ON THE ORIGIN OF W-ALGEBRAS

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ABSTRACT

We show that the complex and projective structures on 2D Riemann surfaces are determined by the solutions to the linear differential equations obtained by the Hamiltonian reduction of $Sl(2, \mathbb{C})$ connections by the gauge parabolic subgroup. The compatibility of complex (μ) and projective (T) structures appears as the associated zero-curvature condition on the reduced symplectic manifold and is nothing but the conformal Ward identity. Generalizing this construction to the reduction of $Sl(n, \mathbb{C})$ connections by the maximal parabolic gauge subgroup, we obtain generalized complex $(\mu, \rho, \dots)$ and projective $(T, W, \dots)$ structures. From their compatibility conditions we explicitly obtain the Ward identities of W_n -gravity and the operator product expansions of the W_n -algebras. The associated linear differential equations (one of which involves the basic differential operator of the n^{th} reduction of the KP hierarchy) allow for a geometric interpretation of the W -symmetries in terms of deformations of flag configurations in the jet bundle $\Gamma^{(n-1)}$. We also show how to derive the W_n -Ward identities from the quantization of the 2 + 1-dimensional Chern-Simons theory.

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1. Introduction

During the last years, W -algebras have turned out to be of central interest in two-dimensional mathematical and theoretical physics. Originally introduced [1] as an extension of the conformal algebra by a spin-3 primary field (describing e.g. the critical point of the 3-state Potts model), it rapidly became clear that W -algebras are related to various other important structures of theoretical physics, such as cosets of affine Lie algebras [2], gauged WZW-models [3,4], reductions of the KP-hierarchy [5,6], Toda field theories [7], integrable IRF lattice models [8] and, more recently, matrix models of random surfaces [9-11].

Whereas the Virasoro algebra has a natural geometric interpretation, we still have no satisfactory geometric understanding of the origin of W_n -algebras. A nice attempt has been made in ref. [12] to identify the geometry underlying W_n -algebras (W_n -geometry for short) as the geometry of certain affine two-dimensional surfaces immersed in n -dimensional affine space.* A related idea is the very recent proposal [14] (for $n = 3$) to connect W -geometry to the geometry of a suitably defined flag bundle over the Riemann surface, with deformations of the flag corresponding to W -transformations.

In the present paper we try to shed some more light on the origin of W -algebras. Developing further the recent work by one of us [15], we show that the (Virasoro) conformal Ward identity expresses the compatibility of complex structures (parametrized by the Beltrami differential μ) with the projective structures (parametrized by the energy-momentum tensor T). The compatibility can be nicely encoded into the Hamiltonian reduction of flat $Sl(2, \mathbb{C})$ connections by the gauge parabolic subgroup. Generalizing this reduction[†] to $Sl(n, \mathbb{C})$ connections, we immediately obtain the Ward identities of W_n -algebras, equivalent to their operator product expansions. Again, these Ward identities express the compatibility of "generalized complex structures" $(\mu, \rho, \dots)$ with "generalized projective structures" $(T, W, \dots)$. The Ward identities appear as the flatness conditions on the reduced phase space of $Sl(n, \mathbb{C})$ connections gauged by the maximal parabolic subgroup. They constitute the compatibility conditions for two linear differential equations. The first involves an ordinary differential operator of order n (in ∂) that only depends on the generalized projective structures parametrized by $T, W, \dots$

* It is interesting to note that it was already realized in ref. 13 that the Toda theories are related to the problem of embedding two-dimensional Riemann surfaces into n -dimensional Riemannian spaces. In view of the relation between Toda theories and W -algebras [7] this statement has an immediate counterpart for the W -geometry. (One of the present authors (A.B.) would like to thank N. Gounalis for pointing out ref. 13 to us.)

† Our Hamiltonian reduction is different from that of ref.[3] since we work with larger initial and final phase spaces.

and coincides with the differential operator $L_{(n)}$ giving rise to the n^{th} reduction of the KP hierarchy. The second equation also involves the generalized complex structures $\mu, \rho, \dots$ and, for $n = 2$, is equivalent to the Beltrami equation defining isothermal complex coordinates. We obtain the explicit form of these differential equations for all n , allowing us to write down all Ward identities and operator product expansions for all n . The n solutions to these linear differential equations are $(-\frac{1}{2}(n-1), 0)$ -differentials. They are well known from the study of Toda field theory [7,16] to be parametrized by $n-1$ functions. For $n = 2$ (Virasoro) there is only one such function which is precisely a (isothermal) complex coordinate. For general n , one has $n-1$ such “generalized complex coordinates” At this point we will make contact with the interpretation of ref. 14 that these generalized coordinates describe the geometry of a suitably defined flag bundle over the Riemann surface. These flags are naturally defined on the jet bundle $\Gamma^{(n-1)}$. We will generalize the results of ref. 14 to any n , thus demonstrating that arbitrary deformations of the flag configurations in the jet bundle $\Gamma^{(n-1)}$ are in one-to-one correspondence with the W -transformations generated by the spin-2 to spin- n fields on $-\frac{n-1}{2}$ -differentials. The flag configurations describe embedding sequences of one to $(n-1)$ -dimensional spaces into the n -dimensional fibers of the jet bundle $\Gamma^{(n-1)}$. This geometric description of W -transformations bears a lot of similarity to the approach of ref. 12 mentioned above. Finally, we discuss a $2+1$ -dimensional quantum field theoretic approach to the W -symmetries. Starting from $2+1$ -dimensional Chern-Simons theory, we show that in a certain $(\mu, \rho, \dots)$ -representation the physical wave functionals are annihilated by constraint operators that coincide with the Ward identities of W_n . Rather surprisingly, we obtain all W_n -Ward identities solely from the (quadratic) constraint expressing the diffeomorphism invariance of the physical Chern-Simons wave functional. Thus we find a direct connection between $SU(n)$ Chern-Simons theory and W_n -algebras.

This paper is organized as follows. In section 2, for completeness, we recall some relatively well known definitions and results on complex and projective structures on Riemann surfaces. The reader familiar with these facts might wish to skip part or all of this section. Section 3 describes the Hamiltonian reduction of $Sl(n, \mathbb{C})$ connections by the gauge parabolic subgroup providing explicit expressions for the two linear differential equations and the Ward identities (equivalent to the operator product expansions) of W_n -algebras as their compatibility condition. In section 4, we describe the solutions to the linear differential equations and discuss their geometric interpretation as related to the flag bundle. Section 5 addresses the $2+1$ -dimensional field-theoretic approach to W -symmetries. Sections 4 and 5 pursue different developments and can be read independently of each other.

2. Some preliminaries on complex and projective structures

In this section, for completeness, we recall some more or less well known facts about complex and projective structures. The reader might wish to skip part or all of this section. Consider a Riemann surface Σ equipped with an atlas of compatible complex analytic coordinate systems, i.e. we have complex coordinates $\xi^{(r)}$ defined on each patch $\Sigma_{(r)}$ and the transition functions ϕ^{rs} between two patches are holomorphic: $\xi^{(r)} = \phi^{rs}(\xi^{(s)})$. We may define another atlas of compatible complex analytic coordinates $\xi'^{(r)}$. The transition functions between the $\xi'^{(r)}$ and the $\xi'^{(s)}$ are again holomorphic, but in general the $\xi'^{(r)}$ are related to the $\xi^{(s)}$ by arbitrary (local) diffeomorphisms, *not* necessarily holomorphic. However, if the $\xi'^{(r)}$ are related to the $\xi^{(s)}$ by holomorphic (local) diffeomorphism they are said to belong to the same complex analytic structure. If the atlas is also projective, i.e. the transition functions ϕ^{rs} are not only holomorphic but also projective, we may define a projective structure in a similar way, replacing everywhere “holomorphic” by “projective” (i.e. Möbius).

Let us (locally) fix some reference coordinates $(z, \bar{z})$ (corresponding to some fixed complex analytic structure). We denote by ∂ and $\bar{\partial}$ the derivatives with respect to these coordinates. The following proposition describes the set of all complex analytic structures in terms of the fixed one.

The set of all *complex analytic structures* (complex structures for short) is in one-to-one correspondence with the set of Beltrami differentials $\mu(z, \bar{z})$. These are $(-1, 1)$ -differentials. i.e. $\mu \equiv \mu_{\bar{z}}^z$, with $|\mu| < 1$. The corresponding complex structure is given by those solutions of the so-called Beltrami equation

$$(\bar{\partial} - \mu\partial) F = 0 . \quad (2.1)$$

which define a local diffeomorphism $(z, \bar{z}) \rightarrow (F(z, \bar{z}), \bar{F}(z, \bar{z}))$.

Indeed, by eq.(2.1) one has $|dF|^2 = |\partial F|^2 |dz + \mu d\bar{z}|^2$. If we define the metric $ds^2 = e^\varphi |dz + \mu d\bar{z}|^2$, then the solutions to (2.1) provide isothermal coordinates, $ds^2 = e^{\varphi'} |dF|^2$. The Beltrami equation (2.1) only determines F up to a holomorphic reparametrization $F \rightarrow G(F)$. Hence, to any given μ we can associate exactly one complex structure, provided by the solutions of (2.1). Conversely, for given F one can define $\mu = \bar{\partial}F/\partial F$, and if G is any holomorphic function of F one has $\bar{\partial}G/\partial G = \bar{\partial}F/\partial F$.

The set of all *projective structures* is in one-to-one correspondence with the set of pairs $(\mu, \tilde{T})$ where μ is the Beltrami differential and $\tilde{T} \equiv \tilde{T}_{zz}$ satisfies (note that $\tilde{T}$ does not exactly transform as a $(2,0)$ -differential but it picks up a Schwarzian derivative

such that equation (2.2) is covariant, see below)

$$(\bar{\partial} - \mu\partial - 2\partial\mu)\tilde{T} = -\frac{1}{2}\partial^3\mu. \quad (2.2)$$

The projective structure corresponding to the pair $(\mu, \tilde{T})$ is given by the ratios of any two solutions ψ of the linear system, where ψ is a $(-\frac{1}{2}, 0)$ -differential

$$\left(\bar{\partial} - \mu\partial + \frac{1}{2}\partial\mu\right)\psi = 0 \quad (2.3)$$

$$(\partial^2 - \tilde{T})\psi = 0. \quad (2.4)$$

Indeed, we will show below that eq.(2.2) is nothing but the compatibility condition for the two equations (2.3) and (2.4). Hence, the dimensionality of the space of solutions of the latter equations is two. Let ψ_1, ψ_2 be two linearly independent solutions of this system. Then by (2.3), $F = \psi_2/\psi_1$ is a solution of the Beltrami equation (2.1). Writing $\psi_2 = \psi_1 F$, it follows from (2.4) that $\psi_1 = (\partial F)^{-\frac{1}{2}}$ and that

$$\tilde{T} = -\frac{1}{2}S_z(F) \quad (2.5)$$

where $S_z(F)$ denotes the Schwarzian derivative of F

$$S_z(F) = \frac{\partial^3 F}{\partial F^3} - \frac{3}{2} \left(\frac{\partial^2 F}{\partial F^2} \right)^2. \quad (2.6)$$

Any other two solutions of (2.3), (2.4) are related to ψ_1, ψ_2 by an $Sl(2, \mathbf{C})$ transformation, inducing a Möbius transformation on the coordinate F . The Schwarzian derivative and hence $\tilde{T}$ is invariant precisely under the Möbius subgroup of holomorphic diffeomorphisms. Thus for given μ , i.e. given complex structure, $\tilde{T}$ serves to distinguish different projective structures. We have shown that the ratios ψ_2/ψ_1 are in one-to-one correspondence with the pairs $(\mu, \tilde{T})$. We will show below that $\tilde{T}$ can be identified - up to a constant - with the energy-momentum tensor T of a conformal field theory.

Remark that it follows from (2.5) that under a holomorphic diffeomorphism $z \rightarrow z' = f(z)$, $\tilde{T}$ transforms as $\tilde{T} \rightarrow \tilde{T}' = (\partial f)^{-2} \left(\tilde{T} + \frac{1}{2}S_z(f) \right)$. On the other hand, μ is a true $(-1, 1)$ -differential, $\mu \rightarrow \mu' = (\partial f / \partial \bar{f})\mu$, and it is easy to show that $\mu\partial\tilde{T} + 2\partial\mu\tilde{T} - \frac{1}{2}\partial^3\mu$ transforms as a true $(2, -1)$ -differential. Thus equation (2.2) is indeed well defined on the Riemann surface (even if the atlas used is not projective). The same can be checked for equations (2.3) and (2.4).

Let us finally demonstrate that (2.2) is the compatibility condition for (2.3) and (2.4). The latter equations are equivalent to^{*}

$$(\bar{\partial} - A_{\bar{z}}) \begin{pmatrix} \psi \\ \partial\psi \end{pmatrix} = 0, \quad A_{\bar{z}} = \begin{pmatrix} -\frac{1}{2}\partial\mu & \mu \\ \mu\tilde{T} - \frac{1}{2}\partial^2\mu & \frac{1}{2}\partial\mu \end{pmatrix} \quad (2.7)$$

and

$$(\partial - A_z) \begin{pmatrix} \psi \\ \partial\psi \end{pmatrix} = 0, \quad A_z = \begin{pmatrix} 0 & 1 \\ \tilde{T} & 0 \end{pmatrix}. \quad (2.8)$$

Compatibility of (2.7) and (2.8) is equivalent to the vanishing of the curvature

$$F_{z\bar{z}} \equiv [\partial - A_z, \bar{\partial} - A_{\bar{z}}] = \begin{pmatrix} 0 & 0 \\ (\bar{\partial} - \mu\partial - 2\partial\mu)\tilde{T} + \frac{1}{2}\partial^3\mu & 0 \end{pmatrix} \quad (2.9)$$

i.e. to condition (2.2).

* Note that since the non-diagonal elements of $A_{\bar{z}}$ connect ψ and $\partial\psi$ they are *not* $(0,1)$ -differentials. The same is true for the elements of A_z . Thus although the matrix elements of $A_{\bar{z}}$ and A_z do not transform as true differentials in general, equations (2.7) and (2.8) are covariant since they are equivalent to (2.3) and (2.4).

