q-Deformations of Virasoro Algebra and Conformal Dimensions

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ABSTRACT

We describe the q -deformations of the realisations of conformal algebra depending on conformal dimension parameter Δ . The particular role of the conformal dimensions $\Delta = 0, 1/2, 1$ is pointed out. The q -deformed central extension terms, describing q -deformation of Virasoro algebra, are derived. In the limit $q \rightarrow 1$ one obtains the usual central term. The transformation properties of q -deformed energy - momentum tensor ($\Delta = 2$) consistent with the q -deformed central extension term are described.

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1. Introduction

Recently a great deal of attention has been paid to the study of quantum groups and algebras [1-5]. These new mathematical objects were applied to some physical models, e.g. in the conformal field theories [6,7] as well as in the vertex and spin models (see e.g. [8,9] and in the quantum optics [10]). The q -deformation of creation and annihilation operators (see e.g. [11-15]) were also applied in constructing different q -(super)algebras and new statistics with violation of Pauli principle [16,17].

The Virasoro algebra as $d = 2$ conformal algebra has played prominent role in physics in the last twenty years, while the study of its q -deformation started very recently. In particular in [18] it was shown how it is possible to construct the q -deformed conformal algebra [19] in terms of q -deformed oscillator. In such a way, one obtains the q -deformation of conformal algebra with conformal dimension $\Delta = 0$. Next in [20] using oscillator realisation, invariant under $q \rightarrow q^{-1}$ and the enforcement of the Jacobi identity a central term for the Virasoro algebra has been obtained. However, it appeared that such a realisation is quite restrictive and it reproduces only a trivial part of the usual central term in the limit $q \rightarrow 1$.

Our starting point here is to consider the q -deformations of the differential realizations of conformal algebra depending on arbitrary conformal dimension Δ . For $q = 1$ all these realization have the same well-known algebraic structure

$$[l_m, l_n] = (m - n)l_{n+m}, \quad (1.1)$$

while, for $q \neq 1$ are characterized for different Δ by different algebraic relations. If we write these algebraic relations in an abstract way we obtain a two-parameter class of conformal dimension - dependent extensions of centerless Virasoro algebras. The results which were given earlier [18-21] can be obtained by the choice of conformal dimension $\Delta = 0$.

In Sect.2 we discuss the algebraic structure of the q -deformed realisations with arbitrary Δ . It appears that for generic $\Delta \neq 0, 1$ the Δ -dependent q -deformation of the commutator $[l_m, l_n]$ is uniquely defined by the requirement of closed algebraic structure. In Sect. 3 the modification of Jacobi identities for arbitrary q -deformed commutators is further discussed. Instead of using algebraic Jacobi identity [20], valid for arbitrary numbers q_1, q_2, q_3 (see also [19], where the case $q_1 = q_3$ was considered)

$$\begin{aligned} & [[A, [B, C]]_{(q_1, q_1^{-1})}]_{(q_1/q_2, q_2/q_1)} \\ & + [[B, [C, A]]_{(q_2, q_2^{-1})}]_{(q_1/q_3, q_3/q_1)} \\ & + [[C, [A, B]]_{(q_3, q_3^{-1})}]_{(q_2/q_1, q_1/q_2)} = 0, \end{aligned} \quad (1.2)$$

where $[A, B]_{(q,q)} = xAB - yBA$, we shall consider the q -deformed Jacobi identities which are equivalent to the Jacobi relations for q -deformed structure constants. In such a way we can obtain the q -deformed Jacobi identities for any value of generic Δ . In Sect. 4 we discuss the case $\Delta = 0, 1$. For $\Delta = 0, 1$ the closed algebraic structure does not determine uniquely the deformation of the commutator in (1.1), and in particular one can also keep undeformed commutators. Further we also consider the interesting case $\Delta = \frac{1}{2}$, the only generic case of Δ with undeformed

commutators. In Sect.5 the infinitesimal q -deformed conformal transformations of energy-momentum tensor, described by the q -deformed $\Delta = 2$ realizations of Virasoro algebra, are discussed. Such an approach provides an alternative method of obtaining the q -deformed central extension term. This technique could be applied to the case of extended super Virasoro algebras with odd q -derivatives describing the roots of q -deformed even derivative.

