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TWO-DIMENSIONAL TOPOLOGICAL GRAVITY
AND INTERSECTION THEORY ON THE
MODULI SPACE OF HOLOMORPHIC BUNDLES

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ABSTRACT

We define a two-dimensional topological Yang-Mills theory for an arbitrary compact simple Lie group. This theory is defined in terms of intersection theory on the moduli space of flat connections on a two-dimensional surface and corresponds physically to a two-dimensional reduction and truncation of four-dimensional topological Yang-Mills theory. Two-dimensional topological Yang-Mills theory defines a topological matter system and may be naturally coupled to two-dimensional topological gravity. This topological Yang-Mills theory is also closely related to Chern-Simons gauge theory in 2+1 dimensions. We also discuss a relation between $SL(2, \mathbf{R})$ Chern-Simons theory and two-dimensional topological gravity.

In the last year significant progress has been made on understanding two-dimensional quantum gravity. The matrix model approach has led to a number of important results [1]-[9]. A quite different approach [10] studies two-dimensional quantum gravity in terms of a topological quantum field theory. In spite of the differences between these two approaches there are a number of arguments which indicate that they are in fact equivalent [10] [11] [12]. Two-dimensional topological gravity is also related to Liouville theory by the argument of [13]. Additional studies of matrix models and topological gravity may be found in [14] [15] [16].

One conclusion which may be drawn from the work of [10] [11] is that two-dimensional quantum gravity coupled to conformal field theory appears to be equivalent to two-dimensional topological gravity coupled to two-dimensional topological quantum field theory, i.e. topological gravity coupled to topological matter. In view of this equivalence it is important to study two-dimensional topological matter systems and their coupling to topological gravity. In this paper we will define a two-dimensional topological matter system associated to an arbitrary compact Lie group G . This theory may be described in terms of intersection theory on the moduli space of flat connections over a compact surface. It is the two-dimensional analogue of four-dimensional Donaldson theory [17] and corresponds physically to the topological quantum field theory obtained by dimensional reduction of the Lagrangian theory described in [18]. We will here describe this topological quantum field theory in a model independent formulation and discuss coupling the theory to two-dimensional quantum gravity. We also show that this theory is closely related to Chern-Simons theory in 2+1 dimensions.

Let Σ be a compact two-dimensional surface and let G be a compact simple Lie group. Fix a principal G -bundle $P \rightarrow \Sigma$. We let $\mathcal{A}$ denote the space of connections on P and $\mathcal{G}$ denote the group of gauge transformations. We shall define our topological quantum field theory in terms of intersection theory on the moduli space $\mathcal{R}_\Sigma$ of flat G -connections on Σ . If $\mathcal{F} = \{A \in \mathcal{A} : F_A = 0\}$ is the space of flat connections on Σ then the moduli space $\mathcal{R}_\Sigma$ is defined to be

$$\mathcal{R}_\Sigma = \mathcal{F}/\mathcal{G}. \tag{1}$$

It is a classical result in differential geometry that a flat G -connection on Σ is determined by a homomorphism $\phi : \pi_1(\Sigma) \rightarrow G$. The moduli space $\mathcal{R}_\Sigma$ may be identified with the space $\text{Hom}(\pi_1(\Sigma), G)/G$, where G acts on $\text{Hom}(\pi_1(\Sigma), G)$ by conjugation. We

note that the space $\mathcal{R}_\Sigma$ may be regarded as the two-dimensional analogue of the space of Yang-Mills instantons in four-dimensions which occupies a central position in Donaldson theory [17].

The definition of $\mathcal{R}_\Sigma$ involves only the topological structure of Σ . If, however, we choose a complex structure σ on Σ (i.e., we make Σ into a Riemann surface Σ_σ) then there is an alternative description of $\mathcal{R}_\Sigma$ in terms of holomorphic bundles over Σ_σ . It follows from the Narasimhan-Seshasdi theorem [19] (also see [20]) that $\mathcal{R}_\Sigma$ may be identified with the moduli space of semi-stable holomorphic $G_{\mathbb{C}}$ -bundles over Σ_σ . Thus $\mathcal{R}_\Sigma$ may be defined geometrically (by (1)) or holomorphically. The space $\mathcal{R}_\Sigma$ has a canonical symplectic structure determined by the topology of Σ [20]. Once a complex structure is chosen on Σ , $\mathcal{R}_\Sigma$ acquires a complex structure, compatible with its symplectic structure, which gives a Kähler structure on $\mathcal{R}_\Sigma$.

We will define natural cohomology classes on $\mathcal{R}_\Sigma$. These classes may be defined from either a geometric or holomorphic point of view corresponding to the geometric or holomorphic definitions of $\mathcal{R}_\Sigma$. The geometric definition of these cohomology classes involves a universal bundle with connection, which has been defined in four-dimensioned gauge theory in [21].