3. Hamiltonian reduction of $Sl(n, \mathbb{C})$ connections: generalizations of complex and projective structures, and W_n -algebras

3.1 THE HAMILTONIAN REDUCTION

In equations (2.7), (2.8) and (2.9) we have written the linear differential equations determining the projective structure and the compatibility condition with the complex structure in a very suggestive form: A_z and $A_{\bar{z}}$ are to be interpreted as components of a flat $Sl(2, \mathbb{C})$ gauge connection. Their very special form, depending only on μ and $\tilde{T}$ can be viewed as resulting from a Hamiltonian reduction as we now proceed to show.

Consider the symplectic manifold $\mathcal{M}$ (phase space) of all 2D $Sl(n, \mathbb{C})$ gauge connections $A = A_z dz + A_{\bar{z}} d\bar{z}$ with symplectic form

$$\Omega = \frac{1}{2\pi\hbar} \int_{\Sigma} \text{tr} \delta A_z \wedge \delta A_{\bar{z}} . \quad (3.1)$$

One could think of this symplectic form as resulting from a Chern-Simons action $S = \frac{1}{2\pi\hbar} \int_{\Sigma} \text{tr} (A_z \wedge \partial_t A_{\bar{z}} + \dots)$. The symplectic form (3.1) implies the Poisson bracket

$$\left\{ A_z^a(z), A_w^b(w) \right\} = 2\pi\hbar \delta^{ab} \delta^{(2)}(z - w) . \quad (3.2)$$

Infinitesimal gauge transformations are generated by the Hamiltonian field

$$H_{\epsilon}(A) = \int_{\Sigma} \text{tr} \epsilon F(A) \quad (3.3)$$

defined on $\mathcal{M}$, where for any $(A_z, A_{\bar{z}}) \in \mathcal{M}$, $F(A)$ is the curvature

$$F(A) \equiv F_{z\bar{z}}(A) = [\partial - A_z, \bar{\partial} - A_{\bar{z}}] . \quad (3.4)$$

We shall now reduce the symplectic manifold $\mathcal{M}$ with respect to a subgroup $\mathcal{G}^P$ of the $Sl(n, \mathbb{C})$ gauge group. As always, a Hamiltonian reduction proceeds in two steps. Let us briefly explain Hamiltonian reductions on a very simple mechanical example^{*}: a two-dimensional point-particle moving in a rotationally symmetrical potential. Coordinates on the phase space $\mathcal{M}$ are (x, y, p_x, p_y) or equivalently (r, ϕ, p_r, p_{ϕ}) . Due

^{*} The reader familiar with Hamiltonian reduction should skip this paragraph.

to the symmetry, p_ϕ is a constant of motion. It generates the symmetry via Poisson brackets: $H_\theta = \theta p_\phi$ generates a rotation $\phi \rightarrow \phi + \theta$. The first step of the Hamiltonian reduction is to fix p_ϕ to some prescribed value (provided by the initial data). The second step is to divide the resulting subspace of $\mathcal{M}$, $\{(r, \phi, p_r, p_\phi), p_\phi = \text{fixed}\}$ by the remaining symmetry group. In this example, this is still the full rotation group $\phi \rightarrow \phi + \theta$. The result is the reduced phase space $\mathcal{M}_{\text{red}} = \{(r, p_r)\}$. In this example, we have chosen the coordinates in such a way that one step of the reduction corresponds to the elimination of a momentum (p_ϕ) and the other step to the elimination of the corresponding coordinate (ϕ). More generally, one introduces the momentum mapping $P : \mathcal{M} \rightarrow g^*$, where g^* is the dual of the symmetry algebra g , associating to any $z \in \mathcal{M}$ an element $P(z) \in g^*$ such that for any $\epsilon \in g$, $\langle P(z), \epsilon \rangle = H_\epsilon(z)$ is the symmetry generator via Poisson brackets. In our example, let $\epsilon_\theta \in g$ denote a rotation by an angle θ . Then $\langle P(r, \phi, p_r, p_\phi), \epsilon_\theta \rangle = \theta p_\phi$. Let $\epsilon^* \in g^*$. Then $\mathcal{M}_{\epsilon^*} \equiv P^{-1}(\epsilon^*)$ is precisely $\{(r, \phi, p_r, p_\phi), p_\phi = \text{fixed}\}$. Let G_{ϵ^*} be the subgroup of the symmetry group mapping $\mathcal{M}_{\epsilon^*}$ into itself. In the example this is just the full rotation group of the plane. The reduced phase space is $\mathcal{M}_{\text{red}} = \mathcal{M}_{\epsilon^*}/G_{\epsilon^*}$. In summary, the two steps of the Hamiltonian reduction are

$$\mathcal{M} \rightarrow \mathcal{M}_{\epsilon^*} \equiv P^{-1}(\epsilon^*) \rightarrow \mathcal{M}_{\epsilon^*}/G_{\epsilon^*} \equiv \mathcal{M}_{\text{red}} .$$

Note that in our example one could as well have first divided by the symmetry group G (eliminating ϕ) obtaining $\mathcal{M}/G$, define the restriction $\bar{P}$ of the canonical momentum mapping P on $\mathcal{M}/G$, and then obtain $\mathcal{M}_{\text{red}}$ as $\bar{P}^{-1}(\epsilon^*)$. In particular, one may take $\epsilon^* = 0$, thus $\mathcal{M}_{\text{red}} = \text{Ker } \bar{P}$. Hence

$$\mathcal{M} \rightarrow \mathcal{M}/G \rightarrow \text{Ker } \bar{P} = \mathcal{M}_{\text{red}} .$$

Let us apply the Hamiltonian reduction to the symplectic manifold $\mathcal{M}$ of 2D $Sl(n, \mathbb{C})$ gauge connections as defined above. We choose as symmetry group[†] the subgroup $\mathcal{G}^P$ of parabolic $Sl(n, \mathbb{C})$ gauge transformations:

$$A \rightarrow g^{-1}Ag + g^{-1}dg \tag{3.5}$$

[†] We denote by G the group $Sl(n, \mathbb{C})$ and by $\mathcal{G}$ the group of $Sl(n, \mathbb{C})$ gauge transformation. The symmetry group relevant for the Hamiltonian reduction is $\mathcal{G}^P$.

where $g \in Sl(n, \mathbf{C})$ such that

$$g = \begin{pmatrix} \times & \dots & \times & 0 \\ \cdot & & \cdot & \cdot \\ \times & \dots & \times & 0 \\ \times & \dots & \times & \times \end{pmatrix}. \quad (3.6)$$

3.2 $n = 2$: THE VIRASORO ALGEBRA

We will first look at $Sl(2, \mathbf{C})$. Then G^P is just the Borel subgroup. Parametrizing $g \in G^P$ as $g = \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}$, it is easy to see that by an appropriate choice of a and b one can always gauge A in a unique way to the form*

$$A = \begin{pmatrix} 0 & 1 \\ \tilde{T} & 0 \end{pmatrix} dz + \begin{pmatrix} \alpha & \mu \\ \gamma & -\alpha \end{pmatrix} d\bar{z} \quad (3.7)$$

with arbitrary $\tilde{T}, \alpha, \mu, \gamma \in \mathbf{C}$. This has used up our G^P -gauge freedom. Thus the space of all $Sl(2, \mathbf{C})$ connections of the form (3.7) is the factorized space $\mathcal{M}/\mathcal{G}^P$, completing the first step of the Hamiltonian reduction. The second step is to restrict this space of gauge orbits $\mathcal{M}/\mathcal{G}^P$ to the kernel of the canonical momentum mapping P . But for $A \in \mathcal{M}/\mathcal{G}^P$, $\epsilon \in \mathfrak{g}^P$ (the Lie algebra of $\mathcal{G}^P$), $P(A)$ is defined as $\langle P(A), \epsilon \rangle = H_\epsilon(A)$. Thus from (3.3), $P(A) = \int_\Sigma \text{tr}(\cdot) F(A)$, and the kernel of P consists of all those A such that $\text{tr} \epsilon F(A) = 0 \forall \epsilon \in \mathfrak{g}^P$. This means that $F(A) = \begin{pmatrix} 0 & 0 \\ \times & 0 \end{pmatrix}$ which determines α and γ in terms of μ and $\tilde{T}$. This completes the second step of the Hamiltonian reduction. The reduced phase space consists of all $Sl(2, \mathbf{C})$ connections of the form

$$A = \begin{pmatrix} 0 & 1 \\ \tilde{T} & 0 \end{pmatrix} dz + \begin{pmatrix} -\frac{1}{2}\partial\mu & \mu \\ \mu\tilde{T} - \frac{1}{2}\partial^2\mu & \frac{1}{2}\partial\mu \end{pmatrix} d\bar{z} \quad (3.8)$$

This is precisely of the form (2.7), (2.8) and the consistency condition (2.2) between the projective and complex structures appears as the flatness condition on $\mathcal{M}_{\text{red}}$, the symplectic manifold of $Sl(2, \mathbf{C})$ connections reduced by the gauge parabolic subgroup $\mathcal{G}^P$.

* If one starts with $A_z = \begin{pmatrix} u & v \\ w & -u \end{pmatrix}$, then a and b are given by $a^2 = v$ and $b = -v^{-1}ua - v^{-1}\partial a$.

Thus g is determined up to an overall sign (as is obvious from (3.5)). However, if we only consider the component of the gauge group that is connected to the identity, we must take $a > 0$, and thus g is determined uniquely.

Clearly, the reduced symplectic manifold is parametrized by $(\mu, \tilde{T})$ with the induced symplectic form being

$$\Omega_{\text{red}} = \frac{1}{2\pi\hbar} \int_{\Sigma} \delta\mu \wedge \delta\tilde{T} . \quad (3.9)$$

Let us note that the flatness condition (2.2) is nothing but the (Virasoro) conformal Ward identity, equivalent to the operator product expansion of the energy-momentum tensor with itself. Indeed, the induced symplectic form (3.9) tells us that μ and $\tilde{T}$ appear as conjugate variables and that in a quantum theory $\tilde{T}$ should be represented as $\tilde{T}(z) = 2\pi\hbar \delta/\delta\mu(z)$ acting on appropriate functionals of μ (partition functions Z). The compatibility condition (2.2) then becomes

$$\left((\bar{\partial} - \mu(z)\partial - 2\partial\mu(z)) \frac{\delta}{\delta\mu(z)} + \frac{1}{4\pi\hbar} \partial^3 \mu(z) \right) Z = 0 . \quad (3.10)$$

This is the conformal Ward identity.[‡] Multiply with $\pi(w-z)^{-1}$, integrate over d^2z , integrate by parts and use $\partial_{\bar{z}}(w-z)^{-1} = -\pi\delta^{(2)}(w-z)$ to obtain

$$\pi^2 \frac{\delta Z}{\delta\mu(w)} + \int d^2z \mu(z) \left(\left(\frac{2}{(w-z)^2} + \frac{\partial_z}{w-z} \right) \pi \frac{\delta Z}{\delta\mu(z)} - \frac{3/(2\hbar)}{(w-z)^4} Z \right) = 0 . \quad (3.11)$$

Take $\delta/\delta\mu(z)$ and then choose the complex structure with $\mu = 0$ (“conformal gauge”):

$$\pi^2 \frac{\delta^2 Z}{\delta\mu(w)\delta\mu(z)} + \left(\frac{2}{(w-z)^2} + \frac{\partial_z}{w-z} \right) \pi \frac{\delta Z}{\delta\mu(z)} - \frac{3/(2\hbar)}{(w-z)^4} Z = 0 . \quad (3.12)$$

Now introduce T and c as

$$c = \frac{3}{\hbar} , \quad T = -\frac{1}{2\hbar} \tilde{T} = -\frac{c}{6} \tilde{T} \simeq -\pi \frac{\delta}{\delta\mu} \quad (3.13)$$

and (3.12) becomes

$$T(w)T(z) = \left(\frac{2}{(w-z)^2} + \frac{\partial_z}{w-z} \right) T(z) + \frac{c/2}{(w-z)^4} \quad (3.14)$$

which is exactly the operator product expansion of an energy-momentum tensor T of a conformal field theory with central charge c . The relation $T = -\frac{c}{6}\tilde{T}$ could already have been obtained from (2.5), since it is $T = \frac{c}{12}S_z(u)$ which transforms as an energy-momentum tensor under a holomorphic diffeomorphism $z \rightarrow z'$, namely $T(z) \rightarrow T'(z') = \left(\frac{\partial z'}{\partial z} \right)^{-2} (T(z) - \frac{c}{12}S_z(z'))$.

[‡] Of course, this is not a new result. To our knowledge, the first derivation of the conformal Ward identity by a reduction of $Sl(2, \mathbb{C})$ -connections appeared in ref. 17.

3.3 $n = 3$: THE W_3 -ALGEBRA

We want to generalize the above construction to the symplectic manifold $\mathcal{M}$ of $Sl(n, \mathbb{C})$ gauge connections reduced by the gauge parabolic group $\mathcal{G}^P$ as defined by (3.6). Unfortunately, we are not aware of a simple general algorithm allowing us to immediately solve the zero-curvature condition to obtain the generic form of the analogue of (3.8) for generic n . However, for any given n , this is a straightforward computation leading to the reduced phase space. We will first consider $n = 3$ and postpone the discussion of the case of general n to the next subsection.

For $n = 3$, an arbitrary $g \in G^P$ has the Gauss decomposition

$$g = \begin{pmatrix} 1 & 0 & 0 \\ \phi_1 & 1 & 0 \\ \phi_3 & \phi_2 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a^{-1}b^{-1} \end{pmatrix} \begin{pmatrix} 1 & F_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.15)$$

and by an appropriate choice of such a g one can gauge the connection A in a unique way to the form

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \tilde{W} & \tilde{T} & 0 \end{pmatrix} dz + \begin{pmatrix} \alpha & \gamma & \rho \\ \delta & \beta & \mu \\ \epsilon & \omega & -\alpha - \beta \end{pmatrix} d\bar{z}. \quad (3.16)$$

This parametrizes the manifold $\mathcal{M}$ factorized by the parabolic gauge group $\mathcal{G}^P$. Computing the curvature form explicitly and restricting us to the kernel of the momentum mapping $\{A : \text{tr } \epsilon F(A) = 0 \ \forall \ \epsilon \in g^P\}$ determines $\alpha, \beta, \gamma, \delta, \epsilon, \omega$ in terms of $\mu, \rho, \tilde{T}, \tilde{W}$:

$$\begin{aligned} \alpha &= -\partial\mu + \frac{2}{3}(\partial^2 - \tilde{T})\rho \\ \gamma &= \mu - \partial\rho \\ \delta &= -\partial^2\mu + \rho\tilde{W} + \frac{2}{3}\partial(\partial^2 - \tilde{T})\rho \\ \beta &= -\frac{1}{3}(\partial^2 - \tilde{T})\rho \\ \epsilon &= -\partial^3\mu + \partial(\rho\tilde{W}) + \mu\tilde{W} + \frac{2}{3}\partial^2(\partial^2 - \tilde{T})\rho \\ \omega &= \frac{1}{3}\partial(\partial^2 - \tilde{T})\rho + \rho\tilde{W} - (\partial^2 - \tilde{T})\mu. \end{aligned} \quad (3.17)$$

The reduced symplectic manifold thus is parametrized by $(\mu, \rho, \tilde{T}, \tilde{W})$ with induced

symplectic from

$$\Omega_{\text{red}} = \frac{1}{2\pi\hbar} \int_{\Sigma} \left(\delta\mu \wedge \delta\tilde{T} + \delta\rho \wedge \delta\tilde{W} \right) . \quad (3.18)$$

Equation (3.17) describes the immersion of the reduced manifold into the non-reduced one.