2. Deformation of conformal algebra realisations with arbitrary conformal dimension

Let $A_\Delta(z)$ be an arbitrary primary field with the conformal dimensions Δ . This means that $A_\Delta(z)$ has the following transformation property:

$$A_\Delta(z) \rightarrow (\phi'(z))^\Delta A_\Delta(\phi(z)), \quad (2.1)$$

where $\phi(z)$ is an arbitrary function of the conformal transformation, prime denotes the derivative and Δ is an arbitrary number. The formula (2.1) in the infinitesimal form reads as

$$\phi(z) = z + \epsilon(z), \quad (2.2)$$

$$\delta(A_\Delta(z)) = \epsilon^{1-\Delta} \partial \cdot \epsilon^\Delta A_\Delta(z). \quad (2.3)$$

If we take $\epsilon = z^{n+1}$, we obtain

$$\delta_n A_\Delta = l_n A_\Delta = (z^{n+1} \partial + \Delta(n+1)z^n) A_\Delta = (z \partial + \Delta(n+1) - n) z^n A_\Delta \quad (2.4)$$

and the generators l_n fulfill the usual commutation relations (1.1) for the conformal algebra.

Let us now construct the q -analog of the formula (2.4). For this purpose let us define it by

$$\delta^q(A_\Delta(z)) = (\epsilon(z))^{1-\Delta} D_q(\epsilon(z)^\Delta A_\Delta(z)), \quad (2.5)$$

where q -deformed derivative is defined as usual:

$$D_q = \frac{1}{z} \frac{q^{z\partial} - q^{-z\partial}}{q - q^{-1}} = \frac{1}{z} [z\partial], \quad (2.6)$$

where $[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$ and q is the deformation parameter. Introducing $\epsilon = z^{n+1}$ into (2.5) we obtain

$$\delta_n^q A_\Delta(z) = L_n^{(\Delta)} A_\Delta(z) = z^{n-\Delta(n+1)} [z\partial] z^{\Delta(n+1)} A_\Delta(z) = [z\partial + \Delta(n+1) - n] z^n A_\Delta(z). \quad (2.7)$$

This formula has also been obtained in [22]. As a result of (2.7) we obtain for $\Delta \neq 0, 1$ the following closed algebraic structure:

$$\begin{aligned} & \left\{ L_n^{(\Delta)}, L_m^{(\Delta)} \right\}_{z_\Delta, y_\Delta} = x_\Delta L_n^{(\Delta)} L_m^{(\Delta)} - y_\Delta L_m^{(\Delta)} L_n^{(\Delta)} \\ & = \frac{1}{q - q^{-1}} \left\{ q^{N_\Delta} (x_\Delta q^{-n} - y_\Delta q^{-m}) - q^{-N_\Delta} (x_\Delta q^n - y_\Delta q^m) \right\} L_{n+m}^{(\Delta)}, \end{aligned} \quad (2.8)$$

where $N_\Delta = z\partial + \Delta$, i.e. $L_0^{(\Delta)} = [N_\Delta]$ and

$$x_\Delta = \frac{[n(\Delta-1)][\Delta m]}{[n][m]}, \quad y_\Delta = \frac{[m(\Delta-1)][\Delta n]}{[n][m]}. \quad (2.9)$$

It is interesting to notice that the operator-valued structure constants in (2.8) depend on the conformal dimensions Δ and the scaling operator $L_0^{(\Delta)}$.

We see that the form of q-deformation of conformal algebra (1.1) depends on the value Δ of the conformal dimension. On the other hand, from the formula (2.7) we can deduce that

$$L_n^{(\Delta')} = z^{-(\Delta'-\Delta)(n+1)} L_n^{(\Delta)} z^{(\Delta'-\Delta)(n+1)}. \quad (2.10)$$

So the generators (2.7) are equivalent to each other for all conformal dimensions Δ . Unfortunately, only for the conventional commutator the algebra is invariant under the mapping (2.10) and for arbitrary two dimensions Δ, Δ' the transformation (2.10) relates two different q-deformed algebras.

