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The fragmentation function for Heavy Quarks in QCD

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Abstract

We present the results of a next-to-leading order QCD calculation of the fragmentation function for heavy quarks. Our formalism can be used to obtain the next-to-leading inclusive heavy quark production cross section in any hard collisions, in the limit when the transverse momentum of the heavy quark is much larger than its mass. We apply our results to the inclusive production of heavy quarks in e^+e^- collisions, and present detailed formulas for the moments of the heavy quark fragmentation function in e^+e^- production processes.

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1. Introduction

Heavy quark production is a calculable process in perturbative QCD. In fact, the heavy quark mass acts as a cut-off for final state collinear singularities, and sets the scale for a perturbative expansion in the strong coupling constant. Next-to-leading QCD results for heavy quark production cross sections have been obtained in the past^[1,2,3,4]. These results are applicable when the transverse momentum of the produced heavy quark is not much larger than its mass. In the case when the transverse momentum is much larger than the mass, large logarithms arise to all orders of the perturbative expansion, so that fixed order perturbation theory breaks down^[2]. It is however possible to resum all these logarithms, and in this paper we describe a method for performing this resummation up to the next-to-leading level of accuracy. The method is based on the factorization theorem, which states that the cross section for the production process of a high $p_\perp$ hadron h can be written in factorized form

$$\frac{Ed^3\sigma}{d^3k} = \sum_i \int \frac{E'd^3\hat{\sigma}_i}{d^3k'}(k', \mu) D_i^{(h)}(z, \mu) \frac{dz}{z^2} \quad (1.1)$$

where $k' = k/z$, $D_i^{(h)}(z, \mu)$ is the fragmentation function for producing the hadron h from parton i and $d^3\hat{\sigma}_i$ is the short distance cross section for producing parton i . The cross section $d^3\hat{\sigma}_i$ is not finite, and it is usually defined with a prescription for the subtraction of collinear singularities, like the $\overline{\text{MS}}$ prescription^[5]. Also $D_i^{(h)}$ is prescription dependent, but the convolution of the two terms in eq. (1.1) is not, so that the physical cross section is prescription independent. Formula (1.1) is appropriate for processes like e^+e^- annihilation, where there are no hadrons involved in the initial state. In processes with hadrons in the initial state, a similar formula holds. For example, in the case of the cross section for producing the hadron h in the collisions of hadrons A and B with momenta p_1, p_2 , we have

$$\begin{aligned} \frac{Ed^3\sigma}{d^3k} = & \sum_{kli} \int \frac{E'd^3\hat{\sigma}_{kl,i}(x_1p_1, x_2p_2, k', \mu', \mu)}{d^3k'} \\ & F_k^A(x_1, \mu') F_l^B(x_2, \mu') D_i^{(h)}(z, \mu) dx_1 dx_2 \frac{dz}{z^2} \end{aligned} \quad (1.2)$$

where $d^3\hat{\sigma}_{kl,i}$ is the short distance cross section for producing parton i from the partons k and l , μ' is the scale associated to the subtraction of the collinear singularities coming from the initial partons, and F_k^A (F_l^B) is the structure function for finding

the parton k (l) in the hadron A (B) with momentum fraction x_1 (x_2).

The fragmentation function $D_i^{(h)}$ is process independent, and it only depends upon the particular hadron h considered. On the other hand, $d^3\hat{\sigma}$ does not depend upon the hadron being produced. It only depends upon the production process. The partonic cross sections $d^3\hat{\sigma}$ is calculable in perturbative QCD. The fragmentation functions, as well as the structure functions, must instead be measured at some scale, and their value at any other scale can be obtained using the Altarelli-Parisi evolution equation.

It turns out, however, that in the case of the production of a heavy flavoured hadron, the fragmentation function is also calculable in perturbation theory, at least in the limit when the mass of the heavy quark can be considered much larger than typical hadronic scales. In the following sections we will describe this calculation at the next-to-leading level. The calculation will be illustrated in the case of heavy quark production in e^+e^- -collisions. Since $D_i(x, \mu)$ is process independent, the result can be used in any other production process.

The rest of this paper is organized as follows. In Section II we present the general kinematic framework for heavy quark inclusive production in e^+e^- -annihilation. In Section III we will illustrate the method for calculating the fragmentation function in the $\overline{\text{MS}}$ scheme. In Section IV, we give a detailed description of the application of our result to the inclusive production of heavy quarks in e^+e^- -annihilation. In Section V, we show how to incorporate effects which are enhanced in the large x region into our formulas. In Section VI we give some physics considerations and conclusions. Some phenomenological results are contained in ref. [6]. A more detailed phenomenological analysis will be given in a forthcoming publication. The application of the fragmentation function formalism to hadroproduction of high $p_\perp$ heavy quarks is in preparation^[7].

2. Kinematics preliminaries

In this section we illustrate the kinematics of the inclusive production of a heavy quark H in e^+e^- -annihilation via γ and Z exchange

$$e^+e^- \rightarrow \gamma, Z \rightarrow H + X \quad (2.1)$$

If we call q and k the four-momentum of the virtual vector boson and the produced heavy quark respectively, then the quantity

$$x = \frac{2qk}{q^2} \quad (2.2)$$

is the energy of the heavy quark expressed as a fraction of the beam energy in the centre-of-mass frame.

The cross section can be written as the product of a leptonic and a hadronic tensor. The hadronic tensor $d\sigma^{\mu\nu}$ is defined by

$$d\sigma^{\mu\nu} = \frac{8\pi}{q^2} \frac{d^3k}{2k^0(2\pi)^3} \sum_X \int \frac{d^4q'}{(2\pi)^4} \langle 0 | J_a^\mu(q') | H + X \rangle \langle H + X | J_b^\nu(q) | 0 \rangle. \quad (2.3)$$

The notation has the following meaning. The intermediate state $H + X$ is a state with the heavy quark H carrying momentum k , plus anything else, which is represented by X . The sum in X means that we sum (and integrate) over all possible X systems, that is to say that we consider fully inclusive heavy quark production. J_a^μ are the currents, and the label a stands for the type of current, either axial or vector or a combination. In what follows, for ease of notation, we will not indicate the dependence of the hadronic tensor upon a, b . The normalization factor $8\pi/q^2$ is chosen in order to simplify subsequent formulas. The integration in q' eliminates the four-dimensional delta function of momentum conservation, so that the hadronic tensor so defined is directly related to a cut blob with a fixed final state heavy quark line, as represented in fig. 1.

