delta W}{\delta S_R^0} = 0. \quad (6.25)$$

Notice that all the coefficients involve the same combination A_R^μ of $S_R^{5\mu}$ and θ_R .

The RGEs for the renormalised 2-point functions of $j_{\mu R}^5$ and Q_R follow as usual from differentiation of eq.(6.25), and simplify using the following (momentum space) Ward identities derived similarly from eq.(6.13),

$$ik^\mu \frac{\delta^2 W}{\delta S_R^{5\nu} \delta S_R^{5\mu}} - \frac{\delta^2 W}{\delta S_R^{5\nu} \delta \theta_R} = 0, \quad (6.26)$$

$$ik^\mu \frac{\delta^2 W}{\delta \theta_R \delta S_R^{5\mu}} - \frac{\delta^2 W}{\delta \theta_R \delta \theta_R} = 0. \quad (6.27)$$

We find,

$$\mathcal{D} \frac{\delta^2 W}{\delta S_R^{5\mu} \delta S_R^{5\nu}} = 2\gamma \frac{\delta^2 W}{\delta S_R^{5\mu} \delta S_R^{5\nu}} - 2\beta^{(T)} (k^2 g_{\mu\nu} - k_\mu k_\nu) - 2\beta^{(L)} k_\mu k_\nu, \quad (6.28)$$

$$\mathcal{D} \frac{\delta^2 W}{\delta \theta_R \delta \theta_R} = 2\gamma \frac{\delta^2 W}{\delta \theta_R \delta \theta_R} + 2\beta^{(L)} k^4, \quad (6.29)$$

the RGE for $\delta^2 W / \delta \theta_R \delta S_R^{5\mu}$ following immediately from eqs.(6.27) and (6.29).

These equations are the realisation in our example of the general RGE (3.6) for 2-point functions. The RG functions γ , $\beta^{(T)}$, $\beta^{(L)}$ are the particular forms of the general coefficients γ^i_j , γ^i_{jk} in the expansion of the anomalous dimension function.

Knowing these coefficients, we can immediately write down the RGEs for the Wilson coefficients in the OPE from the general formula (5.8). Defining,

$$\langle 0 | j_\mu^5 j_\nu^5 | 0 \rangle_R \underset{k \rightarrow \infty}{\sim} \left(c_{jj}^{0(T)}(k^2) (k^2 g_{\mu\nu} - k_\mu k_\nu) + c_{jj}^{0(L)}(k^2) k_\mu k_\nu \right) \mathbf{1} + \dots, \quad (6.30)$$

$$\langle 0 | Q Q | 0 \rangle_R \underset{k \rightarrow \infty}{\sim} c_{QQ}^0(k^2) k^4 \mathbf{1} + \dots, \quad (6.31)$$

we find,

$$\mathcal{D} c_{jj}^{0(T)} = 2\gamma c_{jj}^{0(T)} - 2\beta^{(T)}, \quad (6.32)$$

$$\mathcal{D} c_{jj}^{0(L)} = 2\gamma c_{jj}^{0(L)} - 2\beta^{(L)}, \quad (6.33)$$

$$\mathcal{D} c_{QQ}^0 = 2\gamma c_{QQ}^0 - 2\beta^{(L)}. \quad (6.34)$$

These equations, or equivalently (6.28), (6.29), allow the large momentum behaviour of the Green functions to be deduced from a knowledge of γ , $\beta^{(T)}$ and $\beta^{(L)}$. For example, the solution of eq.(6.33) is

$$c^0_{jj}{}^{(L)}(e^t k, g_R, \mu) = e^2 \int_0^t dt' \gamma(g(t')) \left(c^0_{jj}{}^{(L)}(k, g(t), \mu) - 2 \int_0^t dt'' \beta^{(L)}(g(t'')) e^{-2 \int_0^{t''} dt' \gamma(g(t'))} \right). \quad (6.35)$$

