THE UNRAVELLING OF THE QUANTUM GROUP STRUCTURE
IN THE WZNW THEORY

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ABSTRACT

It is found that the local field $g(x)$ in the WZNW theory is a vertex operator for the Kac - Moody (KM) algebra with a central charge k and for the quantum group with the deformation parameter $q = e^{-i\pi/k+\hbar}$ as well. In some sense the loop group appears to be quantum at each point of the circle. To obtain these results we use:

1. ultralocalization of the KM algebra;
2. lattice regularization.

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INTRODUCTION

We will work in the lagrangian approach to conformal field theory [1,2]. Let us recall some standard notations. The well known action of the WZNW theory can be written in the form:

$$S = \frac{k}{8\pi} \text{Tr} \int [d_\mu g g^{-1}]^2 dx dt + \frac{k}{12\pi} \text{Tr} \int d^{-1} [d g g^{-1}]^3, \quad (1)$$

where $g(x, t)$ takes values in a Lie group G , and d_μ for $\mu = 0, 1$ are t - and x - derivatives, respectively. The coordinate x running from 0 to 2π and the field $g(x)$ is periodical $g(x + 2\pi) = g(x)$. We can rewrite (1) in the first order formalism :

$$S = \frac{k}{4\pi} \text{Tr} \int [J_0 g^{-1} d_0 g] dx dt + \frac{k}{12\pi} \text{Tr} \int d^{-1} [d g g^{-1}]^3 - \int H dt, \quad (2)$$

where

$$H = \frac{k}{8\pi} \text{Tr} \int [J_0^2 + [d_1 g g^{-1}]^2] dx, \quad (3)$$

$$J_0 = d_0 g g^{-1}. \quad (4)$$

Let us denote

$$I = \frac{1}{2} (d_0 g g^{-1} - d_1 g g^{-1}),$$

$$J = \frac{1}{2} (g^{-1} d_0 g + g^{-1} d_1 g). \quad (5)$$

These are the left and right currents respectively. They satisfy the relations of the Kac - Moody algebra:

$$\{I_1(x), I_2(y)\} = \gamma C \delta'(x - y) + \frac{\gamma}{2} [I_1(x) - I_2(x), C] \delta(x - y),$$

$$\{J_1(x), J_2(y)\} = -\gamma C \delta'(x - y) - \frac{\gamma}{2} [J_1(x) - J_2(x), C] \delta(x - y), \quad (6)$$

$$\{I_1(x), J_2(y)\} = 0.$$

Here we have written the standard KM relations in the tensor form. Indices 1 and 2 correspond to the ordering of the matrices I and J in the tensor product. We can get from the formulae (6) the Poisson bracket of any two matrix elements of I and J . The matrix C in the tensor product of the first and second spaces is the tensor Casimir operator. It is equal to

$$C = \sum_i (t^i)_1 (t_i)_2,$$

where t^i are generators of the Lie algebra of the group G and

$$\text{Tr} t^i t_i = 1.$$

It is convenient to use in the classical version instead of the central charge k the parameter $\gamma = \pi/k$ and respectively after quantization $\gamma = \pi/k + \hbar$, where $\hbar$ is the dual Coxeter number for the group G .

We can divide our model into two chiral sectors by means of the change of variables [3]:

$$g = u e^{-iQ} v,$$

$$I = d_1 u u^{-1}, \quad J = -v^{-1} d_1 v. \quad (7)$$

Here Q is an x independent diagonal matrix and we can call it the coordinate of the zero mode of the theory. The matrices $u(x)$ and $v(x)$ correspond to the left and right chiral sectors of the theory. Their correlators are conformal blocks [4]. The solutions of the equations of motion for u and v have the form:

$$u(x, t) = u(x - t), \quad v(x, t) = v(x + t).$$

An interesting technical point is to calculate the Poisson brackets or quantum exchange relations for the fields u and v . Several people treated this problem [5-7] and an analogous

one for the Liouville model [1,2,8]. The main result is the quadratic exchange algebra, where some accessoires of the quantum group appear, i.e., for a particular choice of the boundary conditions we have [7]:

$$\{u_1(x), u_2(y)\} = u_1(x)u_2(y)[r^+\theta(x-y) + r^-\theta(y-x)] , \quad (8)$$

where r^+ and r^- are classical r-matrices which correspond to the quantum group G_q on the classical level. We can consider more natural quasi-periodic boundary conditions:

$$u(x + 2\pi) = u(x)D ,$$

where D belongs to the Cartan subgroup of G and might be interpreted as an exponent of the quasi-momentum of the zero mode $D = \exp(iP)$; P and Q are canonically conjugated variables. The relation (8) is modified as follows [6,7]:

$$\{u_1(x), u_2(y)\} = u_1(x)u_2(y)[r^+(P)\theta(x-y) + r^-(P)\theta(y-x)] . \quad (9)$$

Matrices $r^+(P), r^-(P)$ as well as r^+, r^- are closely related with the quantum group. They are classical analogues of 6-j symbols of the quantum group.