Calling θ the angle of the heavy quark with respect to the beam axis, we have

$$\frac{d^3k}{2k^0(2\pi)^3} = \frac{q^2}{2(4\pi)^2} \sqrt{x^2 - \frac{4m^2}{q^2}} \frac{d\phi}{2\pi} d\cos\theta dx \quad (2.4)$$

and the ϕ integration is irrelevant and can be immediately performed. By Lorentz invariance the hadronic tensor can be written in the form

$$\begin{aligned} \frac{d\sigma^{\mu\nu}}{dx d\cos\theta} = & A \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \\ & + B \left(k_\mu - q_\mu \frac{qk}{q^2} \right) \left(k_\nu - q_\nu \frac{qk}{q^2} \right) + C i \epsilon_{\mu\nu\sigma\rho} q_\sigma k_\rho \\ & + D \frac{q_\mu q_\nu}{q^2} + E (q_\mu k_\nu + k_\mu q_\nu) \end{aligned} \quad (2.5)$$

Since $\sigma^{\mu\nu}$ must be contracted with the leptonic tensor, for small lepton masses the D and E terms can always be neglected. The coefficients A , B and C can be expressed in terms of the transverse, longitudinal, and asymmetric cross section

$$\frac{d\sigma_L}{dx} = \frac{d\sigma^{\mu\nu}}{dx d\cos\theta} \frac{(k_\mu - q_\mu \frac{qk}{q^2})(k_\nu - q_\nu \frac{qk}{q^2})}{|\vec{k}|^2} \quad (2.6)$$

$$\frac{d\sigma_T}{dx} = \frac{1}{2} \left[\frac{d\sigma^{\mu\nu}}{dx d\cos\theta} \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) - \frac{d\sigma_L}{dx} \right] \quad (2.7)$$

$$\frac{d\sigma_A}{dx} = \frac{d\sigma^{\mu\nu}}{dx d\cos\theta} i\epsilon_{\mu\nu\sigma\rho} \frac{q_\sigma k_\rho}{2q^0 |\vec{k}|} \quad (2.8)$$

and

$$A = \frac{d\sigma_T}{dx}, \quad B = \frac{1}{|\vec{k}|^2} \left(\frac{d\sigma_L}{dx} - \frac{d\sigma_T}{dx} \right), \quad C = -\frac{1}{q^0 |\vec{k}|} \frac{d\sigma_A}{dx}. \quad (2.9)$$

Integrating over the angular variables we obtain

$$\frac{d\sigma^{\mu\nu}}{dx} = \frac{2}{3} \left(2 \frac{d\sigma_T}{dx} + \frac{d\sigma_L}{dx} \right) \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \quad (2.10)$$

which is the combination contributing to the total cross section. Defining

$$\frac{d\sigma}{dx} = \frac{d\sigma_T}{dx} + \frac{1}{2} \frac{d\sigma_L}{dx} \quad (2.11)$$

with our normalization, in lowest order, and for $m = 0$ we have

$$\frac{d\sigma^{(0)}}{dx} = \delta(1-x). \quad (2.12)$$

The fragmentation function is defined as

$$D(x) = \frac{1}{\sigma_{\text{tot}}} \frac{d\sigma}{dx} \quad (2.13)$$

$$\sigma_{\text{tot}} = \int \frac{d\sigma}{dx} dx \quad (2.14)$$

In what follows, we always neglect effects which are suppressed by powers of the heavy quark mass. It turns out that in this case the transverse and longitudinal cross sections do not depend upon the type of current. More specifically, axial-axial and vector-vector contributions are equal, and axial-vector interference terms contribute only to the asymmetric cross section.

3. The fragmentation function to next-to-leading QCD order

The cross section for heavy quark production $d\sigma/dx$ is calculable in QCD as a power expansion in terms of the strong coupling constant, since the mass of the quark regulates the collinear divergences which arise when the heavy quark radiates gluons, while soft and other collinear divergences cancel as usual between real and virtual contributions. We have therefore

$$\frac{d\sigma}{dx}(x, Q, m) = \sum_{i=0}^{\infty} a^{(i)}(x, Q, m) \left(\frac{\alpha_S(Q)}{2\pi} \right)^i. \quad (3.1)$$

At this point we introduce the notation

$$\frac{d\sigma^{[1]}}{dx}(x, Q, m) = a^{(0)}(x) + a^{(1)}(x, Q, m) \frac{\alpha_S(Q^2)}{2\pi} \quad (3.2)$$

for the $\mathcal{O}(\alpha_S)$ expression for the cross sections, or for any other quantity having an expansion in α_S (observe that since we are neglecting power suppressed effects in the heavy quark mass the term $a^{(0)}$ does not depend upon m and Q). The diagrams contributing to the terms $a^{(0)}$ and $a^{(1)}$ are shown in fig. 2. Neglecting terms which are suppressed by powers of the heavy quark mass, they give

$$\frac{d\sigma^{[1]}}{dx}(x, Q, m) = \delta(1-x) + \frac{\alpha_S(Q^2)}{2\pi} a^{(1)}(x, Q, m) \quad (3.3)$$

$$\begin{aligned} a^{(1)}(x, Q, m) = & C_F + C_F \left[\log \frac{Q^2}{m^2} \left(\left(\frac{1}{1-x} \right)_+ (1+x^2) + \frac{3}{2} \delta(1-x) \right) \right. \\ & + 2 \frac{1+x^2}{1-x} \log x - \left(\frac{\log(1-x)}{1-x} \right)_+ (1+x^2) \\ & \left. + \frac{1}{2} \left(\frac{1}{1-x} \right)_+ (x^2 - 6x - 2) + \left(\frac{2}{3} \pi^2 - \frac{5}{2} \right) \delta(1-x) \right] \end{aligned} \quad (3.4)$$