Here, $t = \log k^2/\mu^2$, and $g(t)$ is the running coupling constant, which in QCD behaves asymptotically as $g^2(t) \sim -\frac{2}{\beta_1} \frac{1}{t} + O(\frac{\log t}{t^2})$, where β_1 is the leading coefficient in the expansion of the beta function, $\beta(g_R) = \beta_1 g_R^3 + O(g_R^5)$. From ref.[9], the anomalous dimension is $\gamma(g_R) = \gamma_2 g_R^4 + O(g_R^6)$. The simplest diagram which can contribute to the longitudinal part of the current-current Green function is the double triangle diagram of $O(g_R^4)$, so that $\beta^{(L)}(g_R) = \beta_2^{(L)} g_R^4 + O(g_R^6)$, consistent with eq.(6.29) for the $\langle 0|Q Q|0\rangle_R$ Green function. The leading behaviour of the Wilson coefficient $c^0_{jj}{}^{(L)}$, or c^0_{QQ} , is therefore

$$c^0_{jj}{}^{(L)}(k^2) \sim -8 \frac{\beta_2^{(L)}}{\beta_1^2} \frac{1}{t}. \quad (6.36)$$

This is softer than the corresponding result for the transverse part of the Green function, since $\beta^{(T)}(g_R)$ is non-zero even in free field theory, and we have

$$c^0_{jj}{}^{(T)}(k^2) \sim \beta_0^{(T)} t. \quad (6.37)$$

7. Conclusions

To conclude, we have introduced a new formalism, non-linear source renormalisation, to describe the renormalisation of composite operator Green functions. Within this framework, the standard textbook treatment of the renormalisation of insertions of a single composite operator into Green functions of elementary fields appears as the linear approximation to the source renormalisation. The non-linearity is required to renormalise Green functions involving multiple composite operators. The method is therefore conceptually very simple, allowing the renormalisation of Green functions involving arbitrary numbers of composite operators to be described in a unified way, based on a single renormalisation prescription within the generating functional itself.

The NLSR method also allows the constraints on the renormalisation counterterms due to the symmetries of the field theory to be implemented in a simple way, through the Ward identities satisfied by the generating functional. One advantage of the method is that anomalous Ward identities can be applied with no extra difficulties.

The effect of the new divergences in Green functions with two composite operators on the operator product expansion is also described very straightforwardly in the NLSR formalism. A simple RGE for the Wilson coefficients was presented, which, like the RGEs for the Green functions themselves, involves the coefficients of the Taylor expansion of the new anomalous dimension function.

These general features were illustrated with two examples, the first concerning the 2-point Green function for the conserved electromagnetic current in QCD, which allows the asymptotic form of the fundamental ratio $R = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ to be determined from the RGE, and the second involving Green functions of the operators j_μ^5 , FF^* occurring in the anomalous axial-vector Ward identity. These Green functions play an important role in several areas, including the $U(1)$ problem and the η' (see, e.g., refs.[10,7]), the explanation of the EMC measurements of the proton “spin” (see, e.g., ref.[11]), and a generalisation of the Zamolodchikov C -theorem based on the axial-vector current rather than the energy-momentum tensor[12].

Nevertheless, despite their physical importance, these examples are still comparatively elementary from the point of view of renormalisation theory. It would therefore be interesting to exhibit these new techniques in more complicated contexts, such as considered in ref.[3], where, for example, the relations between the primitive divergences and sub-divergences of different Green functions implicit in the NLSR formalism would be manifest. This type of detailed diagrammatic analysis would also be necessary to give a formal proof of renormalisability of composite operator Green functions within the NLSR formalism.

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Appendix A. Equivalence of the RGEs

We show here how the RGE (3.2) in the NLSR formalism is equivalent to the corresponding RGE (2.10) in the formalism where new coupling constants are introduced for each new counterterm.