The gauge group $\mathcal{G}$ acts freely on $P \times \mathcal{A}$ by $(p, A) \mapsto (g(p), A^g)$, for $g \in \mathcal{G}$, and thus defines a principal $\mathcal{G}$ -bundle

$$\mathcal{G} \rightarrow P \times \mathcal{A} \rightarrow \mathcal{Q}, \quad (2)$$

where $\mathcal{Q} = (P \times \mathcal{A})/\mathcal{G}$. The group G also acts on $\mathcal{Q}$. This action is free if we restrict $\mathcal{A}$ to the space of irreducible connections, which we will still denote by $\mathcal{A}$. In this case we obtain a principal G -bundle.

$$G \rightarrow \mathcal{Q} \rightarrow \mathcal{Q}/G, \quad (3)$$

where $\mathcal{Q}/G = \Sigma \times \mathcal{A}/\mathcal{G}$. The G -bundle $\mathcal{Q} \rightarrow \Sigma \times \mathcal{A}/\mathcal{G}$ has a natural connection c , which depends upon the choice of a Riemannian metric on Σ [21]. The bundle $\mathcal{Q}$ with connection c is universal [21]. This means that if $[A] \in \mathcal{A}/\mathcal{G}$ is a gauge equivalence class of connections on P and we let $(\mathcal{Q}|_{[A]}, c|_{[A]})$ denote the restriction of $(\mathcal{Q}, c)$ to $\Sigma \times [A]$, then $c|_{[A]}$ is in the equivalence class of $[A]$.

Restricting the bundle $\mathcal{Q} \rightarrow \Sigma \times \mathcal{A}/\mathcal{G}$ to $\Sigma \times \mathcal{R}_\Sigma$ gives the universal bundle.

$$G \rightarrow \mathcal{P} \rightarrow \Sigma \times \mathcal{R}_\Sigma. \quad (4)$$

Let $x \in \Sigma$ be a point, the restriction of $\mathcal{P}$ to $\{x\} \times \mathcal{R}_\Sigma$ defines a bundle $\mathcal{P}_x$. Now consider the Chern classes $c_i(\mathcal{P}_x) \in H^{2i}(\mathcal{R}_\Sigma; \mathbf{R})$. Thus we obtain natural cohomology classes

$$\kappa_i = c_i(\mathcal{P}_x) \quad (5)$$

in $H^*(\mathcal{R}_\Sigma; \mathbf{R})$. We can generalize this by considering a r -dimensioned homology cycle γ in Σ . If $\mathcal{P}_\gamma$ is the restriction of $\mathcal{P}$ to $\{\gamma\} \times \mathcal{R}_\Sigma$ then by integrating $c_i(\mathcal{P}_\gamma)$ over γ we obtain

$$\kappa_{i,\gamma} = \int_\gamma c_i(\mathcal{P}_\gamma) \quad (6)$$

in $H^{2i-r}(\mathcal{R}_\Sigma; \mathbf{R})$.

The cohomology classes κ_i and $\kappa_{i,\gamma}$ may also be defined from a holomorphic point of view, corresponding to the identification of $\mathcal{R}_\Sigma$ as the moduli space of semi-stable holomorphic $G_{\mathbf{C}}$ -bundles over Σ_σ , as follows. There exists a universal bundle

$$\mathcal{V} \rightarrow \Sigma_\sigma \times \mathcal{R}_\Sigma \quad (7)$$

where $\mathcal{V}$ is a holomorphic $G_{\mathbf{C}}$ -bundle over $\Sigma_\sigma \times \mathcal{R}_\Sigma$ such that for any $\xi \in \mathcal{R}_\Sigma$ the restriction $\mathcal{V}_\xi$ of $\mathcal{V}$ to $\Sigma_\sigma \times \{\xi\}$ is in the isomorphism class represented by ξ [20]. Given a point $x \in \Sigma$, let $\mathcal{V}_x$ be the restriction of $\mathcal{V}$ to $\{x\} \times \mathcal{R}_\Sigma$. The classes κ_i may now be defined as the Chern classes of $\mathcal{V}_x$:

$$\kappa_i = c_i(\mathcal{V}_x) \in H^{2i}(\mathcal{R}_\Sigma; \mathbf{R}). \quad (8)$$

It is proved in [20] that the classes κ_i (together with certain more exotic classes) generate the cohomology ring of $\mathcal{R}_\Sigma$.

If we let $d_1, d_2, \dots, d_n$ be non-negative integers, then the expectation value of the classes κ_i can be defined as

$$\langle \mathcal{O}_{i_1}(x_1) \mathcal{O}_{i_2}(x_2) \cdots \mathcal{O}_{i_n}(x_n) \rangle = \int_{\mathcal{R}_\Sigma} \kappa_{i_1} \wedge \kappa_{i_2} \wedge \cdots \wedge \kappa_{i_n}. \quad (9)$$

The x_i are arbitrary distinct points in Σ and $\kappa_{i_\ell} = c_{i_\ell}(\mathcal{V}_{x_i})$. The dimension of $\mathcal{R}_\Sigma$ is $(2g - 2)\dim(G)$, so the expectation value (9) will vanish unless

$$\sum_{\ell=1}^n i_\ell = (g - 1)\dim(G). \quad (10)$$

The space $\mathcal{R}_\Sigma$ is compact so the integral (9) is finite. The expectation value (9) can also be interpreted as the intersection number of the homology classes in $\mathcal{R}_\Sigma$ Poincaré dual to the cohomology classes κ_i . Sometimes we will denote this intersection number by $\langle \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_n} \rangle$. We also note that as the classes κ_i are even dimensional the ordering of the terms in (9) is unimportant.