The corresponding formulas for the transverse and longitudinal cross sections are given by

$$\begin{aligned} \frac{d\sigma_T^{[1]}}{dx}(x, Q, m) &= \delta(1-x) + \frac{\alpha_S(Q^2)}{2\pi} a_T^{(1)}(x, Q, m) \\ \frac{d\sigma_L^{[1]}}{dx}(x, Q, m) &= \frac{\alpha_S(Q^2)}{2\pi} a_L^{(1)}(x, Q, m) \end{aligned}$$

$$a_L^{(1)} = 2C_F$$

$$a_T^{(1)}(x, Q, m) = a^{(1)}(x, Q, m) - \frac{a_L^{(1)}}{2} \quad (3.5)$$

As we can see, $a^{(1)}$ contains a logarithm of Q^2/m^2 , and therefore, in the high energy limit, becomes large. This is a manifestation of the collinear singularities due to gluon emission. One also expects that collinear singularities will arise to all orders in perturbation theory, and the coefficients $a^{(n)}(x, Q, m)$ will behave as $\log^n(Q^2/m^2)$ for large Q . The factorization theorem, together with the Altarelli-Parisi evolution equation, tells us how the large logarithms are organized to all orders in the perturbative expansion. The factorization formula

$$\frac{d\sigma}{dx}(x, Q, m) = \sum_i \int_x^1 \frac{d\hat{\sigma}_i}{dx}(x/z, Q, \mu) \hat{D}_i(z, \mu, m) \frac{dz}{z} \quad (3.6)$$

states that the physical cross section can be written as a convolution of two terms: a short-distance cross section for producing a parton i (where i can be a heavy or light quark or antiquark, or a gluon) $\frac{d\hat{\sigma}_i}{dx}$, which is insensitive to the low energy features of the process, and therefore does not depend upon the mass of the quark, and a fragmentation function $\hat{D}_i$ for a parton i to go into the heavy quark H , which is insensitive to the high-energy part of the process, and therefore does not depend upon Q . The scale μ is an arbitrary scale that separates the low from the high energy dynamics. The definitions of $\frac{d\hat{\sigma}}{dx}$ and of $\hat{D}$ are not unique. They depend upon the scheme one uses to separate the collinear singularities. For definiteness, we will always refer to the $\overline{\text{MS}}$ factorization scheme^[5].

The fragmentation function $\hat{D}$ obeys the Altarelli-Parisi evolution equation

$$\frac{d\hat{D}_i}{d\log \mu^2}(x, \mu, m) = \sum_j \int_x^1 P_{ij}\left(\frac{x}{z}, \alpha_S(\mu)\right) \hat{D}_j(z, \mu, m) \frac{dz}{z} \quad (3.7)$$

with

$$P_{ij}(x, \alpha_S(\mu)) = \frac{\alpha_S(\mu)}{2\pi} P_{ij}^{(0)}(x) + \left(\frac{\alpha_S(\mu)}{2\pi}\right)^2 P_{ij}^{(1)}(x) + \mathcal{O}(\alpha_S^2). \quad (3.8)$$

$P_{ij}^{(0)}$ are the well-known Altarelli-Parisi splitting functions^[8], and $P_{ij}^{(1)}$ has been computed in refs. [9], [10], [11], and [12].

The evolution equation resums all the large logarithms appearing in $\hat{D}$. More specifically, if we are given the function $\hat{D}(x, \mu_0)$ at some scale μ_0 , then the solution

of eq. (3.7) $\hat{D}(x, \mu)$ has a power expansion in terms of $\alpha_S(\mu)$ which organizes correctly all the powers of $\log(\mu/\mu_0)$ arising in perturbation theory. If we solve the evolution equation using the expression of order α_S for $P(x)$, then $\hat{D}(x, \mu)$ will include correctly all the powers $\alpha_S^n(\mu) \log^n(\mu/\mu_0)$. If we use the order α_S^2 expression for $P(x)$ then also the subleading terms $\alpha_S^{n+1}(\mu) \log^n(\mu/\mu_0)$ will be included correctly in $\hat{D}$. In the next section we will show the details of this procedure.

At this point it is clear that the only missing ingredient that we need in order to calculate $\frac{d\sigma}{dx}$, including the resummation of all the leading and next-to-leading logarithms, is an initial condition for $\hat{D}$ which is accurate to next-to-leading order. When μ_0 is taken to be of the order of the mass m , $\hat{D}_i(x, \mu_0, m)$ cannot contain large logarithms. It must therefore be possible to express it as a perturbative expansion in powers of α_S

$$\hat{D}_i(x, \mu_0, m) = d_i^{(0)}(x) + \frac{\alpha_S}{2\pi} d_i^{(1)}(x, \mu_0, m) + \mathcal{O}(\alpha_S^2). \quad (3.9)$$

To be more precise, when we say that μ_0 is taken to be of the order of the mass, we mean that $\log(\mu_0/m)$ is not large with respect to $1/\alpha_S$, while we always assume that m/μ_0 is small, in order to neglect corrections suppressed by powers of the mass. In practice, after the mass corrections have been dropped, one can safely use also values of μ_0 equal or less than m .

We expect $\hat{D}_g$ to be of order α_S , since in order to produce a heavy quark from a gluon the strong coupling constant must enter at least once, and $\hat{D}_q, \hat{D}_{\bar{H}}$ to be of order α_S^2 . Using the expansions

$$\begin{aligned} \frac{d\sigma}{dx}(x, Q, m) &= \delta(1-x) + \frac{\alpha_S}{2\pi} a^{(1)}(x, Q, m) + \mathcal{O}(\alpha_S^2) \\ \frac{d\hat{\sigma}_H}{dx}(x, Q, \mu) &= \delta(1-x) + \frac{\alpha_S}{2\pi} \hat{a}^{(1)}(x, Q, \mu) + \mathcal{O}(\alpha_S^2) \\ \frac{d\hat{\sigma}_g}{dx}(x, Q, \mu) &= \frac{\alpha_S}{2\pi} \hat{a}_g^{(1)}(x, Q, \mu) + \mathcal{O}(\alpha_S^2). \end{aligned} \quad (3.10)$$

Substituting the above expressions in eq. (3.6), and keeping only the terms up to the order α_S we get

$$\delta(1-x) + \frac{\alpha_S}{2\pi} a^{(1)}(x, Q, m) = \int \delta(1-x/z) d_H^{(0)}(z) \frac{dz}{z}$$

$$+\frac{\alpha_S}{2\pi} \int \delta(1-x/z) d_H^{(1)}(z, \mu, m) \frac{dz}{z} + \frac{\alpha_S}{2\pi} \int \hat{a}_H(x/z, Q, \mu) d_H^{(0)}(z) \frac{dz}{z}. \quad (3.11)$$