The first step is clearly to evaluate the anomalous dimension function $\gamma^i(S_R)$. We do this explicitly up to $O(S_R^2)$, since this is sufficient to see how the equivalence works in general. The bare source in the NLSR method is given by eq.(3.2), viz.

$$S_B^i = \mu^{-\epsilon} Z_l^i (S_R^l + L_{(p,q)}^l{}_{jk} \partial^p S_R^j \partial^q S_R^k + O(S_R^3)). \quad (A.1)$$

Inverting, we have

$$S_R^i = \mu^\epsilon (Z^{-1})_l^i S_B^l - \mu^{2\epsilon} L_{(p,q)}^l{}_{jk} (Z^{-1})_m^j (Z^{-1})_n^k \partial^p S_B^m \partial^q S_B^n + O(S_B^3), \quad (A.2)$$

and so, (omitting the (p, q) label on L^l_{jk} for brevity),*

$$\begin{aligned}\hat{\gamma}^i(S_R) &= \mu \frac{dS_R^i}{d\mu} \\ &= \hat{\gamma}^i_j S_R^j - \left(\mu \frac{dL^l_{jk}}{d\mu} + L^i_{mk} \hat{\gamma}^m_j + L^i_{jn} \hat{\gamma}^n_k - L^l_{jk} \hat{\gamma}^i_l \right) \partial^p S_R^j \partial^q S_R^k + O(S_R^3),\end{aligned}\quad (A.3)$$

where $\hat{\gamma}^i_j = \epsilon \delta^i_j - \gamma^O_{ji}$.

Now return to the extra coupling constant description of section 2. There, we introduce the new coupling $f_R^i_{jk}$ defined by eq.(2.8),

$$f_B^i_{jk} = \mu^\epsilon Z_l^i (Z^{-1})_j{}^m (Z^{-1})_k{}^n (f_R^l_{mn} + L^l_{mn}). \quad (A.4)$$

Since $f_B^i_{jk}$ is μ -independent, we find

$$\mu \frac{d}{d\mu} (f_R^i_{jk} + L^i_{jk}) = \hat{\gamma}^i_l (f_R^l_{jk} + L^l_{jk}) - \hat{\gamma}^m_j (f_R^i_{mk} + L^i_{mk}) - \hat{\gamma}^n_k (f_R^i_{jn} + L^i_{jn}). \quad (A.5)$$

Separating off the term linear in f_R , we therefore have

$$\mu \frac{d}{d\mu} f_R^i_{jk} = -\epsilon f_R^i_{jk} + \gamma^i_l f_R^l_{jk} - \gamma^m_j f_R^i_{mk} - \gamma^n_k f_R^i_{jn} + \beta^i_{jk}, \quad (A.6)$$

as given in eq.(2.9), where

$$\beta^i_{jk} = \hat{\gamma}^i_l L^l_{jk} - \hat{\gamma}^m_j L^i_{mk} - \hat{\gamma}^n_k L^i_{jn} - \mu \frac{dL^i_{jk}}{d\mu}. \quad (A.7)$$

Comparing with eq.(A.3), we therefore see that in the NLSR formalism,

$$\hat{\gamma}^i(S_R) = \hat{\gamma}^i_j S_R^j + \beta^i_{jk} \partial^p S_R^j \partial^q S_R^k + O(S_R^3). \quad (A.8)$$

The equality of the two RGEs, of course for $f_R = 0$, follows if we can show

$$\gamma^i(S_R) \frac{\delta W}{\delta S_R^i} = \gamma^i_j S_R^j \frac{\delta W}{\delta S_R^i} + \beta^i_{jk} \frac{\partial W}{\partial f_R^i_{jk}} + O(S_R^3). \quad (A.9)$$

This is clear at $O(S_R)$. At $O(S_R^2)$, using eq.(A.8), we find as required,

$$\gamma^i(S_R) \frac{\delta W}{\delta S_R^i} - \gamma^i_j S_R^j \frac{\delta W}{\delta S_R^i} = \int \beta^i_{jk} \partial^p S_R^j \partial^q S_R^k O_i^R, \quad (A.10)$$

$$= \beta^i_{jk} \frac{\partial W}{\partial f_R^i_{jk}}. \quad (A.11)$$

* In order to avoid complicating the notation, we have allowed some ambiguity here. The symbol γ^i_j is used in the text exclusively for the functional derivative of the anomalous dimension in the NLSR formalism, i.e. $\gamma^i_j(x, y) = \delta \gamma^i(S_R(x)) / \delta S_R^j(y) |_{S_R=0}$, so that $\gamma^i_j(x, y) = -\gamma^O_{ji} \delta(x, y)$. Below, we also use the notation γ^i_j simply for $-\gamma^O_{ji}$, i.e. as the conventional anomalous dimension matrix for the sources. It should be clear from the context which is intended. Similarly, trivial delta functions of this type have been omitted throughout appendix B.