The zero-curvature condition on the reduced space yields two equations (these are the (3,1) and (3,2) components of $F = 0$, with F defined in (3.4)):^{*}

$$(\bar{\partial} - \mu\partial - 2\partial\mu) \tilde{T} = -2\partial^3\mu + 2\partial(\rho\tilde{W}) + \partial\rho\tilde{W} + \partial^2(\partial^2 - \tilde{T})\rho \quad (3.19)$$

$$(\bar{\partial} - \mu\partial - 3\partial\mu - \rho\partial^2 - 2\partial\rho\partial) \tilde{W} = (\partial^2 - \tilde{T}) \left(-\partial^2\mu + \frac{2}{3}\partial(\partial^2 - \tilde{T})\rho \right) . \quad (3.20)$$

These zero-curvature conditions are the compatibility conditions for the two linear systems

$$(\partial - A_z) \begin{pmatrix} \psi \\ \partial\psi \\ \partial^2\psi \end{pmatrix} = 0 \quad (3.21)$$

and

$$(\bar{\partial} - A_{\bar{z}}) \begin{pmatrix} \psi \\ \partial\psi \\ \partial^2\psi \end{pmatrix} = 0 \quad (3.22)$$

with A_z and $A_{\bar{z}}$ given by (3.17). These linear systems are in fact equivalent to

$$(\partial^3 - \tilde{T}\partial - \tilde{W}) \psi = 0 \quad (3.23)$$

$$\left(\bar{\partial} - \mu\partial + \partial\mu - \rho\partial^2 + \partial\rho\partial - \frac{2}{3}(\partial^2 - \tilde{T})\rho \right) \psi = 0 . \quad (3.24)$$

For $\rho = 0$, eq.(3.24) tells us that the solutions ψ are $(-1,0)$ -differentials, and the ratio of any two solutions (for $\rho = 0$) satisfies the Beltrami equation (2.1). We will discuss these equations (3.23), (3.24) and their solutions in some detail in the next section. Here let us only demonstrate that the compatibility equations (3.19), (3.20) are the Ward identities of W_3 -gravity, encoding the operator product expansions of W_3 -algebras.

^{*} Note that $\partial\rho\partial\tilde{W}$ means $(\partial\rho)(\partial\tilde{W})$, but $\partial(\partial^2 - \tilde{T})\rho$ means $\partial^3\rho - \partial(\tilde{T}\rho)$.

The induced symplectic form (3.18) requires the substitutions $\tilde{T} \rightarrow 2\pi\hbar \delta/\delta\mu$ and $\tilde{W} \rightarrow 2\pi\hbar \delta/\delta\rho$. Proceeding through the same steps as led to (3.11), we obtain from (3.19), (3.20):

$$\begin{aligned} \pi^2 \frac{\delta Z}{\delta\mu(w)} + \int d^2z \mu(z) & \left(\left(\frac{2}{(w-z)^2} + \frac{\partial_z}{w-z} \right) \pi \frac{\delta Z}{\delta\mu(z)} - \frac{6/\hbar}{(w-z)^4} Z \right) \\ & + \int d^2z \rho(z) \left(\left(\frac{3}{(w-z)^2} + \frac{\partial_z}{w-z} \right) \pi \frac{\delta Z}{\delta\rho(z)} \right. \\ & \left. + \frac{2}{(w-z)^3} \pi \frac{\delta Z}{\delta\mu(z)} - \frac{12/\hbar}{(w-z)^5} Z \right) = 0 \end{aligned} \quad (3.25)$$

and

$$\begin{aligned} \pi^2 \frac{\delta Z}{\delta\rho(w)} + \int d^2z \mu(z) & \left\{ \left(\frac{3}{(w-z)^2} + \frac{2\partial_z}{w-z} \right) \pi \frac{\delta Z}{\delta\rho(z)} \right. \\ & \left. - \left(\frac{2}{(w-z)^3} + \frac{2\partial_z}{(w-z)^2} + \frac{\partial_z^2}{w-z} \right) \pi \frac{\delta Z}{\delta\mu(z)} + \frac{12/\hbar}{(w-z)^5} Z \right\} \\ & + \int d^2z \rho(z) \left\{ \left(\frac{2}{(w-z)^2} + \frac{\partial_z}{w-z} \right) \pi \frac{\delta Z}{\delta\rho(z)} \right. \\ & \left. - \left(\frac{8}{(w-z)^4} + \frac{4\partial_z}{(w-z)^3} + \frac{2\partial_z^2}{(w-z)^2} + \frac{2}{3} \frac{\partial_z^3}{w-z} \right) \pi \frac{\delta Z}{\delta\mu(z)} \right. \\ & \left. + \frac{2\hbar}{3} \left(\frac{2}{(w-z)^2} + \frac{\partial_z}{w-z} \right) \pi^2 \frac{\delta^2 Z}{\delta\mu^2(z)} + \frac{40/\hbar}{(w-z)^6} Z \right\} = 0 \end{aligned} \quad (3.26)$$

These equations are the Ward identities of W_3 -gravity. The double functional derivative $\frac{\delta^2 Z}{\delta\mu^2(z)}$ in (3.26) requires some care. It is to be understood as obtained from $\frac{\delta^2 Z}{\delta\mu(z)\delta\mu(z')}$ in the limit $z' \rightarrow z$ by suppressing the short-distance singularity. In the quantum theory this will renormalize the coefficient $2\hbar/3$.

To obtain the operator product expansions (OPEs) of the W_3 -algebra, derive these equations with respect to $\mu(z)$ or $\rho(z)$, and choose the “generalized complex structure” $\mu = \rho = 0$. With the definitions

$$c = \frac{12}{\hbar}, \quad T = -\frac{1}{2\hbar} \tilde{T} = -\frac{c}{24} \tilde{T} \simeq -\pi \frac{\delta}{\delta\mu} \quad (3.27)$$

and

$$\hat{W} = -\frac{1}{2\hbar} \tilde{W} = -\frac{c}{24} \tilde{W} \simeq -\pi \frac{\delta}{\delta\rho}, \quad (3.28)$$

first we obtain again the OPE (3.14) of the energy-momentum tensor T with itself.

Second, we get

$$T(w)\hat{W}(z) = \left(\frac{3}{(w-z)^2} + \frac{\partial_z}{w-z} \right) \hat{W}(z) + \frac{2}{(w-z)^3} T(z) + \frac{c}{(w-z)^5} . \quad (3.29)$$

This tells us that $\hat{W}$ is a quasi-primary field[★] of conformal weight 3. One can define a true primary field as

$$W = \left(-\frac{2}{5} \right)^{\frac{1}{2}} \left(\hat{W} - \frac{1}{2} \partial T \right) \quad (3.30)$$

where we included a numerical factor for later convenience. Then (3.29) becomes

$$T(w)W(z) = \left(\frac{3}{(w-z)^2} + \frac{\partial_z}{w-z} \right) W(z) . \quad (3.31)$$

Third, deriving (3.26) with respect to ρ , we obtain the OPE of $\hat{W}(w)$ with $\hat{W}(z)$. Using (3.14) and (3.29) this can be re-expressed as

$$\begin{aligned} W(w)W(z) = & \frac{c/3}{(w-z)^6} + \left(\frac{2}{(w-z)^4} + \frac{\partial}{(w-z)^3} + \frac{3}{10} \frac{\partial^2}{(w-z)^2} + \frac{1}{15} \frac{\partial^3}{w-z} \right) T(z) \\ & + \beta \left(\frac{2}{(w-z)^2} + \frac{\partial}{w-z} \right) \Lambda(z) \end{aligned} \quad (3.32)$$

where $\beta = \frac{16}{5c}$ and $\Lambda = T^2$. In fact, as already discussed, β and Λ get renormalized. Equations (3.31) and (3.32) are precisely the OPEs of the W_3 -algebra as written in ref. 1, provided we renormalize $\beta \rightarrow \frac{16}{5c+22}$ and $\Lambda \rightarrow T^2 - \frac{3}{10} \partial^2 T$.

★ Deriving (3.25) with respect to μ yields the OPE of $\hat{W}(w)$ with $T(z)$ which is equivalent to (3.29).

3.4 THE CASE OF ARBITRARY n : W_n -ALGEBRAS

As already mentioned, it is straightforward to repeat the analysis of the preceding subsection for any given n . However, as n increases the computations quickly become rather lengthy, and one would like to have general formulas valid for generic n . It is the purpose of the present subsection to provide these formulas as far as possible. In particular, we will explicitly obtain the generic form of the two linear differential equations generalizing (2.3), (2.4) and (3.23), (3.24), give the transformation rules of their solutions ψ under the W -symmetries and display a very simple algorithm to obtain all Ward-identities, resp. all operator product expansions of the W_n -algebras.

We begin by introducing some notation. Let us write $u_2 = \tilde{T}$, $u_3 = \tilde{W}$, etc. Then, by a $g \in \mathcal{G}^P$ (i.e. a g of the form (3.7)) one may gauge A_z to the form

$$A_z = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & & & 1 & 0 \\ 0 & 0 & 0 & & 0 & 1 \\ u_n & u_{n-1} & u_{n-2} & \dots & u_2 & 0 \end{pmatrix}. \quad (3.33)$$

This uses up all our $\mathcal{G}^P$ gauge freedom. We parametrize $A_{\bar{z}}$ as follows:

$$A_{\bar{z}} = \begin{pmatrix} a_{n-1,n-1} & \dots & a_{n-1,1} & a_{n-1,0} \\ \vdots & & \vdots & \vdots \\ a_{1,n-1} & \dots & a_{11} & a_{10} \\ a_{0,n-1} & \dots & a_{01} & a_{00} \end{pmatrix}, \quad (3.34)$$

i.e. $(A_{\bar{z}})_{ij} = a_{n-i,n-j}$. The a_{k0} generalize the μ and ρ . We shall write

$$a_{k0} \equiv \mu_{k+1} \quad (3.35)$$

Thus $a_{10} = \mu_2 \equiv \mu$, $a_{20} = \mu_3 \equiv \rho$, etc. Since $A_{\bar{z}} \in Sl(n, \mathbb{C})$, one has $a_{00} = -\sum_{k=1}^{n-1} a_{kk}$. Of course, the elements a_{ij} of $A_{\bar{z}}$ depend on n , and we write $a_{ij}^{(n)}$ if it is necessary to stress this fact. We further define

$$\mathcal{D}_k = \partial^k - \sum_{l=2}^k u_l \partial^{k-l} \quad (k \leq n) \quad (3.36)$$

and its formal adjoint by

$$\mathcal{D}_k^\dagger \psi = (-)^k \partial^k \psi - \sum_{l=2}^k (-)^{k-l} \partial^{k-l} (u_l \psi) \quad (k \leq n) \quad (3.37)$$

Note that the operator $\mathcal{D}_k$ corresponds to the k^{th} reduction of the KP hierarchy.[†]

Our task is to determine the a_{ij} , $0 \leq i \leq n-1$, $1 \leq j \leq n-1$ in terms of the $a_{k0} \equiv \mu_{k+1}$, $1 \leq k \leq n-1$. The latter parametrize the generalized complex structures, while the u_k , $2 \leq k \leq n$ parametrize the generalized projective structures. The a_{ij} are determined by the restriction to the kernel of the canonical momentum mapping, i.e. by imposing $\text{tr } \epsilon F(A) = 0$, $\forall \epsilon \in g^P$, or in terms of the matrix elements of the curvature: $(F)_{ij} = 0$, $1 \leq i \leq n-2$, $1 \leq j \leq n-1$. This leads to the equations^{*}

$$a_{ij} = \partial a_{i+1,j} + a_{i+1,j+1} + a_{i+1,0} u_{j+1}, \quad 0 \leq i \leq n-2, \quad 0 \leq j \leq n-1. \quad (3.38)$$

For $i = j = 0$, one has to substitute $a_{00} = -\sum_{k=1}^{n-1} a_{kk}$. It is only through this substitution that an n -dependence enters explicitly. Hence for $i > j$, equation (3.38) immediately determines the a_{ij} recursively. The result does not depend on n :

$$\begin{aligned} a_{l,l-p} = & \sum_{k=p}^l (-)^{k-p} \left[\binom{l-p}{k-p} - \binom{l-p-2}{k-p-2} \right] \partial^{k-p} \mu_{k+1} + \sum_{k=p+2}^l \mathcal{D}_{k-p}^\dagger \mu_{k+1} \\ & + \sum_{m=1}^{l-p-2} (-)^m \sum_{k=m+p+2}^{l-1} \binom{m+l-k-1}{m} \partial^m \mathcal{D}_{k-m-p}^\dagger \mu_{k+1}, \end{aligned} \quad (3.39)$$

for $1 \leq p < l \leq n-1$. As usual, $\binom{k}{l}$ denotes the binomial coefficient $\frac{k!}{l!(k-l)!}$ defined to be zero if $l < 0$, and $\binom{0}{0} = 1$.

So far, we have determined the matrix elements of $A_{\bar{z}}$ above the diagonal. The diagonal matrix elements are still determined by eq.(3.38), but now one has to substitute $a_{00} = -\sum_{k=1}^{n-1} a_{kk}$, and solve $n-1$ equations simultaneously. This introduces an

[†] It is sometimes customary to index the u 's differently. Our convention is such that u_l has "naive" dimension l . Thus, if one assigns dimension one to each derivative ∂ , then $\mathcal{D}_k$ has a well-defined dimension equal to k .