In the algebra (2.8) the deformation of the commutator is obtained by introducing the parameters x_Δ, y_Δ , but the operator-valued structure constants depend on N_Δ .

Notice that the formula (2.8) can be cast also in the following form

$$[L_n^{(\Delta)}, L_m^{(\Delta)}]_* = [m-n] L_{n+m}^{(\Delta)}, \quad (2.11)$$

where

$$[L_n^{(\Delta)}, L_m^{(\Delta)}]_* \equiv [L_n^{(\Delta)}, L_m^{(\Delta)}]_{(S_{n,m}^\Delta, S_{m,n}^\Delta)} \equiv S_{n,m}^\Delta L_n^{(\Delta)} L_m^{(\Delta)} - S_{m,n}^\Delta L_m^{(\Delta)} L_n^{(\Delta)} \quad (2.12)$$

with

$$S_{m,n}^\Delta = \frac{q^{m-n} - q^{n-m}}{q^{N_\Delta}(\chi(n, m)q^{-n} - q^{-n}) - q^{-N_\Delta}(\chi(n, m)q^n - q^m)} \quad (2.13)$$

and

$$\chi(n, m) = \frac{[n(\Delta-1)][\Delta m]}{[m(\Delta-1)][\Delta n]}. \quad (2.14)$$

Notice that $S_{m,n}^\Delta$ are in general operators. In such a way we have chosen the deformation of the algebra (1.1) which leads for arbitrary Δ to the conventional r.h.s of q-deformed conformal algebra [18,19].

Using the freedom in the definition of deformed commutator we can make a next step and transform (2.12) to the form

$$[L_n^\Delta, L_m^{(\Delta)}]_{(\tilde{S}_{n,m}^\Delta, \tilde{S}_{m,n}^\Delta)} \equiv \tilde{S}_{n,m}^\Delta L_n^{(\Delta)} L_m^{(\Delta)} - \tilde{S}_{m,n}^\Delta L_m^{(\Delta)} L_n^{(\Delta)} = (m-n) L_{m+n}^{(\Delta)}, \quad (2.15)$$

where $\tilde{S}_{n,m}^\Delta = \frac{S_{n,m}^\Delta(m-n)}{[m-n]}$. As a result we obtain usual conformal algebra, although with the suitably deformed commutator deformed in a complicated way.

3. q-Deformed Jacobi identities

The presence of numerical structure constants even if $S_{m,n}^\Delta$ are operators, permits us to generalize the Jacobi identity to the q-deformed case.

Taking double deformed commutators we see easily that our algebra (2.11) implies the following relation

$$[L_k^{(\Delta)}, [L_n^{(\Delta)}, L_m^{(\Delta)}]_*]_* = [m-n][n+m-k] L_{n+m+k}^{(\Delta)}. \quad (3.1)$$

It is easy to check, using the identity

$$(q^n + q^{-n})[n-m-k][m-k] + \text{cycl. perm.} = 0, \quad (3.2)$$

that the double commutators (3.1) satisfy the following q-deformed Jacobi relation

$$(q^k + q^{-k})[L_k^{(\Delta)}, [L_n^{(\Delta)}, L_m^{(\Delta)}]_*]_* + \text{cycl. perm.} = 0. \quad (3.3)$$

It should be stressed that the relations (3.3) (as well as relations (3.1)) are the consequence of the particular choice of structure constants in q-deformed conformal algebra (2.11). The main merit of the relation (3.3) is found in searching the deformation of the central term. Indeed, let us write down

$$[\hat{L}_n^{(\Delta)}, \hat{L}_m^{(\Delta)}]_* = [m-n] \hat{L}_{n+m}^{(\Delta)} + \hat{c}_{n,m}(q) \quad (3.4)$$

and assume that $\hat{c}_{n,m}(q) = \delta_{n+m,0} \hat{c}(n, q)$, i.e. $\hat{c}$ behaves in the q-deformed commutators as $\hat{L}_0^{(\Delta)}$. Further we assume that

$$[\hat{L}_n^{(\Delta)}, \hat{c}(k, q)]_* \equiv S_{n0}(q) \hat{L}_n^{(\Delta)} \hat{c}(k, q) - S_{0n}(q) \hat{c}(k, q) \hat{L}_n^{(\Delta)} = 0. \quad (3.5)$$