We immediately obtain

$$d^{(0)}(x) = \delta(1-x) \quad (3.12)$$

and

$$d^{(1)}(x, \mu, m) = a^{(1)}(x, Q, m) - \hat{a}^{(1)}(x, Q, \mu). \quad (3.13)$$

We can calculate $\frac{d\hat{\sigma}}{dx}$ from the graphs in Fig. 2 calculated in d dimensions, subtracting the collinear singularity according to the $\overline{\text{MS}}$ prescription. We obtain

$$\begin{aligned} \frac{d\hat{\sigma}_H^{[1]}}{dx}(x, Q, \mu) &= \delta(1-x) + \frac{\alpha_S(Q^2)}{2\pi} \hat{a}^{(1)}(x, Q, \mu) \\ \hat{a}^{(1)}(x, Q, \mu) &= C_F \left[\log \frac{Q^2}{\mu^2} \left(\left(\frac{1}{1-x} \right)_+ (1+x^2) + \frac{3}{2} \delta(1-x) \right) \right. \\ &\quad + \left(\frac{\log(1-x)}{1-x} \right)_+ (1+x^2) - \frac{3}{4} \left(\frac{1}{1-x} \right)_+ (1+x^2) \\ &\quad \left. + \frac{9+5x}{4} + 2 \frac{1+x^2}{1-x} \log x - \frac{7}{2} (1+x) + \left(\frac{2}{3} \pi^2 - \frac{9}{2} \right) \right] \quad (3.14) \end{aligned}$$

where $\hat{a}^{(1)} = \hat{a}_T^{(1)} + \hat{a}_L^{(1)}/2$, and $\hat{a}_L^{(1)} = a_L^{(1)}$. This result agrees with ref. [13], and, after correcting a typing error, with ref. [14]. From eqs. (3.4), (3.13) and (3.14) we obtain the final result

$$\begin{aligned} \hat{D}^{[1]}(x, \mu, m) &= \delta(1-x) + \frac{\alpha_S C_F}{2\pi} \left[\frac{1+x^2}{1-x} \left(\log \frac{\mu^2}{m^2} - 2 \log(1-x) - 1 \right) \right]_+ \\ \hat{D}_{[q\bar{q}H]}(x, \mu, m) &= \mathcal{O}(\alpha_S^2) \\ \hat{D}_g(x, \mu, m) &= \mathcal{O}(\alpha_S). \quad (3.15) \end{aligned}$$

The set of conditions obtained above is independent of the process considered, as implied by the factorization theorem. The same set of initial conditions can be applied to any other heavy quark production process. The only change may be due to the fact that in production processes where the production of a gluon is not suppressed with respect to the production of a quark, one needs an expression for $\hat{D}_g$ which is accurate up to the order α_S . From general considerations one can show that the

correct condition is

$$\hat{D}_g^{[1]}(x, \mu, m) = \frac{\alpha_S T_f}{2\pi} (x^2 + (1-x)^2) \log \frac{\mu^2}{m^2} \quad (3.16)$$

or $\hat{D}_g^{[1]} = 0$ for $\mu = m$. This condition was also verified by comparing the one parton inclusive cross section $q_i g \rightarrow q_j + X$ calculated in ref. [15] with the inclusive cross section calculated in ref. [2] for the same process, in the case when q_j is a massive quark^[7].

In our calculation we will neglect completely the $\hat{D}_g$ and $\hat{D}_{[q\bar{q}H]}$ components of the production process. The justification for such an approximation is that at LEP energies these subprocesses give a negligible contribution. At very high energies, however, one needs to include all the components of the fragmentation function, and to compute correctly their evolution and mixing.

4. Details of the calculation

In this section we describe in detail the procedure we have followed in order to obtain an expression for the fragmentation function which is accurate to next-to-leading order. For simplicity, we work in an approximation in which we neglect all non-singlet evolution, as well as the difference between crossing-even (odd) splitting functions. This approximation is good for the fragmentation function on the Z^0 peak, since the probability to produce a heavy quark indirectly, via a secondary gluon, is very low at these energies.

Instead of working in the x representation we consider the Mellin transform

$$\hat{D}_N(\mu) = \int dx x^{N-1} \hat{D}(x, \mu). \quad (4.1)$$

The next-to-leading order evolution equation of $\hat{D}_N(\mu)$ has a very simple form, and it can be solved analytically. We have

$$\frac{d\hat{D}_N(\mu)}{d\log \mu^2} = \gamma_N^{(2)} \hat{D}_N(\mu) \quad (4.2)$$

$$\gamma_N^{(2)} = \frac{\alpha_S(\mu)}{2\pi} \left[P_N^{(0)} + \frac{\alpha_S(\mu)}{2\pi} P_N^{(1)} \right] \quad (4.3)$$

where $\gamma_N^{(2)}$ is simply the Mellin transform of the Altarelli-Parisi splitting function in next-to-leading order. The expression for $\alpha_S(\mu)$ in eq. (4.3) must be accurate to next-to-leading order

$$\alpha_S(\mu) = \frac{1}{b_0 \log(\mu^2/\Lambda^2)} \left(1 - \frac{b_1 \log(\log(\mu^2/\Lambda^2))}{b_0^2 \log(\mu^2/\Lambda^2)} \right) \quad (4.4)$$

$$b_0 = \frac{33 - 2n_f}{12\pi} \quad (4.5)$$

$$b_1 = \frac{153 - 19n_f}{24\pi^2} \quad (4.6)$$

where n_f is the number of flavours, including the heavy one. Introducing the variable^[14]

$$t = \frac{1}{2\pi b_0} \log \frac{\alpha_S(\mu_0)}{\alpha_S(\mu)} \quad (4.7)$$

we have

$$\alpha_S(t) = \alpha_S(\mu_0) e^{-2\pi b_0 t}. \quad (4.8)$$

Equation (4.2) takes the form

$$\frac{d\hat{D}_N(t)}{dt} = \left[P_N^{(0)} + \frac{\alpha_S(t)}{2\pi} \left(P_N^{(1)} - \frac{2\pi b_1}{b_0} P_N^{(0)} \right) \right] \hat{D}_N(t) \quad (4.9)$$

which can immediately be solved to give

$$\hat{D}_N(\mu) = \hat{D}_N(\mu_0) \exp \left\{ P_N^{(0)} t + \frac{1}{4\pi^2 b_0} (\alpha_S(\mu_0) - \alpha_S(\mu)) \left(P_N^{(1)} - \frac{2\pi b_1}{b_0} P_N^{(0)} \right) \right\}. \quad (4.10)$$