^{*} We have introduced $u_1 = 0$ and $a_{k,n} = 0 \forall k$.

explicit n -dependence. After some algebra, we obtain

$$\begin{aligned}
a_{n-1,n-1}^{(n)} &= -\frac{n-1}{2}\partial\mu_2 \\
&+ \frac{1}{n}\sum_{k=2}^{n-2}(-)^k \left[k \binom{n}{k+1} - k - \sum_{m=1}^{k-2} \sum_{l=1}^{n-1-k} (k+l) \binom{l+m-2}{m-1} \right] \partial^k \mu_{k+1} \\
&+ \sum_{k=2}^{n-1} \frac{k}{n} \mathcal{D}_k^\dagger \mu_{k+1} \\
&+ \frac{1}{n} \sum_{k=3}^{n-2} \sum_{m=1}^{k-2} (-)^m \sum_{l=1}^{n-1-k} (k+l) \binom{l+m-2}{m-1} \partial^m \mathcal{D}_{k-m}^\dagger \mu_{k+1}
\end{aligned} \tag{3.40}$$

The other diagonal elements of $A_{\bar{z}}$ are immediately obtained from this equation and (3.39) by the recurrence relation (3.38) with $i = j$. We will not write them down explicitly. Nor will we do so for the matrix elements below the diagonal. Equations (3.39) and (3.40) are all we will need in the following.

Indeed, the two linear systems

$$(\partial - A_z) \begin{pmatrix} \psi \\ \partial\psi \\ \vdots \\ \partial^{n-1}\psi \end{pmatrix} = 0 \quad , \quad (\bar{\partial} - A_{\bar{z}}) \begin{pmatrix} \psi \\ \partial\psi \\ \vdots \\ \partial^{n-1}\psi \end{pmatrix} = 0 \tag{3.41}$$

are equivalent to

$$\mathcal{D}_n \psi = 0 \tag{3.42}$$

and

$$\left(\bar{\partial} - \sum_{k=0}^{n-1} a_{n-1,n-1-k}^{(n)} \partial^k \right) \psi = 0 \tag{3.43}$$

(with the $a_{n-1,n-1-k}^{(n)}$ given explicitly by (3.39) and (3.40)), provided the compatibility conditions are satisfied. The latter are the zero curvature conditions on the reduced manifold. They read

$$\begin{aligned}
&\bar{\partial} u_{n+1-j} + \sum_{s=1}^{n-1} (a_{n-s,n-j} u_{n+1-s} + a_{ss} u_{n+1-j}) \\
&= \partial a_{0,n-j} + a_{0,n+1-j} \quad , \quad 1 \leq j \leq n-1 .
\end{aligned} \tag{3.44}$$

These $n-1$ equations provide the Ward identities of W_n -gravity and, as shown above, the operator product expansions of the W_n -algebras. We have not found a compact

way to spell out directly eq.(3.44) in terms of the μ_{k+1} for all j . Below however, we will obtain all Ward-identities directly from the differential equations (3.42) and (3.43). For the time being, let us only give the Ward identity resulting from (3.44) for $j = n - 1$:

$$(\bar{\partial} - \mu\partial - 2\partial\mu) \tilde{T} + \frac{1}{2} \binom{n+1}{3} \partial^3 \mu = \sum_{j=3}^n (\partial\mu_j u_j + (j-1)\partial(\mu_j u_j)) + \partial^2 X_n \quad (3.45)$$

where

$$\begin{aligned} X_n = & \frac{1}{2} \sum_{k=2}^{n-1} k(n-k) \mathcal{D}_k^\dagger \mu_{k+1} + \sum_{k=3}^{n-2} \sum_{m=1}^{k-2} (-)^m t_k^m(n) \partial^m \mathcal{D}_{k-m}^\dagger \mu_{k+1} \\ & + \sum_{k=2}^{n-2} (-)^k \left[\frac{k}{2} \binom{n+1}{k+2} - \frac{1}{2} k(n-k) - \sum_{m=1}^{k-2} t_k^m(n) \right] \partial^k \mu_{k+1} \end{aligned} \quad (3.46)$$

and where we have abbreviated $t_m^k(n) = \frac{1}{2} \sum_{j=k+1}^{n-1} j(n-j) \binom{m+j-k-2}{m-1}$.

It is clear from (3.33), (3.34) and (3.1) that the induced symplectic form on the reduced manifold is

$$\Omega_{\text{red}} = \frac{1}{2\pi\hbar} \int_{\Sigma} \sum_{k=2}^n \delta\mu_k \wedge \delta u_k. \quad (3.47)$$

Thus, we obtain the Ward identities from the compatibility conditions (3.44), and in particular from (3.45) via the substitutions

$$u_k \rightarrow 2\pi\hbar \frac{\delta}{\delta\mu_k} \quad (3.48)$$

and following the same steps as before. We learn from (3.45) that the Virasoro central charge is

$$c = \binom{n+1}{3} \frac{3}{\hbar} \quad (3.49)$$

as expected [7]. We also see that u_k is a Virasoro quasi-primary field of spin k as it should be. Let us stress that it is obvious by construction that the operator product expansion (OPE) of any two u_k , u_l , as encoded in (3.44), closes on polynomials of the u_j 's. Remark that the present treatment does not take into account normal ordering effects and thus is really equivalent to (classical) Poisson bracket computations. In other words, in the OPEs, only simple contractions are taken into account.

Let us come back to the problem of writing down *all* Ward identities of W_n -gravity, or equivalently *all* OPEs between the u_j , u_k , for arbitrary n . Since the Ward identities are the compatibility equations of the two linear differential equations we will directly start from the latter. We will also obtain how their solutions ψ transform under the W -generators. The second linear differential equation (3.43) can be written as^{*}

$$\bar{\partial}\psi = \sum_{k=1}^{n-1} \sum_{r=0}^k \sum_{m=0}^{k-r} C_{k,r}^m \left(\partial^m \mathcal{D}_{k-r-m}^\dagger \mu_{k+1} \right) \partial^r \psi \quad (3.50)$$

where the numerical coefficients $C_{k,r}^m$ can be read immediately from equations (3.39) and (3.40) giving the $a_{n-1,n-1-r}^{(n)}$, e.g. $C_{1,1}^0 = 1$, $C_{1,0}^1 = -\frac{1}{2}(n-1)$, $C_{2,2}^0 = 1$, $C_{2,1}^1 = -(n-2)$, $C_{2,0}^0 = \frac{2}{n}$, $C_{2,0}^2 = \frac{2}{n} \left[\binom{n}{3} - 1 \right]$. Going through the now customary steps (multiplying by $(w-z)^{-1}$, integrating over d^2z , multiplying with the partition function Z , replacing $u_l \rightarrow 2\pi\hbar \delta/\delta\mu_l$, differentiating w.r.t. μ_{k+1}) leads to the Ward identities expressing the variation of ψ under a u_{k+1} -transformation:

$$\pi \psi(w) \frac{\delta Z}{\delta \mu_{k+1}(z)} = \sum_{r=0}^k \sum_{m=0}^{k-r} (-)^{k-r} C_{k,r}^m \mathcal{D}_{k-r-m}^{(z)} \partial_z^m \left(\frac{\partial^r \psi(z)}{w-z} Z \right). \quad (3.51)$$

Upon introducing

$$\hat{u}_l = -\frac{1}{2\hbar} u_l \simeq -\pi \frac{\delta}{\delta \mu_l}, \quad l = 2, \dots, n \quad (3.52)$$

(such that $\hat{u}_2 = T$ is the correctly normalized energy-momentum tensor) we obtain the OPE of ψ with $\hat{u}_{k+1}$. It is convenient to write down the transformation with some infinitesimal parameter ϵ :

$$\begin{aligned} \delta_\epsilon^{(k+1)} \psi(z) &\equiv - \oint dw \epsilon(w) \hat{u}_{k+1}(w) \psi(z) \\ &= \sum_{r=0}^k \sum_{m=0}^{k-r} C_{k,r}^m \left(\partial^m \mathcal{D}_{k-r-m}^\dagger \epsilon(z) \right) \partial^r \psi(z). \end{aligned} \quad (3.53)$$

One has e.g.

$$\delta_\epsilon^{(2)} \psi = \epsilon \partial \psi - \frac{n-1}{2} \partial \epsilon \psi \quad (3.54)$$

^{*} We have set $\mathcal{D}_0 = 1$.

telling us that ψ has conformal dimension $-\frac{n-1}{2}$, and

$$\delta_\epsilon^{(3)}\psi = \epsilon\partial^2\psi - (n-2)\partial\epsilon\partial\psi + \frac{2}{n}\left[\mathcal{D}_2\epsilon + \left(\binom{n}{3} - 1\right)\partial^2\epsilon\right]\psi. \quad (3.55)$$

The last equation (and even its full quantum version) was first obtained in ref. 7 by explicitly computing the OPE between $\hat{u}_3$ and ψ . Equations (3.53) to (3.55) show that ψ is a primary field with respect to all W -transformations. For later reference we note that for $u_k = 0 \forall k$, eqs. (3.39), (3.40) and (3.50), (3.53) lead to

$$\begin{aligned} \delta_\epsilon^{(k+1)}\psi &= (-)^k \frac{k}{n} \binom{n}{k+1} \partial^k \epsilon \psi \\ &+ \sum_{p=1}^k (-)^{k-p} \binom{n-1-p}{k-p} \partial^{k-p} \epsilon \partial^p \psi \quad (u_l = 0 \forall l). \end{aligned} \quad (3.56)$$

Following the technique described in ref. 7, these ψ -variations (3.53) now easily lead to the OPEs between any two $\hat{u}$'s. Consider transformations of the first linear differential equation (3.42) by $\oint dw \epsilon(w) \hat{u}_{k+1}(w)$:

$$0 = \oint dw \epsilon(w) \hat{u}_{k+1}(w) \mathcal{D}_n \psi(z). \quad (3.57)$$

Now, $\hat{u}_{k+1}$ transforms ψ - in the way we just computed - but also acts on the u_l 's contained in $\mathcal{D}_n$ - in a way we want to compute. Writing out eq.(3.57) we get

$$\mathcal{D}_n^{(z)} \left(\delta_\epsilon^{(k+1)} \psi(z) \right) = 2\hbar \sum_{r=0}^{n-2} \left(\oint dw \epsilon(w) \hat{u}_{k+1}(w) \hat{u}_{n-r}(z) \right) \partial^r \psi(z). \quad (3.58)$$

It is straightforward to compute the l.h.s. of this equation, using $\mathcal{D}_n \psi = 0$ to eliminate all derivatives of ψ of higher order than $n-1$. (If one wishes, these manipulations can very easily be implemented on an algebraic manipulation program.) One has only to compare the coefficients of $\partial^r \psi$, $r = 0, \dots, n-2$ (the coefficient of $\partial^{n-1} \psi$ on the r.h.s. vanishes automatically) to obtain the OPE of $\hat{u}_{k+1}(w)$ with $\hat{u}_{n-r}(z)$. It is obvious from equations (3.58) and (3.53) that this OPE contains *at most quadratic nonlinearities* in the u 's. As discussed above for the W_3 -algebra, the true Virasoro primary fields are combinations of the u_k 's and their derivatives, e.g. $W_3 \sim u_3 - \frac{n-2}{2} \partial u_2$ and $W_4 \sim u_4 + \alpha \partial u_3 + \beta \partial^2 u_2 + \gamma u_2^2$, etc. In general, these redefinitions are non-linear. Thus it is only when written in terms of the W_k that the W -algebras contain non-linearities of higher than second order.

Before leaving this section, we would like to comment on equations (3.53) - (3.55). We will show directly how they provide a representation of the W -algebra on the primary fields ψ . For the linear Virasoro subalgebra, (3.54) immediately leads to

$$\left[\delta_\epsilon^{(2)}, \delta_{\epsilon'}^{(2)} \right] \psi = \delta_{\tilde{\eta}}^{(2)} \psi \quad , \quad \eta = \epsilon' \partial \epsilon - \epsilon \partial \epsilon' . \quad (3.59)$$

Instead of (3.55) consider the combination

$$\hat{\delta}_\epsilon^{(3)} = \delta_\epsilon^{(3)} + \frac{n-2}{2} \delta_{\partial \epsilon}^{(2)} \quad (3.60)$$

corresponding to the true primary generator $W_3 = \hat{u}_3 - \frac{n-2}{2} \partial \hat{u}_2$. Indeed, using (3.54), (3.55) one finds

$$\left[\delta_\epsilon^{(2)}, \hat{\delta}_{\epsilon'}^{(3)} \right] \psi = \hat{\delta}_{\tilde{\eta}}^{(3)} \psi \quad , \quad \tilde{\eta} = 2\epsilon' \partial \epsilon - \epsilon \partial \epsilon' . \quad (3.61)$$

It is also straightforward to compute $\left[\hat{\delta}_\epsilon^{(3)}, \hat{\delta}_{\epsilon'}^{(3)} \right] \psi$. We only give the result for $n = 3$:

$$\left[\hat{\delta}_\epsilon^{(3)}, \hat{\delta}_{\epsilon'}^{(3)} \right] \psi = \frac{1}{6} \delta_{\tilde{\eta}}^{(2)} \psi - \frac{2}{3} \delta_\omega^{(2)} \psi \quad (3.62)$$

with

$$\begin{aligned} \tilde{\eta} &= \partial^2 (\epsilon' \partial \epsilon - \epsilon \partial \epsilon') - \frac{5}{2} (\partial \epsilon' \partial^2 \epsilon - \partial \epsilon \partial^2 \epsilon') \\ \omega &= u_2 (\epsilon' \partial \epsilon - \epsilon \partial \epsilon') . \end{aligned} \quad (3.63)$$

It is most instructive to compare these commutation relations with the OPE (3.32). Upon multiplying the latter with $\epsilon(w) \epsilon'(z)$ and integrating $\oint dw \oint dz$, it precisely yields (3.62), (3.63), when one uses the following substitutions: $\oint \epsilon W \rightarrow -(-2/5)^{\frac{1}{2}} \hat{\delta}_\epsilon^{(3)}$ and $\oint \epsilon T \rightarrow -\delta_\epsilon^{(2)}$. The only non-trivial point concerns the term $\frac{2\hbar}{3} \oint (\epsilon' \partial \epsilon - \epsilon \partial \epsilon') T^2 \psi$. One of the two T 's must be considered as part of the transformation parameter ($\omega = -2\hbar (\epsilon' \partial \epsilon - \epsilon \partial \epsilon') T$), while the other acts on ψ . However, there are two possible ways to choose which T is which, resulting in an extra factor of 2. This exactly reproduces the term $-\frac{2}{3} \delta_\omega^{(2)} \psi$ in (3.62) with ω , as defined in (3.63), depending on ϵ, ϵ' and u_2 .