The appearance of r-matrices in the WZNW theory is interesting and slightly mysterious. We have many evidences of the presence of the quantum group in the WZNW theory. It is known that R-matrices appear in monodromies of conformal blocks [9,10], invariants of the QG gives us a classification of conformal blocks [11], screening operators form the part of the quantum algebra [12,13], braiding matrices of CFT coincide up to some simple renormalization with 6-j symbols of the quantum group [14], on the classical level we can find the Poisson - Lie symmetry in the theory [6,7,15] and so on. But we have not yet a good explanation why there is a connection between the QG and the WZNW model. To get a QG we need an r-matrix in order to break the classical group symmetry and deform our group. But the WZNW theory is exactly symmetric and there is no natural asymmetric object in it. For example the r-matrix appears in (8,9) after a choice of boundary conditions which break the symmetry.

We will show the more natural (at least for us) approach to the problem of the appearance of the QG in the WZNW model. The key point is the new relation between

the QG and the KM algebra. In some sense it appears that the KM group is quantum from the very beginning. So there is no wonder that the QG plays an important role in the WZNW model which is constructed with the KM group as a symmetry group.

In this paper we will give only a brief sketch of the more physical part of our investigation. Papers with a detailed explanation of all aspects of the problem are in preparation.

1. ULTRALocalization OF THE KAC - MOODY ALGEBRA

We are beginning with a purely mathematical point. We would like to explain what is the cotangent bundle for the Borel subgroup using the language of classical and quantum mechanics. However this simple notion is the key to our further considerations. We will treat only the case $G = SL(n)$. It is not difficult to rewrite all formulae for any semisimple Lie group.

$T^*SL(N)$ can be parametrized by a pair which consists of the group element g and the momentum P . We can choose for example the left invariant momentum P^L . Then the canonical Poisson brackets have the form:

$$\begin{aligned} \{(P^L)_1, g_2\} &= -g_2 C, \\ \{(P^L)_1, (P^L)_2\} &= 1/2[C, (P^L)_1 - (P^L)_2] \end{aligned} \quad (10)$$

$$\{g_1, g_2\} = 0.$$

One can use the same procedure for T^*B where we denote by B the Borel subgroup of $SL(n)$. Let us choose $B = B_-$ (the subgroup of lower triangular matrices) and parametrize T^*B with the pair (t, ω) . Here t belongs to B_- and ω is an element of the algebra of upper triangular matrices b_+ . In this parametrization we have

$$\begin{aligned} \{\omega_1, t_2\} &= -t_2 r^+, \\ \{\omega_1, \omega_2\} &= [r^+, \omega_1 + \omega_2], \end{aligned} \quad (11)$$

$$\{t_1, t_2\} = 0 .$$

r^+ is the standard classical r-matrix:

$$r^+ = \sum_i (h^i)_1 (h_i)_2 + 2 \sum_{\alpha > 0} (t^\alpha)_1 (t_\alpha)_2 , \quad (12)$$

where h^i are generators of the Cartan subalgebra and t^α correspond to positive roots. We see that r^+ plays in (12) the role of the Casimir operator for the group B up to some details.

Simple formulae (12) give us the possibility to construct the KM algebra. To do it we should consider the loop space for T^*B . In the loop space LT^*B variables $t(x)$ and $\omega(x)$ are periodical fields defined on the circle with Poisson brackets:

$$\{\omega_1(x), t_2(y)\} = -t_2(y) r^+ \delta(x-y) ,$$

$$\{\omega_1(x), \omega_2(y)\} = [r^+, \omega_1(x) + \omega_2(y)] \delta(x-y) , \quad (13)$$

$$\{t_1(x), t_2(y)\} = 0 .$$

We can construct from ω and t the following combination:

$$I(x) = \frac{\gamma}{2} \omega t^{-1} + t' t^{-1} , \quad (14)$$

where $t' = dt/dx$. It is easy to check that $I(x)$ satisfies the KM algebra (6). We call this new technical procedure ultralocalization of the KM algebra.