Appendix B. Composite operators occurring in the Lagrangian

In this appendix, we show how our treatment of composite operators which appear in the Lagrangian is related to the approach of ref.[3] using position-dependent couplings, and verify that the anomalous dimension matrix for such operators is related to the beta functions by $\gamma^i_j = \partial_j \beta^i$.

In the position-dependent coupling approach, the total bare coefficient of the operator O_i in the extended Lagrangian, which we denote here by G^i_B , is expanded in powers and derivatives of the renormalised coupling $g_R(x)$. We write

$$G^i_B = G^i(g_R(x)), \quad (B.1)$$

where $G^i_B|_{\partial_\mu g_R=0} = g^i_B$, the usual bare coupling constant. Then, from the μ -independence of G^i_B , and letting $F^i_j \equiv \delta G^i / \delta g^j_R|_{\partial_\mu g_R=0}$, we find

$$\hat{\beta}^i = -(F^{-1})^i_k \mu \frac{\partial G^k}{\partial \mu} \Big|_{\partial_\mu g_R=0}, \quad (B.2)$$

and so,

$$\begin{aligned} \partial_j \hat{\beta}^i &= -(F^{-1})^i_{k,j} \mu \frac{\partial G^k}{\partial \mu} \Big|_{\partial_\mu g_R=0} - (F^{-1})^i_k \mu \frac{\partial}{\partial \mu} F^k_j \\ &= -(F^{-1})^i_k \mu \frac{d}{d\mu} F^k_j \end{aligned} \quad (B.3)$$

In the NLSR formalism of section 3, we keep g_R as a constant parameter of the perturbation expansion, and restrict the x -dependence to the source S^i_R . The coefficient of the operator O_i in the regularised action is written here as

$$\begin{aligned} G^i_B &= g^i_B + S^i_B \\ &= g^i_B(g_R) + \hat{Z}^i_l (S^l_R + L_{(p,q)}^l{}_{jk} \partial^p S^j_R \partial^q S^k_R + \dots) \\ &= G^i(g_R + S_R(x)), \end{aligned} \quad (B.4)$$

(where we have absorbed the factor $\mu^{-\epsilon}$ into $\hat{Z}$). From the definition of the anomalous dimension function as $\hat{\gamma}^i(S_R) = \mu dS^i_R/d\mu$, we have

$$\hat{\gamma}^i(S_R) = -(\hat{Z}^{-1})^i_k \mu \frac{d\hat{Z}^k_l}{d\mu} S^l_R + O(S_R^2), \quad (B.5)$$

and so the anomalous dimension matrix is

$$\hat{\gamma}^i_j = \hat{\gamma}^i_{,j}|_{S_R=0} = -(\hat{Z}^{-1})^i_k \mu \frac{d\hat{Z}^k_j}{d\mu}. \quad (B.6)$$

To relate the two methods, we observe from eq.(B.4) that

$$\hat{Z}_j^i = \frac{\delta G^i(g_R + S_R)}{\delta S_R^j} \Big|_{S_R=0} = F^i_j. \quad (B.7)$$

The coefficients of the terms in eq.(B.4) of $O(S_R^2)$ and higher are given by higher derivatives of G^i . Substituting into eq.(B.6), we find

$$\hat{\gamma}^i_j = -(F^{-1})^i_k \mu \frac{dF^k_j}{d\mu}. \quad (B.8)$$

The equality of (B.8) and (B.3) is then apparent, and we verify

$$\gamma^i_j = \partial_j \beta^i \quad (B.9)$$

for these operators.