4. Solutions of the linear differential equations and geometric interpretation

Recall that for $n = 2$, the two solutions to the linear differential equations (2.3), (2.4) are given by $\psi_1 = (\partial F)^{-\frac{1}{2}}$, $\psi_2 = \psi_1 F$. Then the $(2,0)$ -differential $\tilde{T} \equiv u_2$ is given by $-\frac{1}{2}$ times the Schwarzian derivative of F , and F is a solution to the Beltrami equation. For the complex structure $\mu = 0$, the latter reduces to $\bar{\partial}F = 0$, and F is an arbitrary holomorphic section over the Riemann surface.

This situation generalizes to arbitrary n . Let us first consider the submanifold of the reduced symplectic manifold where the generalized complex structures are $\mu_{k+1} = 0 \forall k$. Then one of the two linear differential equations, namely (3.43), is simply $\bar{\partial}\psi = 0$, and the ψ 's are holomorphic sections. The *holomorphic* solutions to the other linear differential equation (3.42), $\mathcal{D}_n\psi = 0$, are well known [7] to be of the form

$$\psi_0(z) = \prod_{j=1}^{n-1} (\partial F_j(z))^{-\frac{n-j}{n}} \quad (4.1)$$

$$\psi_j(z) = \psi_0(z) \int_{z_0}^z dz_1 \partial F_1(z_1) \int_{z_0}^{z_1} dz_2 \partial F_2(z_2) \int_{z_0}^{z_2} \dots \int_{z_0}^{z_{j-1}} dz_j \partial F_j(z_j) , \quad (4.2)$$

$j = 1, \dots, n-1$, where the F_i 's are holomorphic. The paths of integration start at some arbitrary base-point, and their precise form does not matter since the integrand is holomorphic. In ref. [7] it was shown how to express all u_k 's in terms of the $n-1$ ∂F_j 's : from (4.2) it follows immediately that the ψ_j satisfy the differential equation

$$\partial \frac{1}{\partial F_{n-1}} \partial \dots \frac{1}{\partial F_2} \partial \frac{1}{\partial F_1} \partial \frac{1}{\psi_0} \psi_j = 0 . \quad (4.3)$$

Identifying the differential operator acting on ψ_j with $\mathcal{D}_n$ yields all u_k 's in terms of the $\partial^2 F_i / \partial F_i$, $i = 1, \dots, n-1$. (Note that the choice (4.1) of ψ_0 ensures $u_1 = 0$.) This can be conveniently done by introducing the simple roots e_i of the Lie algebra A_{n-1} , and a $(n-1)$ -component free boson field ϕ by

$$\partial F_i = e^{e_i \cdot \phi} . \quad (4.4)$$

Then we have e.g. for $u_2 \equiv \tilde{T}$:

$$u_2 = \frac{1}{2} (\partial \phi \cdot \partial \phi - \rho_w \cdot \partial^2 \phi) \quad (4.5)$$

where ρ_w denotes the Weyl vector of A_{n-1} . For $n = 2$, one simply has $\partial \phi = \partial^2 F / (\sqrt{2} \partial F)$ and $\rho_w = \sqrt{2}$, and one recovers the Schwarzian derivative $u_2 = -\frac{1}{2} S_z(F)$.

In the present context, it is more convenient to use the alternative form

$$u_2 = \sum_{i,j=1}^{n-1} (K^{-1})_{ij} \left(\frac{1}{2} \phi_i \phi_j - \partial \phi_j \right) \quad (4.6)$$

$$\phi_i = e_i \cdot \phi = \partial^2 F_i / \partial F_i$$

where K is the Cartan matrix of A_{n-1} : $K_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i+1,j}$. The other u_k are similarly expressed as sums of products of derivatives of ϕ . This constitutes a Miura transformation. Note that the solutions ψ_j when expressed in terms of ϕ have the form of vertex operators and screened vertex operators, and that the ψ 's appear as the classical versions of the W_n -generalization of the $\Phi_{(21)}$ primary fields: $\mathcal{D}_n \psi = 0$ is the classical version of the null vector decoupling equation.

The parametrization (4.2) of the ψ_j 's involves multiple integrals, and it will be more advantageous to use a slightly different but equivalent parametrization. Write

$$\psi_j = \psi_0 G_j \quad , \quad j = 1, \dots, n-1 \quad , \quad (4.7)$$

and consider the $n-1$ sections G_j as independent, defining the F_j by

$$F_i = \frac{1}{\partial F_{i-1}} \partial \frac{1}{\partial F_{i-2}} \partial \dots \partial \frac{1}{\partial F_1} \partial G_i \quad , \quad (4.8)$$

in particular $F_1 = G_1$, with ψ_0 still defined by (4.1). It is straightforward to show by induction that ψ_0 , when expressed in terms of the G_i 's, equals the Wronskian of the ∂G 's to the power $-\frac{1}{n}$:

$$(\psi_0)^{-n} = \begin{vmatrix} \partial G_1 & \partial G_2 & \dots & \partial G_{n-1} \\ \partial^2 G_1 & \partial^2 G_2 & \dots & \partial^2 G_{n-1} \\ \vdots & \vdots & & \vdots \\ \partial^{n-1} G_1 & \partial^{n-1} G_2 & \dots & \partial^{n-1} G_{n-1} \end{vmatrix} \quad (4.9)$$

This parametrization has the advantage of involving no integrations and hence no dependence on the base-point or on the integration path (when the integrand is no longer holomorphic).

This parametrization of the ψ 's remains valid on the whole reduced symplectic manifold, i.e. arbitrary μ_k . The u_k 's are still the same functions of the F_i 's or of ϕ which are to be expressed in terms of the G_j 's by (4.8). In this sense, the parametrization (4.7), (4.8) solves the first linear differential equation $\mathcal{D}_n \psi = 0$. The other differential

equation determines the $\bar{z}$ -dependence of the ψ 's and of the G 's and F 's. For $\mu_k = 0 \ \forall k \geq 3$, but $\mu_2 \neq 0$, it reduces to the Beltrami equation $(\bar{\partial} - \mu\partial) G_i = 0 \ \forall i$, and also $(\bar{\partial} - \mu\partial) F_i = 0 \ \forall i$. In this case all G_i and F_i are simply isothermal complex coordinates. For general μ_k , the situation is more complicated, and we do not know the solution to the general problem of determining the G_i and F_i for arbitrary n . For $n = 3$ e.g., F_1 and F_2 satisfy a system of coupled non-linear differential equations:

$$\begin{aligned} \left(\bar{\partial} - \mu\partial + \partial\rho\partial + \frac{1}{3}\rho\partial^2 + \frac{2}{3}\rho\frac{\partial^2 F_2}{\partial F_2}\partial \right) F_1 &= 0 \\ \left(\bar{\partial} - \mu\partial - \frac{1}{3}\rho\partial^2 - \frac{2}{3}\rho\frac{\partial^2 F_1}{\partial F_1}\partial \right) F_2 &= 0 . \end{aligned} \quad (4.10)$$

The coupling between these two equations is due to the appearance of higher order derivatives ∂^2 whenever $\rho \equiv \mu_3 \neq 0$. These cannot be induced by a simple diffeomorphism. Indeed, if one starts at the point $\mu_k = 0 \ \forall k$, one has $\bar{\partial}F_i = 0$. A diffeomorphism $(z, \bar{z}) \rightarrow (z', \bar{z}')$ will transform this equation into $(\bar{\partial} - \mu\partial) F_i = 0$, but it will not generate any higher derivatives.

We can now generalize the propositions of section 2. Consider the two linear differential equations (3.42) and (3.43). Let $\psi_0, \dots, \psi_{n-1}$ be n linearly independent simultaneous solutions. They are numbered in some fixed but arbitrary way. We define $G_1, \dots, G_{n-1}$ by (4.7), and thus $F_1, \dots, F_{n-1}$ by (4.8). Since $u_1 = 0$, $\mathcal{D}_n\psi = 0$ implies that the Wronskian of the ψ_j 's is constant. Upon multiplying ψ_0 by some appropriate constant, we may take the Wronskian equal to unity. This implies that ψ_0 is given by (4.1) in terms of the F_i 's. Thus, given any n linearly independent solutions $\psi_0, \dots, \psi_{n-1}$, we can unambiguously define a set of F_i 's, $i = 1, \dots, n-1$. Choosing a different set of ψ 's induces a non-trivial transformation of the F 's. As shown above, the F 's allow us to reconstruct the u_k 's. Different sets of F 's coming from different sets of solutions ψ to the same differential equation obviously lead to the same u_k 's. Thus we have the following generalizations of the results of section 2.

The $Sl(n, \mathbb{C})$ -generalized *complex* structures are in one-to-one correspondence with the $\mu_k(z, \bar{z})$, $k = 2, \dots, n$. The latter are $(-1, k-1)$ -differentials. The corresponding generalized complex structure is determined by the ratios of the solutions of

$$\left(\bar{\partial} - \sum_{k=0}^{n-1} a_{n-1, n-1-k}^{(n)} \partial^k \right) \psi = 0 . \quad (4.11)$$

with $a_{n-1, n-1-k}^{(n)} \equiv a_{n-1, n-1-k}^{(n)}(\mu_2, \dots, \mu_n)$ given by equations (3.39) and (3.40).

The $Sl(n, \mathbb{C})$ -generalized *projective* structures are in one-to-one correspondence with the (μ_k, u_k) satisfying the conditions resulting from (3.44) (i.e. the Ward identities, e.g. (3.19) and (3.20) for $n = 3$). The corresponding generalized projective structure is determined by the ratios of simultaneous solutions to (4.11) and

$$\mathcal{D}_n \psi \equiv \left(\partial^n - \sum_{k=2}^n u_k \partial^{n-k} \right) \psi = 0 . \quad (4.12)$$

What is the geometric interpretation of these $Sl(n, \mathbb{C})$ -generalized structures? This question is twofold: first, starting from the subspace $\mu_k = 0 \ \forall \ k$, where (4.11) reads $\bar{\partial}\psi = 0$, what transformation takes us to a generic point $\mu_k \neq 0$? The other question is simpler: starting from the trivial point $\mu_k = u_k = 0 \ \forall \ k$, where also (4.12) simplifies to $\partial^n \psi = 0$, what transformation takes us to a generic point $u_k \neq 0$ within the subspace $\mu_k = 0$? The latter question was partially (i.e. for $n = 3$) answered in ref. 14. Starting from the observation that, for $n = 2$, diffeomorphisms can be considered as deformations of some flags in the jet space $\Gamma^{(1)}$, the authors of ref. 14 argue that, for $n = 3$, W_3 -transformations can be realized as deformations of flags in the jet space $\Gamma^{(2)}$. We shall generalize their argument to all n . This answers our second question. It also gives a partial answer to our first question. Indeed, the flag configurations being parametrized by the sections G_i , non-holomorphic G_i 's take us away from the point $\mu_k = 0$ where $\bar{\partial}\psi = 0$. However, this does not really solve the first problem since we do not know how to determine the G_i for given non-zero μ_k , see the discussion above equation (4.10).

Before entering into details, let us roughly explain the basic ideas and concepts involved. For $n = 2$, instead of considering the diffeomorphisms as acting on the coordinates defined on the Riemann surface, it is more fruitful to consider their action directly on $(\Delta, \bar{\Delta})$ -differentials f . (For the time being, we only consider holomorphic diffeomorphisms.) It is given by $\delta_\epsilon^{(2)} f = \epsilon \partial f + \Delta \epsilon f$. Thus, we consider the representations of the transformation group instead of the group itself. The transformation $\delta^{(2)}$ is precisely given by the Lie derivative. It involves f itself and its first derivative ∂f . We will consider the bundle whose fiber for any point z on the Riemann surface is a certain two-dimensional space, with components corresponding, roughly speaking, to $f(z)$ and $\partial f(z)$. This is called the jet bundle $\Gamma^{(1)}$. A diffeomorphism changes f into some combination of f and ∂f and will be interpreted as a transformation acting on the fiber of $\Gamma^{(1)}$. In the present context, a flag of length $k + 1$ is an embedding sequence $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_k$ of subspaces of dimension 1 to $k + 1$, such that the embedding $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ is an embedding as a (local) direct sum. Since the fiber of $\Gamma^{(1)}$

is two-dimensional, we can define the flag $\mathcal{F}_0 \subset \mathcal{F}_1$ of one- and two-dimensional subspaces, with, roughly speaking, $\mathcal{F}_0(z)$ being spanned by $f(z)$ and $\mathcal{F}_1(z)$ spanned by $f(z)$ and $\partial f(z)$. Since a diffeomorphism changes f into some combination of f and ∂f , it changes the way $\mathcal{F}_0$ is embedded into $\mathcal{F}_1$. Thus a diffeomorphism will be described as a deformation of the flag in the jet bundle $\Gamma^{(1)}$. Generalizing to arbitrary n , we know from eq. (3.53) that $\delta^{(k+1)}$ -transformations, generated by u_{k+1} , involve f and its first k derivatives ($k = 1, \dots, n-1$). One then considers the jet bundle $\Gamma^{(n-1)}$ whose fibers are n -dimensional spaces with components corresponding to $f(z), \partial f(z), \dots, \partial^{n-1} f(z)$. The $\delta^{(k+1)}$ -transformation only involves up to $\partial^k f(z)$. It is equivalent to changing a whole embedding sequence of one to k -dimensional subspaces (flag). They are thus described as deformations of some flags in the jet bundle $\Gamma^{(n-1)}$ (effectively only $\Gamma^{(k)}$ for $\delta^{(k+1)}$).