Relations (6) include the Schwinger term with $\delta'(x-y)$. It follows that we can not consider such a theory at a point. The δ' function gives us a connection between the variables at different points. From a more physical point of view there is not any scaling parameter in the theory and so we should have some problems in the coinciding points. And indeed we do have. For example in the standard formalism one cannot calculate the Poisson brackets of the chiral components $u(x)$ and $v(y)$ if x is equal to y . The same is true for the monodromy of the current.

$$M = P \exp \int_0^{2\pi} I(x) dx . \quad (15)$$

The only method we know of to avoid these difficulties at coinciding points is to regularize our theory somehow. The regularized theory (RT) should not be conformal and should include characteristic mass and characteristic distance. In the RT all problems of uncertainty of the Poisson brackets would disappear. It is clear that we can obtain the r -matrix and symmetry breaking in this approach. There are many different ways to regularize the theory. Different ways could give different r -matrices from the same class. The invariant of the regularization is the parameter q of the quantum group.

We would like to suggest a reasonable way to regularize the WZNW model. It is a lattice regularization. The step of the lattice is a natural distance parameter. This approach will give us all necessary results. And we think that any other regularization will give the same answer.

The ultralocalization shows us a natural way to the lattice. The LT^*B can be considered locally by definition. It is easy to rewrite it on the lattice using the experience of the investigation of integrable models [16]. This procedure will give us a lattice analogue of the KM algebra [17].

2. LATTICE AND QUANTIZATION

The lattice analogue of the relations (13) is the following

$$\begin{aligned} \{(\Omega^i)_1, (T^i)_2\} &= \frac{\gamma}{2} (T^i)_2 r^+ (\Omega^i)_1 , \\ \{(\Omega^i)_1, (\Omega^i)_2\} &= \frac{\gamma}{2} [r^+, (\Omega^i)_1 (\Omega^i)_2] . \end{aligned} \quad (16)$$

The gauge transformation on the lattice has the form:

$$L^i = T^{i+1} \Omega^i (T^i)^{-1} , \quad (17)$$

where L is the lattice analogue of the Kac - Moody current. Let us denote the step of the lattice by ϵ . In the continuous limit our variables T and Ω will be connected with t and ω on the circle by means of the formulae:

$$\Omega^i = 1 + \epsilon \omega(x^i), \quad T^i = t(x^i). \quad (18)$$

Here x^i is the coordinate of the i -th point of the lattice.

We can see from the formulae (18) that Ω is the group element and the corresponding algebra element for it is the conjugated momentum ω for the group B . This construction is well known in the theory of quantum groups [18]. In this theory we should consider the group B itself and the space of conjugated momenta b^* on the same footing. The space b^* becomes the Lie algebra which corresponds to the Lie group B^* . In the particular case under consideration it is a very simple construction. B is the group of lower triangular matrices and B^* is the group of upper triangular matrices. Now it is not surprising that we would like to suggest new brackets for T :

$$\{(T^i)_1, (T^i)_2\} = \gamma/2 [r^+, (T^i)_1 (T^i)_2]. \quad (19)$$

So we have regularized the theory.

The formulae (17) and (19) lead to the following Poisson brackets for the current L on the lattice [17]:

$$\begin{aligned} \{(L^i)_1, (L^i)_2\} &= \frac{\gamma}{2} (r^+ (L^i)_1 (L^i)_2 + (L^i)_1 (L^i)_2 r^-), \\ \{(L^i)_1, (L^{i-1})_2\} &= -\frac{\gamma}{2} (L^i)_1 r^+ (L^{i-1})_2. \end{aligned} \quad (20)$$

The Poisson brackets between currents standing at not nearest neighbour points are equal to zero. The current L behaves in the continuous limit:

$$L^i = 1 + \epsilon I(x^i). \quad (21)$$

and analogously for Ω . It is easy to check that in the continuous limit when ϵ goes to zero (20) and (21) give us the usual KM algebra (6) for the current $I(x)$. The r -matrices will combine into the Casimir operator C ,

$$C = 1/2(r^+ - r^-), \quad r^- = -Pr^+P. \quad (22)$$

Here P is the permutation of the first and the second spaces in the tensor product.

It means that the lattice model (20) is the necessary regularization because the symmetry is broken in the model and we can get the symmetry back in the continuous limit. Let us remark that the lattice regularization of the KM algebra (20) is probably closely connected with the lattice regularization of the Toda theory [19].