In particular, if $S_{n0} = q^n$ and $S_{0n} = q^{-n}$ corresponding to $\Delta = 0$ (see Sect. 4), the operator part of the solution of (3.5) does not depend on k , i.e. one can write

$$\hat{c}(k, q) = c(k, q) \cdot q^{2N_\Delta}, \quad (3.6)$$

where $[N_\Delta, L_n^{(\Delta)}] = n L_n^{(\Delta)}$. In the general case the operator N_Δ can be regarded as an additional generator. However, in our differential realisation (2.7) (equivalent to oscillator realisation with the invariance $q \rightarrow q^{-1}$, it is a function of $L_0^{(\Delta)}$, $[N_\Delta] = L_0^{(\Delta)}$).

For such a case the substitution of (3.4)-(3.6) into (3.3) gives

$$(q^m + q^{-m})[m-n]c(k, q)\delta_{k+m+n,0} + \text{cycl. perm.} = 0. \quad (3.7)$$

The solution of (3.7) can be written down in the following compact form

$$c(n, q) = c(q) \frac{[n-1][n][n+1]}{q^n + q^{-n}}, \quad (3.8)$$

where $c(q)$ is an arbitrary function of q .

The q -deformed Jacobi identity (3.3) can be understood as a special case of the following general scheme. Let us write

$$[L_n, L_m]_* = f_{n,m} L_{n+m}, \quad (f_{n,m} = -f_{m,n}), \quad (3.9)$$

and introduce new structure constants g_{kmn} satisfying the relations

$$g_{kmn} f_{k,n+m} f_{n,m} + \text{cycl. perm.} = 0. \quad (3.10)$$

For the algebra (3.9) the q -deformed Jacobi identity due to (3.10) takes the form

$$g_{kmn} [[L_k, [L_m, L_n]]_* + \text{cycl. perm.}] = 0. \quad (3.11)$$

It should be stressed that for a given choice of g_{kmn} one obtains the class of associative algebras (3.9) with q -deformed Lie bracket given by the $*$ -commutator $[,]_*$. If $g_{kmn} = 1$, one obtains as a solution the conventional relation for the structure constants, $f_{n,m} = m - n$; if $g_{kmn}(q) = q^k + q^{-k}$, one obtains the q -deformed solution $f_{n,m}(q) = [m - n]$.

4. q - Deformation of conformal algebra realisations with $\Delta = 0, 1, 1/2$ and central extension

Let us write the realizations (2.7) for $\Delta = 0, 1$,

$$L_n^{(0)} = [z\partial - n]z^n, \quad (4.1)$$

$$L_n^{(1)} = [z\partial + 1]z^n, \quad (4.2)$$

from which we have

$$L_n^{(0)} L_m^{(0)} = [z\partial - n] L_{n+m}^{(0)}, \quad (4.3)$$

$$L_n^{(1)} L_m^{(1)} = [z\partial + 1 - n] L_{n+m}^{(1)}. \quad (4.4)$$

From the last formulas it follows directly that

$$[L_n^{(\Delta)}, L_m^{(\Delta)}]_{x,y} = \frac{1}{q - q^{-1}} \left\{ q^{N_\Delta} (xq^{-n} - yq^{-m}) - q^{-N_\Delta} (xq^n - yq^m) \right\} L_{n+m}^{(\Delta)}, \quad (\Delta = 0, 1), \quad (4.5)$$

where $N_\Delta = z\partial + \Delta$ i.e. $L_0^{(\Delta)} = [N_\Delta]$ and x, y are arbitrary numbers. In particular, choosing $xq^{-n} - yq^{-m} = 0$ we obtain

$$[L_n^{(\Delta)}, L_m^{(\Delta)}]_{(1,q^{m-n})} = [m - n] q^{-N_\Delta + m} L_{n+m}^{(\Delta)}, \quad (\Delta = 0, 1). \quad (4.6)$$