The further terms in the RGEs (3.6), (3.7) show similar simplifications in the case where all the indices refer to operators in the Lagrangian. For example, to second order in S_R^i , we have

$$\hat{Z}_l^i L^l_{jk} = \frac{\delta^2 G^i}{\delta S_R^j \delta S_R^k} \Big|_{S_R=0} = F^i_{j,k}, \quad (B.10)$$

since for such operators there are no derivatives in the $O(S_R^2)$ counterterms on dimensional grounds. Differentiating eq.(B.10) gives

$$\mu \frac{dL^i_{jk}}{d\mu} = \hat{\gamma}^i_m L^m_{jk} - \hat{\gamma}^m_j L^i_{mk} - \hat{\gamma}^n_k L^i_{jn} - \hat{\beta}^i_{j,k}. \quad (B.11)$$

Comparing with eq.(A.7), we verify that where i, j, k all refer to operators in the Lagrangian, we have

$$\gamma^i_{jk} = \beta^i_{jk} = \partial_j \partial_k \beta^i. \quad (B.12)$$

Of course, other components of γ^i_{jk} , notably γ^0_{jk} , are not so simple.

Using these results, it is now a straightforward exercise to compare our RGEs (3.6), (3.7) with the corresponding RGEs of ref.[3] (eqs.(2.20), (2.21)) in the case they consider, viz. operators occurring in the Lagrangian, and no VEVs except of course for the unit operator.

Appendix C. RGE for the Wilson coefficients in the OPE

We discuss here the subtle questions in the derivation of the RGE for the Wilson coefficients $c^i_{jk}(x-y)$ which were deferred in section 5, and also show the relation between the corresponding RGEs in the NLSR and extra coupling formalisms.

We start from the RGE (5.7) for the Green functions after use of the OPE, viz.

$$\begin{aligned} \mathcal{D}c^i_{jk}(x-y) W_{iS\dots S}(y,\dots) - \left(\int dz \gamma^i_l(z,y) c^l_{jk}(x-y) - \int dz \gamma^l_j(z,x) c^i_{lk}(z-y) \right. \\ \left. - \int dz \gamma^l_k(z,y) c^i_{jl}(x-z) - \int dz \gamma^i_{jk}(z,x,y) \right) W_{iS\dots S}(z,\dots) = 0, \end{aligned} \quad (C.1)$$

writing all the spacetime dependence explicitly. Now from the definition of the anomalous dimension function $\gamma^i(S_R)$ and with the notation of eq.(A.8) we have

$$\gamma^i_j(z,x) = -\gamma^O_{j^i} \delta(z,x), \quad (C.2)$$

$$\gamma^i_{jk}(z,x,y) = \beta^i_{jk} \partial_x^p \partial_y^q (\delta(z,x) \delta(x,y)) + j \leftrightarrow k. \quad (C.3)$$

The anomalous dimension terms are straightforward. For example, from eq.(C.2),

$$\int dz \gamma^i_l(z,y) c^l_{jk}(x-y) W_{iS\dots S}(z,\dots) = -\gamma^O_{l^i} c^l_{jk}(x-y) W_{iS\dots S}(y,\dots). \quad (C.4)$$

The inhomogeneous term is more subtle. We have

$$\begin{aligned} \int dz \gamma^i_{jk}(z,x,y) W_{iS\dots S}(z,\dots) &= \int dz \beta^i_{jk} \partial_x^p \partial_y^q (\delta(z,x) \delta(x,y)) W_{iS\dots S}(z,\dots) + j \leftrightarrow k \\ &= \beta^i_{jk} \partial_x^p \partial_y^q (\delta(x,y) W_{iS\dots S}(x,\dots)) + j \leftrightarrow k. \end{aligned} \quad (C.5)$$