Let us now explain these ideas in more detail, thus generalizing the approach of ref. 14 to arbitrary n . Consider some power of the (holomorphic) canonical line bundle. Its local sections are simply some (holomorphic) Δ -differentials $f(z)$. Two sections f and g are said to be k -equivalent at the point z if $f(z) = g(z)$ and $\partial^j f(z) = \partial^j g(z)$, $j = 1, \dots, k$. The corresponding equivalence class containing f is called a k -jet of f at z . (In fact, this is nothing but a coordinate-free formulation of the Taylor-polynomial of order k of f around z .) The k^{th} jet manifold is the set of all k -jets, such that z is some point on the Riemann surface and f some section. The k^{th} jet manifold is naturally a bundle over the Riemann surface which we denote by $\Gamma^{(k)}$. Its fibers are $k+1$ -dimensional spaces. By definition, for *fixed* z there exists a section f such that we may write the fiber coordinates as $(f(z), \partial f(z), \dots, \partial^k f(z))$. In fact, for a given section f , one defines its prolongation as the section $(f, \partial f, \dots, \partial^k f)$ of $\Gamma^{(k)}$. This just means that at each point of the Riemann surface one considers simultaneously $f(z)$ and its first k derivatives. However, it is clear that not every section of $\Gamma^{(k)}$ is a prolongation of some section f .

In the following we are interested in the case $k = n-1$. We introduce a basis $\{e_i\}_{i=0, \dots, n-1}$ in the fiber of $\Gamma^{(n-1)}$. Any section f can be lifted to a section of $\Gamma^{(n-1)}$ by

$$f^\Gamma = f(z)e_0 + \partial f(z)e_1 + \dots + \partial^{n-1} f(z)e_{n-1} . \quad (4.13)$$

Under a diffeomorphism, $f(z) = J^\Delta f'(z')$, $J = \partial z' / \partial z$, the basis $\{e_i\}$ is transformed into a basis $\{e'_i\}$, the transformation matrix being upper triangular. For $n = 2$, e.g.

$$\begin{pmatrix} e'_0 \\ e'_1 \end{pmatrix} = \begin{pmatrix} J^\Delta & \partial J^\Delta \\ 0 & J^{\Delta+1} \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \end{pmatrix} . \quad (4.14)$$

Thus the basis $\{e_i\}$ depends on the choice of coordinates. We will need to consider also a different basis of $\Gamma^{(n-1)}$, namely the one provided by the n solutions ψ_j of $\mathcal{D}_n\psi = 0$, see (4.1), (4.2). According to (4.13), we lift each of the ψ_j to a section ψ_j^Γ of $\Gamma^{(n-1)}$. Then the change of basis is

$$\psi_j^\Gamma = \sum_{i=0}^{n-1} M_{ji} e_i \quad , \quad M_{ji} = \partial^i \psi_j . \quad (4.15)$$

It is instructive to consider the special point of the symplectic manifold where $\mu_k = u_k = 0 \ \forall \ k$. We denote all quantities at this point by an extra asterisk. As already mentioned, at this point the F 's and G 's are holomorphic and can be chosen as

$$G_j^* = \frac{z^j}{j!} \Rightarrow F_j^* = z, \ \psi_0^* = 1, \ M_{ji}^* = \frac{z^{j-i}}{(j-i)!} \delta_{j \geq i} . \quad (4.16)$$

Any section f^Γ of $\Gamma^{(n-1)}$ can be written in either of the two bases:

$$f^\Gamma = \sum_{i=0}^{n-1} f^{(i)} e_i = \sum_{j=0}^{n-1} f_j \psi_j^\Gamma \quad (4.17)$$

with

$$f^{(i)} = \sum_{j=0}^{n-1} f_j M_{ji} = \sum_{j=0}^{n-1} f_j \partial^i \psi_j . \quad (4.18)$$

For fixed f^Γ , the $f^{(i)}$ and f_j provide two different dual bases in $\Gamma^{(n-1)}$, related by (4.18).

Following ref. 14 let us first consider the case $n = 2$ in some detail. For $n = 2$, $\psi_0 = (\partial F)^{-\frac{1}{2}}$, $\psi_1 = \psi_0 F$ and

$$M = (\partial F)^{-\frac{1}{2}} \begin{pmatrix} 1 & -\frac{1}{2} \frac{\partial^2 F}{\partial F} \\ F & \partial F - \frac{1}{2} \frac{\partial^2 F}{\partial F} F \end{pmatrix} . \quad (4.19)$$

Thus

$$\begin{aligned} f^{(0)} &= (\partial F)^{-\frac{1}{2}} (f_0 + F f_1) \\ f^{(1)} &= -\frac{1}{2} (\partial F)^{-\frac{3}{2}} \partial^2 F (f_0 + F f_1) + (\partial F)^{\frac{1}{2}} f_1 . \end{aligned} \quad (4.20)$$

Defining

$$\hat{f}_0 = f_0 + F f_1 , \quad \hat{f}_1 = f_1 \quad (4.21)$$

this can be written as

$$\begin{aligned} f^{(0)} &= (\partial F)^{-\frac{1}{2}} \hat{f}_0 \\ f^{(1)} &= (\partial F)^{\frac{1}{2}} \hat{f}_1 - \frac{1}{2} (\partial F)^{-\frac{3}{2}} \partial^2 F \hat{f}_0 . \end{aligned} \quad (4.22)$$

Clearly, for fixed z , $\hat{f}_0$ and $\hat{f}_1$ each span one-dimensional subspaces in the fiber of $\Gamma^{(1)}$. Let $\mathcal{F}_0(z)$ be the one-dimensional subspace spanned by $\hat{f}_0$, and $\mathcal{F}_1(z)$ the two-dimensional subspace spanned by $\hat{f}_0$ and $\hat{f}_1$. Denote the corresponding bundles simply by $\mathcal{F}_0$ and $\mathcal{F}_1$. We have the following flag^{*}:

$$\mathcal{F}_0 \subset \mathcal{F}_1 \quad (4.23)$$

and the way $\mathcal{F}_0$ is embedded into $\mathcal{F}_1$ is described by $F(z)$, see (4.21). Changing $F(z)$ results in a deformation of this flag. Consider a deformation of the flag given by $F \rightarrow \tilde{F} = F + \delta F$. This implies $\hat{f}_0 \rightarrow \tilde{\hat{f}}_0 = \hat{f}_0 + \tilde{F} \hat{f}_1 = \hat{f}_0 + \delta F \hat{f}_1$ while $\hat{f}_1$ remains unchanged. Thus by (4.22), $f^{(0)} \rightarrow \tilde{f}^{(0)} = f^{(0)} - \frac{1}{2} (\partial F)^{-\frac{3}{2}} \partial \delta F \hat{f}_0 + (\partial F)^{-\frac{1}{2}} \delta F \hat{f}_1 + \mathcal{O}((\delta F)^2)$. Using again (4.22) one has

$$\tilde{f}^{(0)} = f^{(0)} - \frac{1}{2} f^{(0)} \partial \left(\frac{\delta F}{\partial F} \right) + \frac{\delta F}{\partial F} f^{(1)} . \quad (4.24)$$

Clearly, upon writing $\delta F = \epsilon \partial F$ and projecting back on the zero component of the section (i.e. $f^{(0)} \rightarrow f$, $f^{(1)} \rightarrow \partial f$) one recovers the transformation of $-\frac{1}{2}$ differentials under a diffeomorphism $z \rightarrow z + \epsilon$ (cf. (3.54)). Of course, this was to be expected since $z \rightarrow z + \epsilon$ precisely induces $\delta F = \epsilon \partial F$.

We will now generalize the preceding discussion to arbitrary n . It is straightforward to check, using (4.7) and (4.18), that equations (4.20), (4.21) are generalized as

$$\begin{aligned} f^{(0)} &= \psi_0 \hat{f}_0 \\ f^{(1)} &= \partial \psi_0 \hat{f}_0 + \psi_0 \partial G_1 \hat{f}_1 \\ &\vdots \\ f^{(j)} &= \partial^j \psi_0 \hat{f}_0 + \sum_{l=1}^j \sum_{k=0}^{j-l} \binom{j}{k} \partial^k \psi_0 \tilde{g}_{l,l}^{j-k} \hat{f}_l \end{aligned} \quad (4.25)$$

* Note that we index the $\mathcal{F}$'s differently from ref. 14.

with

$$\begin{aligned}
 \hat{f}_0 &= f_0 + \sum_{i=1}^{n-1} f_i G_i \\
 \hat{f}_1 &= f_1 + \sum_{i=2}^{n-1} f_i \frac{\partial G_i}{\partial G_1} \\
 &\vdots \\
 \hat{f}_j &= f_j + \sum_{i=j+1}^{n-1} f_i g_i^j
 \end{aligned} \tag{4.26}$$

We have introduced g_i^j which is recursively defined by

$$g_i^0 = G_i, \quad g_i^{j+1} = \frac{\partial g_i^j}{\partial g_{j+1}^j} \tag{4.27}$$

(note that $g_i^{i-1} = F_i$, cf. (4.8)), and $\tilde{g}_{i,m}^j$ defined by

$$\tilde{g}_{i,1}^j = \partial^j G_i, \quad \tilde{g}_{i,m+1}^j = \tilde{g}_{i,m}^j - \tilde{g}_{m,m}^j g_i^m \quad \forall m \geq 1. \tag{4.28}$$

Let $\mathcal{F}_j(z)$ denote the $(j+1)$ -dimensional subspace of the fiber of Γ^{n-1} at z spanned by $\hat{f}_0, \hat{f}_1, \dots, \hat{f}_j$. We then have the flag configuration

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_j \subset \mathcal{F}_{j+1} \subset \dots \subset \mathcal{F}_{n-1}. \tag{4.29}$$

This flag configuration is described by the n sections $G_j(z)$. Changing $G_j \rightarrow G_j + \delta G_j$ will deform this flag. We will show that these deformations generate W -transformation, thus generalizing the results of ref. 14. In particular, we will identify which deformations δG_j generate a specific $\delta^{(k)}$ -transformation.

Consider the deformation $G_j \rightarrow \tilde{G}_j = G_j + \delta G_j$. Thus by (4.26)

$$\delta \hat{f}_0 = \sum_{i=1}^{n-1} \epsilon_i \hat{f}_i \tag{4.30}$$

where the ϵ_i are defined in terms of the δG_j as follows:

$$\epsilon_i = e_i^i, \quad e_i^1 = \delta G_i, \quad e_i^{j+1} = e_i^j - g_i^j e_j^j, \tag{4.31}$$

with g_i^j given by (4.27), in particular one has $\epsilon_1 = \delta G_1$ and $\epsilon_2 = \delta G_2 - \frac{\partial G_2}{\partial G_1} \delta G_1$. One

can invert these relations and express the δG_i in terms of the ϵ_j :

$$\delta G_i = \epsilon_i + \sum_{j=1}^{i-1} g_i^j \epsilon_j . \quad (4.32)$$

In the special case that all $\epsilon_j = 0$ except $\epsilon_1 \equiv \partial G_1 \epsilon$, eq. (4.32) implies $\delta G_i = \partial G_i \epsilon$. This is precisely the transformation induced by a diffeomorphism $z \rightarrow z + \epsilon$. Let us show that $f^{(0)} \rightarrow \tilde{f}^{(0)}$ reproduces the $\delta_\epsilon^{(2)}$ -transformation: since one has $\delta F_j = \partial F_j \epsilon$, it follows from (4.1) that $\delta \psi_0 = -\frac{n-1}{2} \psi_0 \partial \epsilon + \partial \psi_0 \epsilon$ and thus

$$\begin{aligned} \tilde{f}^{(0)} &= f^{(0)} + \delta \psi_0 \hat{f}_0 + \psi_0 \delta \hat{f}_0 + \mathcal{O}(\epsilon^2) \\ &= f^{(0)} \left(1 - \frac{n-1}{2} \partial \epsilon + \frac{\partial \psi_0}{\psi_0} \epsilon \right) + \psi_0 \hat{f}_1 \partial G_1 \epsilon + \mathcal{O}(\epsilon^2) \end{aligned} \quad (4.33)$$

and using (4.25):

$$\tilde{f}^{(0)} = f^{(0)} - \frac{n-1}{2} \partial \epsilon f^{(0)} + \epsilon f^{(1)} . \quad (4.34)$$

Projecting back on the zero component ($f^{(0)} \rightarrow f$, $f^{(1)} \rightarrow \partial f$) we have recovered the $\delta_\epsilon^{(2)}$ -transformation of $-\frac{n-1}{2}$ -differentials f .

In order to study the general transformations (arbitrary ϵ_j , $j = 1, \dots, n-1$) it is instructive to consider first the point $\mu_k = u_k = 0 \ \forall \ k$ where G_i , F_i are given by (4.16). One has $g_i^{*j} = \frac{z^{i-j}}{(i-j)!}$. Thus $f^{*(j)} = \hat{f}_j^* = \sum_{l=j}^{n-1} f_l^* \frac{z^{l-j}}{(l-j)!}$ which is inverted as $f_j^* = \sum_{l=j}^{n-1} f^{*(l)} \frac{(-z)^{l-j}}{(l-j)!}$. Similarly from (4.32) one obtains

$$\delta G_i^* = \epsilon_i + \sum_{j=1}^{i-1} \frac{z^{i-j}}{(i-j)!} \epsilon_j \quad (4.35)$$

and thus by (4.8)

$$\delta F_i^* = \sum_{j=1}^i \binom{i-1}{j-1} \partial^{j-1} \epsilon_j . \quad (4.36)$$

Furthermore, since

$$\frac{\delta \psi_0}{\psi_0} = - \sum_{i=1}^{n-1} \frac{n-i}{n} \frac{\partial \delta F_i}{\partial F_i} \quad (4.37)$$

we have

$$\frac{\delta\psi_0^*}{\psi_0^*} = - \sum_{i=1}^{n-1} \frac{n-i}{n} \sum_{j=1}^i \binom{i-1}{j-1} \partial^j \epsilon_j = - \sum_{j=1}^{n-1} \frac{1}{n} \binom{n}{j+1} \partial^j \epsilon_j, \quad (4.38)$$

and thus

$$\begin{aligned} \delta f^{*(0)} &= \frac{\delta\psi_0^*}{\psi_0^*} f^{*(0)} + \sum_{j=1}^{n-1} \epsilon_j \hat{f}_j^* \\ &= \sum_{j=1}^{n-1} \left(f^{*(j)} \epsilon_j - \frac{1}{n} \binom{n}{j+1} f^{*(0)} \partial^j \epsilon_j \right). \end{aligned} \quad (4.39)$$

Comparing with (3.56) and using standard identities between binomial coefficients, we see that (upon projecting $f^{*(0)} \rightarrow f^*$, $f^{*(j)} \rightarrow \partial^j f^*$) this is the following combination of transformations

$$\delta f^* = \sum_{j=1}^{n-1} \delta_{\tilde{\epsilon}_j}^{(j+1)} f^* \quad (4.40)$$

where

$$\tilde{\epsilon}_j = \sum_{i=j}^{n-1} \binom{n-1-j}{n-1-i} \partial^{i-j} \epsilon_i. \quad (4.41)$$

This relation is immediately inverted as

$$\epsilon_j = \sum_{i=j}^{n-1} (-)^{i-j} \binom{n-1-j}{n-1-i} \partial^{i-j} \tilde{\epsilon}_i. \quad (4.42)$$

Thus, if for fixed k only $\tilde{\epsilon}_k \neq 0$, we see that the transformation $\delta_{\tilde{\epsilon}_k}^{(k+1)}$, generated by the spin- $(k+1)$ operator, corresponds to $\epsilon_j = (-)^{k-j} \binom{n-1-j}{n-1-k} \partial^{k-j} \tilde{\epsilon}_k$ for $j \leq k$ and $\epsilon_j = 0$ for $j > k$. In fact it was already obvious from eqs. (4.30) and (4.25) that a $\delta^{(k+1)}$ -transformation can only involve ϵ_j 's with $j \leq k$, since it only involves derivatives up to order k .