If we redefine

$$L_n = q^{-N} L_n^{(\Delta)}, \quad (\Delta = 0, 1), \quad (4.7)$$

we obtain

$$[L_n, L_m]_{(q^{m-n}, q^{n-m})} = [m - n] L_{n+m}, \quad (4.8)$$

i.e. the q -deformed conformal algebra considered in [18-21] and abstracted from the oscillator realisation. The advantage of the algebra (4.8) is that the parameters $S_{n,m}$ deforming the commutator as well as the q -deformed structure constants are numerical. In such a case the derivation of the central extension term in previous section give the formula

$$[\tilde{L}_n, \tilde{L}_m]_{(q^{m-n}, q^{n-m})} = [m - n] \tilde{L}_{n+m} + c(n, q) q^{2N} \delta_{n+m,0}, \quad (4.9)$$

where $c(n, q)$ is given by (3.8). It is interesting to note that by introducing

$$\hat{\tilde{L}}_n = q^{-N} \tilde{L}_n = q^{-2N} \tilde{L}_n^{(\Delta)}, \quad (4.10)$$

the relation (4.9) can be rewritten as follows:

$$[\hat{\tilde{L}}_n, \hat{\tilde{L}}_m]_{(q^{m-n}, q^{n-m})} = [m - n] q^{-N+n+m} \hat{\tilde{L}}_{n+m} + c(n, q) \delta_{n+m,0}, \quad (4.11)$$

i.e. an algebra with numerical central term but with operator-valued structure constants.

Let us go back, however, to the general formula (4.5). As x and y can take an arbitrary value, an interesting case is $x = y = 1$. For this, formula (4.5) reduces to

$$[L_n^{(\Delta)}, L_m^{(\Delta)}] = \left[\frac{m - n}{2} \right] \left\{ q^{N_\Delta - \frac{n+m}{2}} + q^{-N_\Delta + \frac{n+m}{2}} \right\} L_{n+m}^{(\Delta)}, \quad (\Delta = 0, 1). \quad (4.12)$$

It is interesting to note that if $\Delta = \frac{1}{2}$ one should choose

$$x_{\frac{1}{2}} = y_{\frac{1}{2}} = - \left(q^{\frac{n}{2}} + q^{-\frac{n}{2}} \right) \left(q^{\frac{m}{2}} + q^{-\frac{m}{2}} \right) \quad (4.13)$$

and one obtains

$$\left[L_n^{(\frac{1}{2})}, L_m^{(\frac{1}{2})} \right] = \left[\frac{m - n}{2} \right] \left\{ q^{N_{\frac{1}{2}} - \frac{n+m}{2}} + q^{-N_{\frac{1}{2}} + \frac{n+m}{2}} \right\} L_{n+m}^{(\frac{1}{2})}. \quad (4.14)$$

The relations (4.12) and (4.14) were also written down recently in [22,24].

Let us point out that the realisations (4.1)-(4.2) for the algebra (4.12) obviously satisfy the usual Jacobi identity:

$$[L_k^{(\Delta)}, [L_n^{(\Delta)}, L_m^{(\Delta)}]] + \text{cycl.perm.} = 0, \quad (4.15)$$

If we wish to find the central terms using the Jacobi identity (4.15) due to the operator form of the structure constants in (3.7), we need to assume the following two commutation relations ($\Delta = 0, 1$):

$$[\tilde{L}_k^{(\Delta)}, \tilde{L}_m^{(\Delta)}] = \left[\frac{m - n}{2} \right] \left\{ q^{N_\Delta - \frac{k+m}{2}} + q^{-N_\Delta + \frac{k+m}{2}} \right\} \tilde{L}_{k+m}^{(\Delta)} + c(k, m; q), \quad (4.16)$$

$$\begin{aligned}
& [\tilde{L}_k^{(\Delta)}, \tilde{L}_m^{(\Delta)}]_{q^{-\frac{\Delta}{2}}, q^{\frac{\Delta}{2}}} \\
&= \frac{1}{q - q^{-1}} \left\{ q^{N_\Delta} \left(q^{-\frac{\Delta}{2}k} - q^{\frac{\Delta}{2}-m} \right) - q^{-N_\Delta} \left(q^{\frac{\Delta}{2}} - q^{\frac{\Delta}{2}+m} \right) \right\} \tilde{L}_{n+m}^{(\Delta)} \\
&+ \frac{1}{2} \left(q^{\frac{\Delta}{2}} + q^{-\frac{\Delta}{2}} \right) \tilde{c}(k, m; q).
\end{aligned} \tag{4.17}$$