It is not difficult to rewrite the relations (16), (19), and (20) in the quantum case:

$$\begin{aligned}(\Omega^i)_1(\Omega^i)_2 R^+ &= R^+(\Omega^i)_2(\Omega^i)_1, \\(T^i)_1(T^i)_2 R^+ &= R^+(T^i)_2(T^i)_1,\end{aligned}\tag{23}$$

$$(\Omega^i)_1(T^i)_2 R^+ = (T^i)_2(\Omega^i)_1;$$

$$\begin{aligned}(L^i)_1(L^i)_2 &= R^+(L^i)_1(L^i)_2 R^-, \\(L^i)_1 R^+(L^{i-1})_2 &= (L^{i-1})_2(L^i)_1.\end{aligned}\tag{24}$$

Here R^+, R^- are quantum R-matrices with the deformation parameter q equal to

$$q = e^{-i\gamma} = e^{-i\pi/(k+\hbar)}.\tag{25}$$

This is the relation between the deformation parameter of the quantum group and the central charge of the KM algebra, the latter being determined by formulae (16),(19).

Now we are ready to calculate the exchange relation for the chiral component at one point and the for monodromy of the current. The lattice analogue of the equation

$$du/dx = I(x)u$$

is the following

$$u^i = L^i u^{i-1}.$$

We can choose for simplicity the initial boundary condition $u^0 = 1$. It leads to the formula for u^i :

$$u^i = L^i \dots L^1. \quad (26)$$

If we denote by N the total number of points in the lattice we can write the lattice analogue of the monodromy:

$$M = u^N = L^N \dots L^1. \quad (27)$$

There exists a very useful relation for the currents L^i which helps us in calculations of the exchange algebra of the chiral components (26) and the monodromy matrix (27):

$$(L^i)_1((L^{i+1})_2(L^i)_2(L^{i-1})_2) = ((L^{i+1})_2(L^i)_2(L^{i-1})_2)(L^i)_1. \quad (28)$$

Using (24) we have after straightforward calculations

$$\begin{aligned} (u^i)_1(u^i)_2 &= R^+(u^i)_2(u^i)_1 R^-, \\ (u^i)_1(u^k)_2 &= (u^k)_2(u^i)_1 R^+, \quad i < k, \\ (u^i)_1(u^k)_2 &= (u^k)_2(u^i)_1 R^-, \quad i > k. \end{aligned} \quad (29)$$

We see that the second and the third relations give us the usual chiral exchange algebra for different points. The interesting one is the first relation which is an exchange relation at a single point. The R -matrix appears in it not only from the right hand side as in (8),(9) but from the left hand side too. Thus the local gauge symmetry of the regularized theory:

$$(L^g)^i = g^{i+1} L^i (g^i)^{-1}, \quad (u^g)^i = g^i u^i$$

is in fact a quantum gauge symmetry with g^i taking values in the quantum group, for every point i . In other words we have the nontrivial exchange relations at a single point:

$$(g^i)_1 (g^i)_2 R^+ = R^+ (g^i)_2 (g^i)_1 \quad (30)$$

and g^i and g^j commute for different points i and j . In the continuous limit it leads to the following strange Poisson brackets for $g(x)$:

$$\{g_1(x), g_2(x)\} = \frac{\gamma}{2} [r^+, g_1(x)g_2(x)] , \quad (31)$$

$$\{g_1(x), g_2(y)\} = 0 .$$

The relation for the monodromy is the following:

$$M_1 R^+ M_2 (R^+)^{-1} = R^- M_2 (R^-)^{-1} M_1 . \quad (32)$$

3. THE QUANTUM GROUP IN THE WZNW MODEL

Now we can explain the appearance of the quantum group in the WZNW model using at least three different ways. All of them will give us the same answer.

1. We did see in the previous section that the elements of the gauge group have nontrivial exchange relations with themselves after the regularization. Let us discuss the Poisson brackets (31) in detail. On the one hand, we can think that these brackets are exactly equal to zero because in the continuous limit we should take into account only terms with the δ -function and its derivatives. However our r.h.s. in (31) is nontrivial only at one point and it is finite. Thus if we consider integrals of the matrix elements of $g(x)$ with some smooth weights then their Poisson brackets will be equal to zero. They do not feel our finite nonzero bracket (31) at the coinciding points. But when we are interested in the value of $g(x)$ at the point we should consider it as an element of the quantum group. So our conclusion is the following. If we consider the KM group with a central charge k then its value at a point belongs to the quantum group G_q with the deformation parameter $q = \exp(-i\pi/k + \hbar)$.

We have seen that to get this result we should regularize the theory. When the parameter of regularization ϵ goes to zero the quantum group appears on very short distances of the order of the step of the lattice. In the continuous limit it appears at a single point.