Similarly, for general u_k , i.e. general G_i, F_i , one may check that the deformation (4.30), (4.32) of the flag configuration generates some combination of W -transformations $\delta_{\tilde{\epsilon}_j}^{(j+1)}$ on $-\frac{n-1}{2}$ -differentials. Again, if $\epsilon_i = 0 \forall i > k$, only $\delta^{(j+1)}$ -transformations with $j \leq k$ are present. As a special case, we have already discussed the diffeomorphisms where all $\epsilon_i = 0$ except ϵ_1 (see (4.33), (4.34)). In general, the corresponding computations are rather tedious, and we will only sketch them in the appendix for non-zero ϵ_1 and ϵ_2 (leading to a combination of $\delta^{(2)}$ - and $\delta^{(3)}$ -transformations).

We have thus generalized the results of ref. 14 to arbitrary n : the W -transformations $\delta^{(k)}$ are in one-to-one correspondence with deformations of the flag bundle. However, as already noted, these results are valid only on the subspace $\mu_k = 0$. Elucidating the *full* geometric meaning of the data $(\{\mu_k\}, \{u_k\})$, analogous to the results we recalled in section 2 for $n = 2$, remains still a challenge for future work.

In this respect it is worthwhile to make the following remark. Remember from section 3 that the μ_k and u_k parametrize the gauge connection A on the reduced symplectic manifold. The Ward identities are equivalent to the flatness condition, i.e. $A = g^{-1}dg$, $g \in G \equiv Sl(n, \mathbf{C})$. One may decompose g as $g = \tilde{g}h$ with $\tilde{g} \in G^P$ (the gauge parabolic subgroup, cf. (3.6)) and $h \in G/G^P$. Thus h is parametrized by $n - 1$ sections $H_1, \dots, H_{n-1}$. They can be considered as coordinates on $\mathbf{CP}^{n-1} \simeq G/G^P$. Therefore, the solutions to the compatibility conditions are (locally) parametrized by the mapping of a curve X into $\mathbf{CP}^{n-1}$ (up to $\mathbf{CP}^{n-1}$ -automorphisms). The solutions of (3.42), (3.43) may be interpreted as liftings of sections of a hyperplane line bundle from $\mathbf{CP}^n$ to X , the sections of this line bundle being of the form $(a_0 + a_1 H_1 + \dots + a_{n-1} H_{n-1})(dH_1 \wedge \dots \wedge dH_{n-1})^{-\frac{1}{n-1}}$.

5. Field theoretic description

In this chapter we give an - admittedly incomplete - $2 + 1$ -dimensional quantum field theoretic description of the W -symmetries. To formulate this approach, let us recall how the W_2 (Virasoro) Ward identity can be derived from the $2 + 1$ -dimensional Chern-Simons (CS) theory [18]. We start from the theory given by the action

$$S_{\text{CS}} = \frac{k}{\pi} \text{tr} \int_{\mathcal{M}} \left(A dA + \frac{2}{3} A^3 \right) \quad (5.1)$$

where $\mathcal{M} = \Sigma \times \mathbf{R}$, Σ being a two-dimensional Riemann surface, and A is a G -valued gauge field, where G is some compact Lie group. This action implies the canonical equal-time commutation relations (cf. (3.2))

$$\left[\bar{A}^a(z), A^b(w) \right] = \frac{\pi}{k} \delta^{ab} \delta^{(2)}(z - w) . \quad (5.2)$$

We have written $A \equiv A_z$ and $\bar{A} \equiv A_{\bar{z}}$. If we choose the A^a as coordinates in the wave functionals, $\bar{A}^a$ is represented as $\frac{\pi}{k} \frac{\delta}{\delta A^a}$.

Using the commutation relations (5.2) it is easy to see that infinitesimal gauge transformations with parameter ϵ^a are generated by

$$H_\epsilon^{(1)} = -\frac{k}{\pi} \int_{\Sigma} d^2 z \text{tr} \epsilon(z) \epsilon^{ij} F_{ij}(z) = -\frac{k}{\pi} \int_{\Sigma} d^2 z \epsilon^a \left(\partial \bar{A}^a - \bar{\partial} A^a + f^{abc} A^b \bar{A}^c \right) . \quad (5.3)$$

Any physical wave functional $\Psi[A] \in \mathcal{H}_{\text{CS}}\{A\}$ should be gauge invariant and obey

$$\begin{aligned} 0 &= \delta_\epsilon \Psi[A] = \left[H_\epsilon^{(1)}, \Psi[A] \right] \\ &= -\frac{k}{\pi} \int_{\Sigma} d^2 z \epsilon^a \left(\frac{\pi}{k} \partial \frac{\delta}{\delta A^a} - \bar{\partial} A^a + \frac{\pi}{k} f^{abc} A^b \frac{\delta}{\delta A^c} \right) \Psi[A] . \end{aligned} \quad (5.4)$$

Let us recall that in (5.1) the field A_0 appears as a Lagrange multiplier leading to the classical constraints $\epsilon^{ij} F_{ij} = 0$. In section 3, these constraints were interpreted as the Ward identities of the two-dimensional theory. Here they become operator constraints that annihilate the physical wave functionals.

At first sight, we cannot get any information about the two-dimensional geometry from the theory, because the CS action does not depend on the 2D geometry at all.

However, following reference 18, we will show how one can nevertheless obtain information about this geometry. As above for the gauge transformations, one easily checks that a diffeomorphism with parameter ($v^z \equiv v, v^{\bar{z}} \equiv \bar{v}$) is generated by

$$H_\epsilon^{(2)} = -\frac{k}{2\pi} \int_\Sigma d^2z \operatorname{tr} A \wedge L_v A = \frac{k}{\pi} \int_\Sigma d^2z (v A^a + \bar{v} \bar{A}^a) (\bar{\partial} A^a - \partial \bar{A}^a) \quad (5.5)$$

where L_v denotes the Lie derivative. Note that $H_\epsilon^{(2)}$ can also be written as

$$H_\epsilon^{(2)} = -\frac{k}{\pi} \int_\Sigma d^2z (v A^a + \bar{v} \bar{A}^a) \epsilon^{ij} F_{ij}^a \quad (5.6)$$

since the term $\sim f^{abc}$ vanishes by antisymmetry. The geometry-independence of the theory (5.1), i.e. its diffeomorphism invariance, implies again that physical wave functionals must be annihilated by $H_\epsilon^{(2)}$:

$$\int_\Sigma d^2z \left(v A^a + \frac{\pi}{k} \bar{v} \frac{\delta}{\delta A^a} \right) \left(\bar{\partial} A^a - \frac{\pi}{k} \partial \frac{\delta}{\delta A^a} \right) \Psi[A] = 0 \quad (5.7)$$

(where some normal ordering is understood). As it is written here, this constraint does not give us the information about the complex structure on Σ , which we need to get the W_2 -Ward identity. However, we can introduce a new quantum representation for the wave functional. Consider the basis of functionals

$$\chi_\mu[A] = \exp \left(-\frac{k}{2\pi} \int \mu \operatorname{tr} A^2 \right) . \quad (5.8)$$

(We omit a term linear in A , which was used in ref. 18 to derive the Kac-Moody Ward identity, and which is irrelevant here.) It is clear from (5.8) that μ must be a $(-1, 1)$ differential in order that μA^2 be a $(1, 1)$ differential as it should. We may use the basis (5.8) to “Fourier-transform” the wave-functional $\Psi[A]$:

$$\Psi[\mu] = \int \mathcal{D}A \Psi[A] \chi_\mu[A] . \quad (5.9)$$

Now, if we consider the constraint (5.7) in the μ -representation, we get (see [18] for details)

$$\int_\Sigma d^2z (v + \mu \bar{v}) (\bar{\partial} - \mu \partial - 2\partial\mu) \frac{\delta}{\delta \mu(z)} \Psi[\mu] = -\frac{c}{12\pi} \int_\Sigma d^2z (v + \mu \bar{v}) \partial^3 \mu \Psi[\mu] \quad (5.10)$$

where the term $c \partial^3 \mu$ on the r.h.s. appears due to the regularization of the quadratic term A^2 in (5.8), c being some constant. If we add to (5.10) the Kac-Moody Ward

identity we get the Sugawara construction as the compatibility condition for these two constraints (and thus a definite value $c = \frac{k \dim G}{k + c_v}$), but here we will not do so, and, for the time being, c should be considered as some arbitrary constant. Comparing with equation (3.10), we see that the constraint (5.10) is precisely the Ward identity for the Virasoro symmetry provided we identify μ with the Beltrami differential. Thus μ gives us the information about the complex structure (cf. (2.1)). The stress tensor for a given complex structure μ then is (cf. (3.27))

$$T_\mu(z) = -\pi \frac{1}{\Psi[\mu]} \frac{\delta}{\delta \mu(z)} \Psi[\mu] . \quad (5.11)$$

What have we done? The main idea we used is that the geometry appears when we consider some special representation of the wave functional, namely the μ -representation (5.9). In some sense, we can say that the CS theory and its Hilbert space do not depend on the geometry, but if we wish to define some special basis for the wave functionals we cannot do it without making reference to the geometry. It is impossible to define the integral of A^2 , or more generally of A^k , without introducing some appropriate geometric structure.

Let us now try to give the analogous interpretations of the W_n -symmetries for $n > 2$. The reader should be aware that part of the following has a somewhat formal character, since we will make certain assumptions about regularizations which we have not proved for the time being. We first consider W_3 for simplicity. Later on, we show how to generalize to W_n . For W_3 we have the two constraints (3.19) and (3.20) (i.e. the W_3 -Ward identities) which we rewrite, using (3.27) and (3.28), as

$$\begin{aligned} \mathcal{C}_2 &\equiv (\bar{\partial} - \mu \partial - 2\partial \mu) \frac{\delta}{\delta \mu} - 2\partial \left(\rho \frac{\delta}{\delta \rho} \right) - \partial \rho \frac{\delta}{\delta \rho} \\ &= -\partial^2 \left(\rho \frac{\delta}{\delta \mu} \right) - \frac{c}{12\pi} \left(\partial^3 \mu - \frac{1}{2} \partial^4 \rho \right) , \end{aligned} \quad (5.12)$$

and

$$\begin{aligned} \mathcal{C}_3 &\equiv (\bar{\partial} - \mu \partial - 3\partial \mu) \frac{\delta}{\delta \rho} - \frac{16\pi}{c} \frac{\delta}{\delta \mu} \partial \left(\rho \frac{\delta}{\delta \mu} \right) \\ &= (\rho \partial^2 + 2\partial \rho \partial) \frac{\delta}{\delta \rho} + \left(\partial^2 \mu - \frac{2}{3} \partial^3 \rho \right) \frac{\delta}{\delta \mu} - \frac{2}{3} \partial^3 \left(\rho \frac{\delta}{\delta \mu} \right) \\ &\quad - \frac{c}{24\pi} \left(\partial^4 \mu - \frac{2}{3} \partial^5 \rho \right) \end{aligned} \quad (5.13)$$

where $\frac{\delta}{\delta \mu}$ and $\frac{\delta}{\delta \rho}$ are supposed to act on appropriate functionals of μ and ρ . We will show that the latter are precisely the physical wave functionals in a μ/ρ -representation.

Let us try to reverse the reasoning outlined above for the Virasoro case. Starting from the Ward identities (5.12) and (5.13) we will get the formulas which are analogous to (5.7). Note that we have written these Ward identities such that all terms of first order in the derivatives ∂ and $\bar{\partial}$ are on the l.h.s. while the higher order derivatives are on the r.h.s. In the above derivation of (5.10) the higher derivative term $c\partial^3\mu$ only occurred due to regularization effects. A naive computation would have only produced the l.h.s. of (5.10). Now we will similarly neglect the right-hand sides of (5.12) and (5.13) and interpret the left-hand sides as the naive forms of the constraints in the μ/ρ -representation. Our goal is to find the corresponding constraints in the A -representation.