Using (4.16), (4.17) we obtain from the Jacobi identity (4.15) the following equation for $c(k, m; q)$:

$$\left(q^{\frac{\Delta}{2}} + q^{-\frac{\Delta}{2}} \right) \left[\frac{m-n}{2} \right] c(k, m+n; q) + \text{cycl. perm.} = 0 \tag{4.18}$$

and an identical to (3.13) equation for $\tilde{c}$. The solutions of these equations are

$$c(k, m; q) = \tilde{c}(k, m; q) = c \frac{\left[\frac{k}{2} \right] \left[\frac{m-1}{2} \right] \left[\frac{k+m-1}{2} \right]}{\left(q^{\frac{\Delta}{2}} + q^{\frac{\Delta}{2}} \right)} \delta_{m+k,0}, \tag{4.19}$$

where c is an arbitrary function of q .

Because in our considerations we use the commutations relations (4.16) - (4.17) the product of two generators has the form

$$\tilde{L}_m^{(\Delta)} \tilde{L}_k^{(\Delta)} = [N_\Delta - m] \tilde{L}_{m+k}^{(\Delta)} - \frac{c(k, m; q)}{2}. \tag{4.20}$$

Here we should like to draw attention to the fact that the product of two generators as given by (4.20) implies that the generators of the algebra do not possess the associativity property, due to the presence of the central term (see also [23]).

5. q -Deformed $\Delta = 2$ Virasoro realisation and q -deformed energy-momentum tensor

Let us consider the algebra (2.8) for $\Delta = 2$, described by the generators

$$L_n^{(2)} = z^{-(n+1)} D_q z^{2(n+1)}, \tag{5.1}$$

which reads as follows

$$[L_n^{(2)}, L_m^{(2)}]_{(q^m + q^{-m}, q^n + q^{-n})} = [m - n] (q^{N_2} + q^{-N_2}) \cdot L_{n+m}^{(2)}. \tag{5.2}$$

It is well known [27, 28] that the energy-momentum tensor $T(z)$ has the conformal dimension two but it does not transform as a primary field (see (2.1)). Indeed if $q = 1$ (undeformed case) it transforms as follows:

$$T(z) \rightarrow (\phi'(z))^2 T^\phi(z) + c \left(\frac{\phi'''(z)}{\phi'(z)} - \frac{3}{2} \left(\frac{\phi''(z)}{\phi'(z)} \right)^2 \right), \tag{5.3}$$

where c is the central charge. In the infinitesimal form we have

$$\delta_\epsilon T(z) = (\epsilon(z) \partial + 2\epsilon'(z)) T(z) + c \epsilon'''(z). \tag{5.4}$$

After the substitution $\epsilon(z) = z^{n+1}$, formula (5.4) reduces to

$$\delta_n T(z) = L_n T(z) + c g(n) z^{n-2}, \tag{5.5}$$

where L_n is given by (2.4) with $\Delta = 2$ and $g(n) = (n-1)n(n+1)$.

If the central term is not known, its value can be obtained if we assume that the variations (5.5) satisfy the algebra (1.1), i.e.

$$[\delta_m, \delta_n] T(z) = (m-n) \delta_{m+n} T(z). \tag{5.6}$$

Using (5.5), we obtain

$$\begin{aligned}
& [L_n, L_m] T(z) + c (L_n g(m) z^{m-2} - L_m g(n) z^{n-2}) \\
&= (m-n) (L_{n+m} T(z) + c g(n+m) z^{n+m-2}).
\end{aligned} \tag{5.7}$$

Eq. (1.1) and the explicit realization of L_n give the equation for the function $g(n)$:

$$g(m)(2n+m) - g(n)(2m+n) = (m-n)g(n+m), \tag{5.8}$$

with the general solution

$$g(m) = c_1 m^3 - c_2 m, \tag{5.9}$$

where c_1 and c_2 are arbitrary constants. Using the redefinition of L_n , we can fix $c_1 = c_2 = 1$ and we arrive to the known solution $g(n) = (n+1)(n-1)$.