According to the discussion given in section 2, one must choose the initial value of μ to be of the order of m , and then use for $\hat{D}(x, \mu_0)$ its order α_S expression $\hat{D}^{[1]}$, given in eq. (3.15). In order to obtain the physical fragmentation function, one must convolute the $\overline{\text{MS}}$ fragmentation function with the hard, $\overline{\text{MS}}$ subtracted cross section, evaluated at the scale μ

$$\begin{aligned} \sigma_N(Q) &= \hat{\sigma}_N(Q, \mu) \exp \left\{ P_N^{(0)} t \right. \\ &\quad \left. + \frac{1}{4\pi^2 b_0} (\alpha_S(\mu_0) - \alpha_S(\mu)) \left(P_N^{(1)} - \frac{2\pi b_1}{b_0} P_N^{(0)} \right) \right\} \hat{D}_N^{[1]}(\mu_0, m). \end{aligned} \quad (4.11)$$

where σ_N is

$$\sigma_N(Q) = \int \frac{d\sigma}{dx}(x, Q) x^{N-1} dx, \quad (4.12)$$

$\hat{\sigma}_N$ is the Mellin transform of the quantity defined in eq.(3.14)

$$\hat{\sigma}_N(Q, \mu) = \int \frac{d\hat{\sigma}}{dx}(x, Q, \mu) x^{N-1} dx = 1 + \frac{\alpha_S(\mu)}{2\pi} \hat{a}_N^{(1)}(Q, \mu). \quad (4.13)$$

and $\hat{a}_N^{(1)}(Q, \mu)$ is the Mellin transform of the $\hat{a}^{(1)}(x, Q, \mu)$ defined in eq. (3.14). Setting $\mu = Q$, neglecting next-to-next-to-leading terms and using the equation

$$\sigma_N^{[1]}(\mu_0, m) = \hat{D}_N^{[1]}(\mu_0, m) + \frac{\alpha_S(\mu_0)}{2\pi} \hat{a}_N^{(1)}. \quad (4.14)$$

which follows from eq. (3.13), we can rewrite eq.(4.11) as

$$\sigma_N(Q) = \exp \left\{ P_N^{(0)} t + \frac{1}{4\pi^2 b_0} (\alpha_S(\mu_0) - \alpha_S(Q)) \left(P_N^{(1)} - \hat{a}_N^{(1)} - \frac{2\pi b_1}{b_0} P_N^{(0)} \right) \right\} \sigma_N^{[1]}(\mu_0, m). \quad (4.15)$$

We have introduced the notation $\hat{a}_N^{(1)} = \hat{a}_N^{(1)}(\mu_0, \mu_0)$ since for $Q = \mu$ $\hat{a}_N^{(1)}$ does not depend upon μ . As we can see, eq. (4.15) differs from the next-to-leading $\overline{\text{MS}}$ evolution equation only by a correction to the next-to-leading evolution kernel, and it has as initial condition the $\mathcal{O}(\alpha_S)$ formula for the cross section for producing a massive quark. Eqs. (4.11) and (4.15) are equivalent at the next-to-leading level, and they are all we need in order to compute the inclusive cross section for heavy quark production in e^+e^- collisions, accurate to the next-to-leading level. Explicit expressions for the various quantities entering in eqs. (4.11) and (4.15) are given in appendix A. The fragmentation function in x space can be obtained by inverting the N space formulas numerically, by contour integration in the complex N plane.

5. The $x \rightarrow 1$ region

In the region $x \rightarrow 1$ one is no longer confident that the fragmentation function is correctly predicted by the next-to-leading order calculation. Terms proportional to $\log(1-x)$ arise to any given order in perturbation theory, and in order to obtain a reliable result in the large x region one must resum these logarithms. This feature of the perturbative expansion is due to the fact that in the limit of $x \rightarrow 1$ the phase

space for gluon radiation is reduced, so that the usual cancellation of soft divergences is incomplete.

The region $x \rightarrow 1$ for the evolution of the fragmentation function is relatively well understood^[16]. A modified evolution kernel allows one to resum all leading logarithms of $1 - x$ in the anomalous dimension of the fragmentation function. We have

$$\begin{aligned} \frac{dD(x, Q^2)}{d \log Q^2} &= \int_x^1 \frac{C_F}{\pi} \left[\frac{\alpha_S(Q^2)}{1-z} - \frac{b_0 \alpha_S^2(Q^2) \log(1-z)}{1-z} + \dots \right]_+ D\left(\frac{x}{z}, Q^2\right) \frac{dz}{z} \\ &= \int_x^1 \frac{C_F}{\pi} \left[\frac{\alpha_S(Q^2)}{(1-z)(1 + b_0 \alpha_S(Q^2) \log(1-z))} \right]_+ D\left(\frac{x}{z}, Q^2\right) \frac{dz}{z} \\ &= \int_x^1 \frac{C_F}{\pi} \left[\frac{\alpha_S(Q^2(1-z))}{1-z} \right]_+ D\left(\frac{x}{z}, Q^2\right) \frac{dz}{z} \end{aligned} \quad (5.1)$$

where one uses the usual one loop expression for α_S

$$\alpha_S(\mu) = \frac{1}{b_0 \log \frac{\mu^2}{\Lambda^2}}. \quad (5.2)$$

In discussing the large x limit, we do not need to distinguish between $d\sigma/dx$ and $D(x)$, since they differ by a factor $1 + \alpha_S/\pi$, which carries no enhancement in the large x region. Eq. (5.1) allows us to find all the leading logarithmic behaviour in the $x \rightarrow 1$ region, provided we already know the leading logarithmic behaviour of $D(x, \mu)$ at some initial scale μ . It would seem natural to assume that the correct initial condition is that $D(x, \mu) = \delta(1 - x)$ for $\mu = m_Q$. This is not correct, as one can see from eq. (3.4). In fact, there we can see that there is a term of order $\alpha_S \log(1 - x)/(1 - x)$, which contains a leading singularity as $x \rightarrow 1$, and should be included in the initial condition. This fact makes us suspect that probably this type of terms will arise also in higher orders of α_S , and that we should resum them also if we want a correct result.