Introduce the basis of functionals

$$\chi_{\mu,\rho}[A] = \exp\left(-\frac{k}{2\pi} \int \mu \operatorname{tr} A^2 - \alpha \frac{k}{3\pi} \int \rho \operatorname{tr} A^3\right) \quad (5.14)$$

(where α is some numerical constant to be determined below) and “Fourier-transforms” of the wave functionals with respect to this basis

$$\Psi[\mu, \rho] = \int \mathcal{D}A \Psi[A] \chi_{\mu,\rho}[A] . \quad (5.15)$$

Consider first the constraint (5.12) without its r.h.s., i.e. $\mathcal{C}_2$ as acting on $\Psi[\mu, \rho]$:

$$\int \mathcal{D}A \Psi[A] \mathcal{C}_2 \chi_{\mu,\rho}[A] = 0 . \quad (5.16)$$

Now one uses the following identities valid when acting on $\chi_{\mu,\rho}[A]$

$$\begin{aligned} -\frac{\pi}{k} A^a \partial \frac{\delta}{\delta A^a} &= \operatorname{tr} A \partial(\mu A) + \alpha \operatorname{tr} A \partial(\rho A^2) \\ &= \partial \mu \operatorname{tr} A^2 + \frac{1}{2} \mu \partial \operatorname{tr} A^2 + \alpha \partial \rho \operatorname{tr} A^3 + \frac{2}{3} \alpha \rho \partial \operatorname{tr} A^3 \\ &= -\frac{\pi}{k} \left(2\partial \mu \frac{\delta}{\delta \mu} + \mu \partial \frac{\delta}{\delta \mu} + 3\partial \rho \frac{\delta}{\delta \rho} + 2\rho \partial \frac{\delta}{\delta \rho} \right) , \end{aligned} \quad (5.17)$$

as well as $A^a \bar{\partial} A^a = -\frac{\pi}{k} \bar{\partial} \frac{\delta}{\delta \mu}$. Integrating $\frac{\delta}{\delta A^a}$ by parts, (5.16) becomes

$$\int \mathcal{D}A \chi_{\mu,\rho}[A] A^a \left(\bar{\partial} A^a - \frac{\pi}{k} \partial \frac{\delta}{\delta A^a} \right) \Psi[A] = 0 \quad (5.18)$$

or upon “Fourier-transforming”

$$A^a \left(\bar{\partial} A^a - \frac{\pi}{k} \partial \frac{\delta}{\delta A^a} \right) \Psi[A] = 0 \quad (5.19)$$

which is the holomorphic part ($\bar{v} = 0$) of the constraint (5.7). Thus, as was to be

expected, the first constraint just translates the general coordinate invariance of the $2 + 1$ -dimensional field theory. This holds true not only for W_3 but for all W_n .

However, this is not the whole story. As we just remarked, (5.19) is the holomorphic part of the constraint (5.7) in the A -representation. It is natural to ask what the antiholomorphic part ($\sim \bar{v}$) corresponds to. In the case of the W_2 (Virasoro) case it simply gave rise to $\mu\bar{v}$ multiplying the same constraint as v , see equation (5.10). For W_3 however, this cannot be simply the case, due to the presence of ρ . Nevertheless, we do not expect any new constraints to appear. Indeed, starting from the full constraint (5.7) in the A -representation, and using similar manipulations to those above, as well as a simple identity for the d -symbols involved in $\text{tr } A^3 = d^{abc} A^a A^b A^c$, namely

$$d^{abc} d^{ade} A^b A^c A^d = -\frac{2}{3} A^a A^a A^e, \quad (5.20)$$

valid for $G = SU(3)$, one obtains

$$\int_{\Sigma} d^2 z ((v + \mu\bar{v})\mathcal{C}_2 + \rho\bar{v}\mathcal{C}_3) \Psi[\mu, \rho] = 0, \quad (5.21)$$

provided we choose the numerical constant

$$\alpha = (6k/c)^{\frac{1}{2}}. \quad (5.22)$$

Something rather surprising at first sight has happened. The constraint (5.7) that expresses the diffeomorphism invariance in the A -representation generates both constraints $\mathcal{C}_2$ and $\mathcal{C}_3$ once we go to the μ/ρ -representation. The constraints $\mathcal{C}_2$ and $\mathcal{C}_3$, however, constitute the full set of W_3 -Ward-identities, encoding the spin-2 *and* the spin-3 transformations.* Thus the full structure of W_3 is already contained in the $2 + 1$ -dimensional Chern-Simons theory and its diffeomorphism invariance. In a sense, it is hidden in the A -representation and made apparent when one goes to the μ/ρ -representation. Let us stress that in order to obtain the W_3 -constraints (5.21) we had also to make the specific choice of the Lie group $G = SU(3)$. Below we will briefly discuss the general case of W_n .

* Of course, our present discussion only produces the part of the Ward identities containing only one derivative ∂ or $\bar{\partial}$, assuming that the terms containing more than one derivative ∂ are correctly reproduced by regularization effects.

We have seen that the constraint $\mathcal{C}_3$ is already contained in the “quadratic” constraint (5.7). However, one might have thought that it is related to some cubic constraint. This is indeed also the case. When one considers the constraint $\mathcal{C}_3$ alone, proceeding through the analogous steps as above, one easily shows that it gives rise to the following cubic constraint in the A -representation

$$d^{abc} A^b A^c \left(\bar{\partial} A^a - \frac{\pi}{k} \partial \frac{\delta}{\delta A^a} \right) \Psi[A] = 0 , \quad (5.23)$$

Again, this can be interpreted as the holomorphic part of a new constraint in the A -representation

$$\int_{\Sigma} d^2 z \operatorname{tr} \left(w A^2 + \bar{w} \left(\frac{\pi}{k} \right)^2 \frac{\delta^2}{\delta A^2} \right) \left(\bar{\partial} A - \frac{\pi}{k} \partial \frac{\delta}{\delta A} \right) \Psi[A] = 0 . \quad (5.24)$$

where $w \equiv w^{zz}$ and $\bar{w} \equiv w^{\bar{z}\bar{z}}$ and $\operatorname{tr} A^2 \frac{\delta}{\delta A} \equiv d^{abc} A^a A^b \frac{\delta}{\delta A^c}$ etc. If we (naively) compute the “anti-holomorphic” part of this constraint ($\sim \bar{w}$) in the μ/ρ -representation we obtain[†]

$$\int_{\Sigma} d^2 z \bar{w} \left(\left(\mu^2 - \frac{8\pi}{c} \rho^2 \frac{\delta}{\delta \mu} \right) \mathcal{C}_3 + \frac{8\pi}{c} \rho \left(2\mu \frac{\delta}{\delta \mu} + 3\rho \frac{\delta}{\delta \rho} \right) \mathcal{C}_2 \right) \Psi[\mu, \rho] = 0 . \quad (5.25)$$

Thus we only find again some combination of the constraints $\mathcal{C}_2$ and $\mathcal{C}_3$.

It is straightforward to generalize the above discussion from W_3 to W_n . Equation (5.14) is replaced by

$$\chi_{\mu_2, \dots, \mu_n}[A] = \exp \left(-\frac{k}{\pi} \sum_{j=2}^n \frac{\alpha_j}{j} \int \mu_j \operatorname{tr} A^j \right) , \quad (5.26)$$

where the α_j are some numerical constants depending on k and c . Starting from the quadratic constraint (5.7) one proceeds through exactly the same steps as for W_3 in order to obtain (for $\bar{v} = 0$)

$$\begin{aligned} & \int d^2 z v \mathcal{C}_2 \Psi[\mu_2, \dots, \mu_n] \\ & \equiv \int d^2 z v \left((\bar{\partial} - 2\partial\mu_2 - \mu_2\partial) \frac{\delta}{\delta \mu_2} - \sum_{j=3}^n (j\partial\mu_j - \mu_j\partial) \frac{\delta}{\delta \mu_j} \right) \Psi[\mu_2, \dots, \mu_n] = 0 . \end{aligned} \quad (5.27)$$

This is precisely one of the Ward identities of W_n , cf. equation (3.45), provided one

[†] “Naive” means that we do not justify by any regularization why we drop $\delta(0)$ -type infinities that occur as $\frac{\delta}{\delta A(z)} A(z)$ or $\frac{\delta}{\delta \mu(z)} \mu(z)$ etc.

neglects all terms containing more than one derivative ∂ . One obtains the other Ward identities when considering non-zero $\bar{v}$. One has to choose the α_j appropriately and use several identities between the different d -tensors involved in $\text{tr } A^j$. At this point one has to use that the gauge group is $G = SU(n)$. We do not present a general proof but it is rather obvious that the generalization of (5.21) reads

$$\int_{\Sigma} d^2z \left((v + \mu_2 \bar{v}) \mathcal{C}_2 + \bar{v} \sum_{j=3}^n \mu_j \mathcal{C}_j \right) \Psi[\mu_2, \dots, \mu_n] = 0 , \quad (5.28)$$

where $\mathcal{C}_2, \dots, \mathcal{C}_n$ are the $n - 1$ Ward identities of W_n (again with all terms with more than one derivative ∂ dropped).

Remember that the diffeomorphism invariance led to the constraint (5.7) and that the constraint corresponding to the Kac-Moody symmetry (i.e. W_1) is nothing but the gauge transformation law

$$\int_{\Sigma} d^2z \epsilon^a \left(\bar{\partial} A^a - \frac{\pi}{k} \partial \frac{\delta}{\delta A^a} - \frac{\pi}{k} f^{abc} A^b \frac{\delta}{\delta A^c} \right) \Psi[A] = 0 . \quad (5.29)$$

Thus, for W_n , it is plausible that we have (using a classical notation $\bar{A} = \frac{\pi}{k} \frac{\delta}{\delta A}$) the following sequence of symmetries that also act as constraint operators annihilating physical wave functionals $\Psi[A]$

$$\begin{aligned} I_1 &= \int_{\Sigma} d^2z \text{tr } \epsilon (\bar{\partial} A - \partial \bar{A} + [\bar{A}, A]) \\ I_2 &= \int_{\Sigma} d^2z \text{tr } (vA + \bar{v}\bar{A}) (\bar{\partial} A - \partial \bar{A}) \\ I_3 &= \int_{\Sigma} d^2z \text{tr } (wA^2 + \bar{w}\bar{A}^2) (\bar{\partial} A - \partial \bar{A}) \\ &\vdots \\ I_n &= \int_{\Sigma} d^2z \text{tr } (wA^{n-1} + \bar{w}\bar{A}^{n-1}) (\bar{\partial} A - \partial \bar{A}) . \end{aligned} \quad (5.30)$$

Thus, the W -symmetries could be described in terms of these additional symmetries of a $2 + 1$ -dimensional field theory, although we have seen that the constraint I_2 in fact contains already *all* W_n -Ward identities. The transformation law for A under

the additional symmetries I_k can be easily derived by acting with I_k on A using the standard CS commutators (5.2). Then

$$\begin{aligned}
 \delta_1 A &= \frac{k}{\pi} [I_1, A] = \partial \epsilon + [A, \epsilon] \\
 \delta_2 A &= \frac{k}{\pi} [I_2, A] = \partial(vA + \bar{v}\bar{A}) + \bar{v}(\bar{\partial}A - \partial\bar{A}) \\
 \delta_3 A &= \frac{k}{\pi} [I_3, A^a] = d^{abc} \left(\partial \left(wA^b A^c + \bar{w}\bar{A}^b \bar{A}^c \right) + 2\bar{w}\bar{A}^b (\bar{\partial}A^c - \partial\bar{A}^c) \right) \\
 &\vdots
 \end{aligned} \tag{5.31}$$

It is an open question which field theoretic action obeys all these additional symmetries. The flag construction of the last section might suggest considering an induced CS theory in some n -dimensional space.

APPENDIX

We sketch the proof for general n that deformations (4.32) of the flag configurations with only ϵ_1 and ϵ_2 non-zero correspond to spin-2 and spin-3 transformations $\delta_{\epsilon_1}^{(2)}$ and $\delta_{\epsilon_2}^{(3)}$. Equation (4.32) implies

$$\delta F_i = \partial F_i \left(\hat{\epsilon}_1 + (i-1)\partial\hat{\epsilon}_2 + \sum_{j=1}^{i-1} (\phi_j + \phi_{j+1})\hat{\epsilon}_2 \right) \quad (\text{A1})$$

where we introduced $\hat{\epsilon}_1 = \partial F_1 \epsilon_1$, $\hat{\epsilon}_2 = \partial F_1 \partial F_2 \epsilon_2$ and $\phi_j = \partial^2 F_j / \partial F_j$ (cf. (4.7)). Inserting (A1) into (4.37) yields $\delta\psi_0$. From equations (4.25) and (4.30) one has

$$\delta f^{(0)} = \frac{\delta\psi_0}{\psi_0} f^{(0)} + \epsilon_1 \psi_0 \hat{f}_1 + \epsilon_2 \psi_0 \hat{f}_2 . \quad (\text{A2})$$

Next, one uses (4.25) to re-express $\hat{f}_1$ and $\hat{f}_2$ in terms of $f^{(2)}$, $f^{(1)}$ and $f^{(0)}$. This involves first and second derivatives of ψ_0 and $G_1 = F_1$:

$$\begin{aligned} \delta f^{(0)} = & \left[\frac{\delta\psi_0}{\psi_0} - \frac{\partial\psi_0}{\psi_0} \hat{\epsilon}_1 + \left(\frac{\partial\psi_0}{\psi_0} \phi_1 + 2 \left(\frac{\partial\psi_0}{\psi_0} \right)^2 - \frac{\partial^2\psi_0}{\psi_0} \right) \hat{\epsilon}_2 \right] f^{(0)} \\ & + \left[\hat{\epsilon}_1 - \left(2 \frac{\partial\psi_0}{\psi_0} + \phi_1 \right) \hat{\epsilon}_2 \right] f^{(1)} + \hat{\epsilon}_2 f^{(2)} . \end{aligned} \quad (\text{A3})$$

Projecting $f^{(i)} \rightarrow \partial^i f$, using (3.54), (3.55) for $\delta^{(2)}$ and $\delta^{(3)}$ and eq. (4.5) for u_2 , it is not very difficult to show that

$$\delta f = \left(\delta_{\tilde{\epsilon}_1}^{(2)} + \delta_{\tilde{\epsilon}_2}^{(3)} \right) f \quad (\text{A4})$$

with ^{*}

$$\tilde{\epsilon}_2 = \hat{\epsilon}_2 , \quad \tilde{\epsilon}_1 = \hat{\epsilon}_1 - \left(2 \frac{\partial\psi_0}{\psi_0} + \phi_1 \right) \hat{\epsilon}_2 + (n-2)\partial\hat{\epsilon}_2 . \quad (\text{A5})$$

Of course, when $\psi_0 = 1$, $F_i = z$, $\phi_i = 0$, one has $\hat{\epsilon}_i = \epsilon_i$ and this equation reduces to eq. (4.41).

^{*} The relation between $\tilde{\epsilon}_1$ and $\hat{\epsilon}_1$, $\tilde{\epsilon}_2$ is “field-dependent”. Note however that $\frac{\partial\psi_0}{\psi_0}$ is expressible entirely in terms of the ϕ_i which in turn can be expressed through the u_k . Thus this relation is well-defined at any point of our reduced symplectic manifold.

ACKNOWLEDGEMENTS

It is a pleasure to thank A. Gerasimov and A. Marshakov for discussions about their work [14]. I.K. thanks the CERN Theory Division for the kind hospitality, and A.B. is particularly grateful to R. Stora for many illuminating discussions.

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