Now we would like to apply this approach to the q -deformed algebra (5.2). The q -analogue of the formula (4.5) is

$$\delta_n^q T_q(z) = L_n^{(2)} T_q(z) + c(q, n) z^{n-2} \tag{5.10}$$

where $T_q(z)$ denotes the q -deformed energy-momentum tensor and $L_n^{(2)}$ is defined by (5.1).

Now we would like to consider $c(q, n)$ as the central extension term for the q -deformed conformal algebra. Taking into account the q -commutator (5.2) we rewrite the consistency condition (5.6) in the deformed form as follows:

$$[\delta_m^q, \delta_n^q]_{(q^m + q^{-m}, q^n + q^{-n})} T_q(z) = [m-n] \left\{ q^{N_2} + q^{-N_2} \right\} \delta_{m+n}^q T_q(z). \tag{5.11}$$

Substituting (4.10) into (4.11) and using (4.2), we obtain

$$\begin{aligned}
& (q^m + q^{-m}) [2n+m] c(q, m) - (q^n + q^{-n}) [2m+n] c(q, n) \\
&= [m-n] (q^{n+m} + q^{-n-m}) c(q, n+m).
\end{aligned} \tag{5.12}$$

Let us consider in some detail the derivation of the solution of (5.12). Substituting into (5.12) $m = 0$ and $m = -n$ we conclude immediately that $c(q, 0) = 0$

and $c(q, n) = -c(q, -n)$. Then similarly as in the nondeformed case we can put $c(q, 1) = 0$. Indeed we can use the redefinition of $T_q(z)$ to redefine $c(q, 1)$. For this let us notice that shifting $T_q(z)$ as

$$\tilde{T}_q(z) = T_q(z) + \frac{\alpha}{[2]} z^{-2}, \quad (5.13)$$

where α is an arbitrary constant function of q , the function $c(q, n)$ changes as

$$\tilde{c}(q, n) = c(q, n) + \alpha \frac{[2n]}{[2]} \quad (5.14)$$

and hence we can put $\tilde{c}(q, 1) = 0$ by choosing $\alpha = -c(q, 1)$. Next, putting $n = m + 1$, we obtain from (5.12)

$$(q^m + q^{-m}) [m - 2] \tilde{c}(q, m) = (q^{m-1} + q^{-m+1}) [m - 1] \tilde{c}(q, m - 1). \quad (5.15)$$

The equation (5.15) has as its solution the q -deformed central extension term given by (3.8).

6. Final remarks

Our aim in this work has been to present different derivations of the q -extension of the n^3 central term in the Virasoro algebra. It appears that not all the usual methods of obtaining the Virasoro central term have a q -analogue. Let us mention, in particular, that all the cohomological derivations (see, e.g., [29]) would need the q -extension of cohomology theory which leads to noncommutative geometry (see, e.g., [4]) and is an interesting problem in itself. The derivation based on the oscillator realisation of Virasoro algebra in the string theory as bilinears in terms of infinite number of oscillator modes, also has not been succeeded yet because of the complicated character of coupling among different q -oscillator modes [13]. The only method which we have found working for the q -analogue of Virasoro algebra is to use the q -deformation of Jacobi identity, as well as the closure of q -deformed transformation laws. We would like to mention that the derivation of q -extension of central term in the Virasoro algebra from a q -deformed Jacobi identity was also proposed in [23], but unnecessarily with in the framework of nonassociative algebra. We have also presented several realisation-dependent q -deformations of the conformal algebra as a step towards the final aim of conformal-dimension-independent q -deformation of the Virasoro algebra.

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