Eq. (5.1) is essentially the usual Altarelli-Parisi equation in the limit of $x \rightarrow 1$. The only change is that the argument in α_S is taken to be the maximum virtuality allowed kinematically in the gluon line. Eq. (5.1) can be written in integral form

$$D(x, Q^2) = \frac{C_F}{\pi} \int_x^1 dz \int_{V_0}^{Q^2} \frac{dV}{V} \left[\frac{\alpha_S(V(1-z))}{1-z} \right]_+ D\left(\frac{x}{z}, V\right) + D(x, V_0). \quad (5.3)$$

Eq. (5.3) is represented diagrammatically in fig. 3, where $V = k'^2$ represents the virtuality of the intermediate fermion line. From this representation we can understand the modifications one needs in order to correctly include the effect of the mass of the quark line. First of all, the virtuality V of the intermediate lines is now given by $k'^2 - m^2$. We find that because of the mass term, the virtuality of the intermediate line cannot be lower than $(1 - z)m^2$. In fact, one has

$$\begin{aligned} V &= k'^2 - m^2 = (k' + l)^2 - m^2 \\ &= 2k' \cdot l + (k'^2 - m^2) \\ &= 2k'^0 l^0 (1 - \sqrt{1 - m^2/k'^0{}^2} \cos \theta) + (k'^2 - m^2). \end{aligned} \quad (5.4)$$

Since l^0 is of the order of $|k'|(1 - z)$, we get

$$V > (1 - z)m^2 + (k'^2 - m^2). \quad (5.5)$$

In fig. 3, k'^2 is integrated from its minimum up to its maximum value. For the minimum virtuality $V_0 = 0$ we get the equation

$$D(x, Q^2) = \frac{C_F}{\pi} \int_x^1 dz \int_{(1-z)m^2}^{Q^2} \frac{dV}{V} \left[\frac{\alpha_S(V(1-z))}{1-z} \right]_+ D\left(\frac{x}{z}, V\right) + \delta(1-x) \quad (5.6)$$

where now the limit of the V integration has become z dependent.

Of course, the above derivation cannot be considered as a solid proof. Eq. (5.6), however, reproduces a leading singularity of the term of order α_S in the first iteration, which is in agreement with eq. (3.4). Furthermore, it certainly sums up correctly the leading logarithms arising in the region of virtualities larger than m . In the following we will therefore assume that it is correct.

We now turn our attention to how to incorporate correctly the leading $x \rightarrow 1$ singularities. The Mellin transform of eq. (5.6) is

$$\frac{dD_N(Q^2)}{d \log Q^2} = H_N(Q^2) D_N(Q^2) \quad (5.7)$$

where

$$H_N(V) = \frac{C_F}{\pi b_0} \int \frac{dz}{1-z} z^{N-1} \alpha_S(V(1-z)) \theta(V - m^2(1-z)). \quad (5.8)$$

The integral can be easily performed. We obtain:

$$\begin{aligned} H_N(V) &= \frac{C_F}{\pi b_0} \log \frac{\log \frac{V}{N\Lambda^2}}{\log \frac{V^2}{\Lambda^2 m^2}} \quad \text{for } m^2 > V > \frac{m^2}{N} \\ H_N(V) &= \frac{C_F}{\pi b_0} \log \frac{\log \frac{V}{N\Lambda^2}}{\log \frac{V}{\Lambda^2}} \quad \text{for } V > m^2. \end{aligned} \quad (5.9)$$

In order to solve eq. (5.7) it is convenient to split the evolution into the two ranges from m^2/N to m^2 and from m^2 to Q^2 . The solution is be given by

$$D_N(Q^2) = D_N^{(\text{S, ev})}(Q^2) D_N^{(\text{S, in})} \quad (5.10)$$

where

$$\begin{aligned} D_N^{(\text{S, ev})}(Q^2) &= \exp \left\{ \frac{C_F}{\pi b_0} \left[\log \frac{Q^2}{N\Lambda^2} \log \log \frac{Q^2}{N\Lambda^2} - \log \frac{Q^2}{\Lambda^2} \log \log \frac{Q^2}{\Lambda^2} \right. \right. \\ &\quad \left. \left. - \log \frac{m^2}{N\Lambda^2} \log \log \frac{m^2}{N\Lambda^2} + \log \frac{m^2}{\Lambda^2} \log \log \frac{m^2}{\Lambda^2} \right] \right\} \end{aligned} \quad (5.11)$$

$$\begin{aligned} D_N^{(\text{S, in})} &= \exp \left\{ \frac{C_F}{\pi b_0} \left[\log \frac{m^2}{\Lambda^2 N} \log \log \frac{m^2}{\Lambda^2 N} \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \log \frac{m^2}{\Lambda^2 N^2} \log \log \frac{m^2}{\Lambda^2 N^2} - \frac{1}{2} \log \frac{m^2}{\Lambda^2} \log \log \frac{m^2}{\Lambda^2} \right] \right\}. \end{aligned} \quad (5.12)$$