The fact that the derivatives now act on $W_{iS\dots S}$ prevents us from simply factoring off the Green function to obtain a RGE for the c^i_{jk} . However, the terms where some of the derivatives in eq.(C.5) act on $W_{iS\dots S}$ are less singular than those where all the derivatives act on the delta function. In the spirit of the original OPE approximation (5.1) used to derive eq.(C.1), such terms should be dropped. We are then left simply with

$$\int dz \gamma^i_{jk}(z,x,y) W_{iS\dots S}(z,\dots) \sim \beta^i_{jk} \partial_x^p \partial_y^q (\delta(x,y)) W_{iS\dots S}(y,\dots) + j \leftrightarrow k. \quad (C.6)$$

Factoring off the Green functions from eq.(C.1), we therefore find

$$\begin{aligned} \mathcal{D}c^i_{jk}(x-y) + \gamma^O_{l^i} c^l_{jk}(x-y) - \gamma^O_{j^l} c^i_{jk}(x-y) - \gamma^O_{k^l} c^i_{jl}(x-y) \\ + (\beta^i_{jk} \partial_x^p \partial_y^q \delta(x,y) + j \leftrightarrow k) = 0. \end{aligned} \quad (C.7)$$

The equivalence with the notation of eq.(5.8) in the text is clear.

It is interesting to repeat the derivation of eq.(C.7) in the formalism of section 2, where we introduce extra coupling constants and write the RGE as in eq.(2.10). The analogues of eqs.(5.5) and (5.6) are then,

$$\begin{aligned} \mathcal{D}W_{jkS\dots S} \sim \mathcal{D}c^i_{jk} W_{iS\dots S} + c^i_{jk} n\gamma^\phi W_{iS\dots S} + c^i_{jk} \gamma^O_{i^l} W_{lS\dots S} \\ - c^i_{jk} B^l_{mn} \frac{\partial}{\partial f_{R^l_{mn}}} W_{iS\dots S}, \end{aligned} \quad (C.8)$$

and

$$\begin{aligned} \mathcal{D}W_{jkS\dots S} \sim & c_{jk}^i n\gamma^\phi W_{iS\dots S} + \gamma_j^{O_i} c_{ik}^l W_{lS\dots S} + \gamma_k^{O_i} c_{ji}^l W_{lS\dots S} \\ & - c_{jk}^i B_{mn}^l \frac{\partial}{\partial f_{R^l mn}} W_{iS\dots S} - B_{mn}^l \left(\frac{\partial}{\partial f_{R^l mn}} c_{jk}^i \right) W_{iS\dots S}. \end{aligned} \quad (C.9)$$

The final term in eq.(C.9) is the analogue of the γ_{jk}^i term in the NLSR formalism. Notice that because of the different structure of the RGE, the OPE has been applied to obtain this term, and the Green function $W_{iS\dots S}$ factorises straightforwardly. This is to be contrasted with the NLSR formalism, where the OPE was not used in the final term of eq.(C.1) and we had subsequently to argue that the terms which prevented a simple factorisation were non-leading in the OPE sense. The extra coupling method therefore supports this argument, and presents it in a simple way.

Comparing eqs.(C.8) and (C.9), we find the RGE

$$\mathcal{D}c_{jk}^i = -\gamma_l^{O_i} c_{jk}^l + \gamma_j^{O_l} c_{lk}^i + \gamma_k^{O_l} c_{jl}^i - B_{mn}^l \frac{\partial}{\partial f_{R^l mn}} c_{jk}^i. \quad (C.10)$$

In this formalism, the Wilson coefficients are functions of the new couplings as well, i.e. $c_{jk}^i(g_R, f_{R^i}^{jk})$, and the RGE is homogeneous.

To compare with eq.(C.7), we need the $f_R = 0$ limit of eq.(C.10). From eq.(2.9), we have already $B_{mn}^l = \beta_{mn}^l + O(f_R)$. Equivalence with the NLSR RGE is therefore established if the f_R -dependence of the Wilson coefficients is as follows:

$$c_{jk}^i(g_R, f_{R^i}^{jk}) = c_{jk}^i(g_R) + f_{R^i}^{jk} \partial_x^p \partial_y^q \delta(x, y) + j \leftrightarrow k + O(f_R^2). \quad (C.11)$$

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