Now let us consider the KM algebra of the current $I(x)$. In our approach it should become the quantum algebra at a single point. But we know that in the continuous limit

$$L = 1 + \epsilon I(x) .$$

It means that we can find the quantum properties of the current algebra only for very big currents of the order of $1/\epsilon$.

2. Formula (32) for the exchange relation for the monodromy of the current gives us an independent way to explain the quantum group structure. It appears that the relation (32) is equivalent to the standard commutation relations for the quantum algebra [19]. So we can construct generators of the quantum algebra from the matrix elements of the matrix M . They will act in the Hilbert space of the left sector of the WZNW theory. The monodromy of the right current will form quantum group generators for the right sector of the theory.

3. The third way is to regularize the exchange relations for u and v with quasi-periodic boundary conditions directly. Using the experience from the previous section and starting with formula (9) we can write for $u(x)$:

$$(R^+)^{-1} u_1(x) u_2(x) = u_2(x) u_1(x) R^+(P)$$

and for $v(x)$:

$$R^+(P) v_1(x) v_2(x) = v_2(x) v_1(x) (R^+)^{-1} , \quad (33)$$

where $R^+(P)$ is the 6-j symbol of the quantum group G_q . On the one hand, these formulae are very natural in the spirit of the section 2. On the other hand we can write the lattice analogue not only for one chiral sector but also for the full WZNW model. In this case we will get formulae (33) as well. We will give the exact derivation in a more detailed paper. Now let us discuss the following surprising fact: the field g at the point $x = 0$ and the monodromy M form a closed algebra. Let us recall that

$$g(0) = u(0) e^{-iQ} v(0) ,$$

$$M = u(2\pi) (u(0))^{-1} = u(0) e^{iP} (u(0))^{-1} .$$

It is easy to check using the technique of [4] that

$$g_1 g_2 R^+ = R^+ g_2 g_1 ,$$

$$M_1 R^+ g_2 = R^- g_2 M_1 , \quad (34)$$

$$M_1 R^+ M_2 (R^+)^{-1} = R^- M_2 (R^-)^{-1} M_1 .$$

It appears that formulae (34) give us the analogue of the relations (10) for the quantum group. In other words, we can call a system formed by g and M the quantum cotangent bundle or, more physically, the quantum top. Here the field g is the coordinate on the quantum group and the monodromy M is an analogue of the conjugated momentum. Let us remark that the field $g(0)$ is the value of the KM group in the point and in agreement with the previous consideration it takes values in the quantum group.

In the theory of the cotangent bundle of the group it is natural to call the coordinate on the group vertex operator. For example, the WZNW model is the theory of the cotangent bundle of the KM group. And the local field $g(x)$ is the coordinate on the KM group and the vertex operator in the theory. It acts in the Hilbert space of the theory:

$$H = \oplus_i H^i \otimes H_i ,$$

where H^i, H_i are irreducible representations of the left and right currents respectively. The field $g(0)$ realizes the representation of the left and right current algebras as well:

$$\{I_1(x), g_2(0)\} = C g_2(0) \delta(x) ,$$

$$\{J_1(x), g_2(0)\} = -g_2(0) C \delta(x) . \quad (35)$$

The field $g(0)$ realizes the tensor product which people examining CFT called fusion. Formulae (34) for the quantum group are analogues of (35) for the KM algebra. The operator $g(0)$ realizes the tensor product for the quantum groups and the fusion algebra for the WZNW model as well. Let us remark here that the question about the restriction

of the class of representations of the quantum group is not clear yet in our approach and we will return to this item. However now it is clear why we have R-matrices in braidings of WZNW vertex operators. These vertex operators realize the quantum group tensor product and the braiding constants for the QG are exactly R-matrices and quantum 6-j symbols.

PERSPECTIVES

This paper is only a brief sketch of the results about the relationship between the WZNW, the QG and the KM algebra. We are going to publish a series of articles devoted to these problems. Now let us announce some questions which we will discuss:

1. The quantum cotangent bundle as a toy model of CFT. The WZNW model as a top for the Kac-Moody group [4].
2. The correspondence between the tensor algebra of representations of the KM algebra with the central charge k and the quantum group with the deformation parameter $q = e^{-i\pi/k+h}$ based on the exchange algebra of the monodromy matrix (32).
3. Lattice WZNW theory as an integrable model.
4. Ultralocalization of the full WZNW model.
5. Representation theory of the lattice Kac-Moody algebra (24).

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