The above solution contains all the leading $x \rightarrow 1$ singularities of the fragmentation function. We would like now to combine this solution with the next-to-leading solution in eq. (4.11), taking care to avoid overcounting, and obtain an expression which is correct in the next-to-leading sense, and furthermore incorporates correctly the leading singular terms in the limit $x \rightarrow 1$. In order to do this, we must isolate and drop the leading and next-to-leading order terms contained in eqs. (5.11) and (5.12), since they are already included correctly in eq. (4.11). We obtain

$$\begin{aligned} D_N^{(\text{S, ev})}(Q^2) &= \exp \left\{ \frac{C_F}{\pi b_0} \left[-\log(N) \log \frac{\log \frac{Q^2}{\Lambda^2}}{\log \frac{m^2}{\Lambda^2}} + \left(\frac{1}{\log \frac{Q^2}{\Lambda^2}} - \frac{1}{\log \frac{m^2}{\Lambda^2}} \right) \frac{\log^2 N}{2} \right] \right. \\ &\quad \left. + \mathcal{O}(\alpha_S^2) \right\} \end{aligned} \quad (5.13)$$

$$D_N^{(\text{S, in})} = \left(1 - \frac{C_F}{2\pi b_0} \frac{\log^2 N}{\log \frac{m^2}{\Lambda^2}} \right) + \mathcal{O}(\alpha_S^2) \quad (5.14)$$

where $\mathcal{O}(\alpha_S^2)$ indicates terms suppressed by two inverse powers of $\log(Q^2/\Lambda^2)$ or $\log(m^2/\Lambda^2)$. We define therefore

$$C_N^{(S)}(Q^2) = D_N^{(S, \text{ev})}(Q^2) \exp \left\{ -\frac{C_F}{\pi b_0} \left[-\log(N) \log \frac{\log \frac{Q^2}{\Lambda^2}}{\log \frac{m^2}{\Lambda^2}} + \left(\frac{1}{\log \frac{Q^2}{\Lambda^2}} - \frac{1}{\log \frac{m^2}{\Lambda^2}} \right) \frac{\log^2 N}{2} \right] \right\} \quad (5.15)$$

where $C_N^{(S)}$ differs from one by terms of second order in α_S . Comparing eqs. (5.13), (5.14), (5.15) with eq. (4.15), we see that eq. (4.15) corrected for large x effects becomes

$$\begin{aligned} \sigma_N(Q) = & \exp \left\{ P_N^{(0)} t + \frac{1}{4\pi^2 b_0} (\alpha_S(\mu_0) - \alpha_S(Q)) \left(P_N^{(1)} - \hat{a}_N^{(1)} - \frac{2\pi b_1}{b_0} P_N^{(0)} \right) \right\} \\ & \left(\sigma_N^{[1]}(\mu_0, m) + \frac{C_F}{2\pi b_0} \frac{\log^2 N}{\log \frac{m^2}{\Lambda^2}} \right) D_N^{(S, \text{in})} C_N^{(S)}(Q^2). \end{aligned} \quad (5.16)$$

6. Conclusions

In this paper, we have exposed the concepts and the formulas which describe the production of heavy flavour at very large transverse momenta, with a next-to-leading order accuracy. In order to make contact with phenomenological issues, one should clearly understand what are the limitations of this approach, what kind of non-perturbative corrections come into play, and what we expect from the next-to-next-to-leading effects. One clear indication of a possible problem is given by the formulas for the Sudakov effects, where we see that scales of the order of m^2/N^2 come into play. We expect our formulas to enter a non-perturbative regime (i. e. to lose their validity) when $m/N \approx \Lambda$, or, in x space when $m(1-x) \approx \Lambda$. For bottom production, we see that this condition is met for $x \approx 0.9$. Other types of considerations about non-perturbative effects in the fragmentation function of heavy quarks give the same kind of effect, that is to say, an alteration of the x scale by a factor of the order of Λ/m . Monte Carlo models of fragmentation exhibit similar behaviour. On the other hand, we know that the value of α_S at the scale of the b quark mass is large, and effects of order α_S^2 are probably not negligible. There is clearly an interplay between non-perturbative effects and higher order effects. A parametrization of non-perturbative effects will therefore depend upon the specific choice for the parameters of the perturbative part, like the initial scale μ_0 and the

value of Λ . A phenomenological analysis of the fragmentation function physics should address in detail all these problems.

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APPENDIX A: Analytical results in the moments representation

The final formula for the fragmentation function (eq. 5.16) is

$$\begin{aligned} \sigma_N(Q) = & \exp \left\{ P_N^{(0)} t + \frac{1}{4\pi^2 b_0} [\alpha_S(\mu_0) - \alpha_S(Q)] \left[P_N^{(1)} - \hat{a}_N^{(1)} - \frac{2\pi b_1}{b_0} P_N^{(0)} \right] \right\} \\ & \left[\sigma_N^{[1]}(\mu_0, m) + \frac{C_F}{2\pi b_0} \frac{\log^2 N}{\log \frac{m^2}{\Lambda^2}} \right] D^{(S, \text{in})} C_N^{(S)}(Q^2) \end{aligned} \quad (\text{A.1})$$

where

$$t = \frac{1}{2\pi b_0} \log \frac{\alpha_S(\mu_0)}{\alpha_S(Q)} \quad (\text{A.2})$$

and one must use the two-loop expression for α_S given in eq. (4.4). We introduce the notation

$$S_1(N) = \psi_0(N+1) - \psi_0(1) \quad (\text{A.3})$$

$$S_2(N) = -\psi_1(N+1) + \psi_1(1) \quad (\text{A.4})$$

$$S_3(N) = \frac{1}{2}(\psi_2(N+1) - \psi_2(1)) \quad (\text{A.5})$$

$$\zeta(2) = \frac{\pi^2}{6} \quad (\text{A.6})$$

where ψ_m are the poligamma functions, defined as

$$\psi_m(x) = \frac{d^m \log \Gamma(x)}{dx^m}. \quad (\text{A.7})$$

In order to perform the Mellin transform we have used the following identities

$$\int \left(\frac{1}{1-x} \right)_+ x^{N-1} dx = -S_1(N-1) \quad (\text{A.8})$$

$$\int \frac{x^{N-1} \log^m x}{1-x} dx = -\psi_m(N) \quad \text{for } m > 0 \quad (\text{A.9})$$

$$\int \left(\frac{\log(1-x)}{1-x} \right)_+ x^{N-1} dx = \frac{1}{2} (S_1^2(N-1) + S_2(N-1)) \quad (\text{A.10})$$

$$\int \left(\frac{\log(1-x)}{1-x} \right)_+ \log x x^{N-1} dx = \psi_1(N) S_1(N-1) - \frac{1}{2} \psi_2(N). \quad (\text{A.11})$$

We have

$$\begin{aligned} \hat{a}_N^{(1)}(Q, \mu) = & C_F \left[\log \frac{Q^2}{\mu^2} \left(\frac{3}{2} + \frac{1}{N(N+1)} - 2S_1(N) \right) \right. \\ & + \frac{1}{N} - 2 \frac{2N+1}{(N(N+1))^2} - 4\psi_1(N+1) + S_1^2(N) \\ & - \frac{1}{N(N+1)} S_1(N) + \frac{1}{(N+1)^2} + S_2(N) \\ & \left. + \frac{3}{2} S_1(N) - \frac{3}{2} \frac{1}{N+1} + \frac{2}{3} \pi^2 - \frac{9}{2} \right] \end{aligned} \quad (\text{A.12})$$

Furthermore

$$\begin{aligned} d_N^{(1)}(\mu, m) = & C_F \left[\log \frac{\mu^2}{m^2} \left(\frac{3}{2} + \frac{1}{N(N+1)} - 2S_1(N) \right) \right. \\ & - 2S_1^2(N) + \frac{2}{N(N+1)} S_1(N) \\ & \left. - \frac{2}{(N+1)^2} - 2S_2(N) + 2 - \frac{1}{N(N+1)} + 2S_1(N) \right] \end{aligned} \quad (\text{A.13})$$

and

$$a_N^{(1)}(Q, m) = \hat{a}_N^{(1)}(Q, \mu) + d_N^{(1)}(\mu, m) \quad (\text{A.14})$$

Finally

$$\sigma_N^{[1]}(\mu_0, m) = 1 + \frac{\alpha_S(\mu_0^2)}{2\pi} a_N^{(1)}(\mu_0, m). \quad (\text{A.15})$$

A formula for $P_{qq}^{(1)}(x)$ is given in ref. [9]. We give here the Mellin transform of $P_{qq}^{(1)}(x)$. We regularize the singularities at $x = 1$ by an overall $+$ prescription, in such

a way that its first moment $P_1^{(1)} = 0$, which is consistent with our approximation of neglecting the creation of more than one heavy quark pair. Defining

$$\begin{aligned}
 P_F(N) = & \left(2S_1(N) - \frac{1}{N(N+1)} \right) [2S_2(N) - 2\zeta(2)] \\
 & - \frac{2(2N+1)}{N^2(N+1)^2} S_1(N) + 4S_3(N) - 3S_2(N) + 3\zeta(2) \\
 & + \frac{3N^3 + N^2 - 1}{N^3(N+1)^3} - \frac{23}{8}
 \end{aligned} \tag{A.16}$$

$$P_{NF}(N) = \frac{20}{9}S_1(N) - \frac{4}{3}S_2(N) - \frac{1}{6} - 2\frac{11N^2 + 5N - 3}{9N^2(N+1)^2} \tag{A.17}$$

$$\begin{aligned}
 P_G(N) = & -P_F(N) + S_1(N) \left[-\frac{134}{9} - 2\frac{2N+1}{N^2(N+1)^2} \right] + 4S_1(N)S_2(N) \\
 & + S_2(N) \left[\frac{13}{3} - \frac{2}{N(N+1)} \right] + \frac{43}{24} \\
 & + \frac{151N^4 + 263N^3 + 97N^2 + 3N + 9}{9N^3(N+1)^3}
 \end{aligned} \tag{A.18}$$

$$\Delta(N) = 2 \left[-2S_1(N) + \frac{3}{2} + \frac{1}{N(N+1)} \right] \left[2S_2(N) - \frac{1}{3}\pi^2 - \frac{2N+1}{N^2(N+1)^2} \right]. \tag{A.19}$$

We have then

$$P_N^{(1)} = C_F^2(P_F(N) + \Delta(N)) + \frac{1}{2}C_FC_AP_G(N) + C_FN_FT_FP_{NF}(N) \tag{A.20}$$

where

$$C_F = \frac{4}{3}, \quad C_A = 3, \quad T_F = \frac{1}{2} \tag{A.21}$$

and N_F is the number of flavours (including the heavy ones, up to the scale Q). An expression for the Mellin transform can also be extracted from ref. [9], but unfortunately there is a typo there in the expression for $P_F(N)$.

REFERENCES

1. P. Nason, S. Dawson and R. K. Ellis, *Nucl. Phys.* **B303**(1988)607.

2. P. Nason, S. Dawson and R. K. Ellis, *Nucl. Phys.* **B327**(1989)49.
3. W. Beenakker *et al.*, *Phys. Rev.* **D40**(1989)54.
4. W. Beenakker *et al.*, DESY 90/064 (1990).
5. W. A. Bardeen *et al.*, *Phys. Rev.* **D18**(1978)242.
6. B. Mele and P. Nason, *Phys. Lett.* **B245**(1990)635.
7. P. Chiappetta, J. P. Guillet and P. Nason, in preparation.
8. G. Altarelli and G. Parisi, *Nucl. Phys.* **B126**(1977)298.
9. G. Curci, W. Furmanski and R. Petronzio, *Nucl. Phys.* **B175**(1980)27.
10. W. Furmanski and R. Petronzio, *Phys. Lett.* **97B**(1980)437.
11. E. G. Floratos, D. A. Ross and C. T. Sachrajda, *Nucl. Phys.* **B152**(1979)493;
Gonzales-Arroyo and C. Lopez, *Nucl. Phys.* **B166**(1980)429;
E. G. Floratos, R. Lacaze and C. Kounnas, *Nucl. Phys.* **B192**(1981)417.
12. J. Kalinowski, K. Konishi, P. N. Scharbach and T. R. Taylor, *Nucl. Phys.* **B181**(1981)253.
13. G. Altarelli, R. K. Ellis, G. Martinelli and So-Young Pi, *Nucl. Phys.* **B160**(1979)301.
14. W. Furmanski and R. Petronzio, *Z. Phys.* **C11**(1982)293.
15. F. Aversa, P. Chiappetta, M. Greco and J. P. Guillet, *Nucl. Phys.* **B327**(1989)105.
16. A. Bassetto, M. Ciafaloni and G. Marchesini, *Phys. Rep.* **100**(1983)201.

Figure Captions

- Fig. 1: Diagrammatic representation of the hadronic tensor for heavy quark inclusive production.
- Fig. 2: The diagrams contributing to the fragmentation function to order α_S .
- Fig. 3: Diagrammatic representation of the evolution equation for the fragmentation function in the large x limit.

Fig. 1

Fig. 2

Fig. 3