In general, there will be reducible flat connections in $\mathcal{R}_\Sigma$ which will lead to subtleties in the evaluation of the intersection members defined by (9), which we will not enter into here. If G is simply connected then any G -bundle $P \rightarrow \Sigma$ is trivial, and the space of connections on P will always contain the trivial flat connection, which is reducible. If, however, G is not simply connected then all the connections on a G -bundle $P \rightarrow \Sigma$ may be irreducible. For example, consider $G = SO(3)$. An $SO(3)$ -bundle $P \rightarrow \Sigma$ is classified topologically by the second Stiefel-Whitney class $w_2(P) \in H^2(\Sigma; \mathbf{Z}_2) = \mathbf{Z}_2$. If P is the $SO(3)$ -bundle over Σ with $w_2(P) \neq 0$ then all of the connections on P are irreducible and there are no problems associated with the evaluation of (9).

If γ is an r -dimensional homology cycle in Σ then the classes κ_i may be generalized to

$$\kappa_{i,\gamma} = \int_{\gamma} c_i(\mathcal{V}_\gamma) \in H^{2i-r}(\mathcal{R}_\Sigma; \mathbf{R}), \quad (11)$$

where $\mathcal{V}_\gamma$ is the restriction of $\mathcal{V}$ to $\{\gamma\} \times \mathcal{R}_\Sigma$. Given n homology cycles $\gamma_1, \gamma_2, \dots, \gamma_n$ of dimensions $r_1, r_2, \dots, r_n$ respectively we may define the expectation value of the classes κ_{i,γ_i} in exact analogy with (9).

As the cohomology classes $\kappa_i \in H^{2i}(\mathcal{R}_\Sigma; \mathbf{R})$ can be defined using only that $\mathcal{R}_\Sigma$ is the moduli space of flat connections on Σ , and as this characterization of $\mathcal{R}_\Sigma$ depends only on the topology of Σ , it is clear that the expectation values defined by (9) depend only on the topology of Σ . As the notation indicates, the intersection number defined by (9) may be realized as the correlation functions of a Lagrangian topological quantum field theory on Σ . We will now make a few brief comments on such a Lagrangian realization.

Let A be a gauge connection on Σ with curvature F and let φ be a Lie algebra valued zero-form on Σ which transforms in the adjoint representation. One of the simplest two-dimensional topological quantum field theories is defined by the action (see [22], for example)

$$S_{\text{top}} = \int_{\Sigma} \text{Tr}(F \wedge \varphi) \quad (12)$$

Witten has considered the following variant of this action [23] (also see [24]). In addition to A and φ we introduce a zero-form χ and one-form ψ on Σ , also in the adjoint representation of the gauge group G , with the fermionic symmetry

$$\begin{aligned} \delta\varphi &= 0, & \delta\chi &= \epsilon\varphi \\ \delta\psi_i &= 0, & \delta A_i &= i\epsilon\psi_i. \end{aligned} \quad (13)$$

Now consider the action :

$$\hat{S}_{\text{top}} = \int_{\Sigma} \text{Tr}(F \wedge \varphi - i\chi \wedge D\psi). \quad (14)$$

It is argued in [23] that (14) can be regarded as a dimensional reduction and truncation of the Lagrangian introduced in [18] to define four-dimensional topological Yang-Mills theory.

We can now consider the gauge invariant observables $\mathcal{O}_i(x) = \text{Tr } \varphi^{2i}(x)$. In the quantum field theory defined by (14), it follows from the fermionic symmetry (13), that the semiclassical approximation is exact. Thus the calculation of the general correlation function $\langle \mathcal{O}_{i_1}(x_1) \mathcal{O}_{i_2}(x_2) \dots \mathcal{O}_{i_n}(x_n) \rangle$ reduces exactly to a computation on the space of “instantons” $\mathcal{R}_\Sigma$, just as in the four-dimensional case discussed in [18]. In exact analogy with [18] the correlation functions of the operators $\mathcal{O}_i(x)$ may be written as integrals of differential

forms over the moduli space $\mathcal{R}_\Sigma$; the differential forms are the κ_i defined by (8). This explains the quantum field theoretic origin of the intersection numbers (9).

Let us now consider coupling the topological gauge theory defined above to topological gravity [10]. Let $\mathcal{M}_{g,n}$ be the moduli space of Riemann surfaces of genus g with n marked points. The compactification of $\mathcal{M}_{g,n}$ consisting of stable curves is denoted by $\bar{\mathcal{M}}_{g,n}$. Let Σ be a stable curve with n marked points $x_1, x_2, \dots, x_n$. For each x_i the cotangent space $T_{x_i}^* \Sigma$ is a complex line. As Σ varies over $\bar{\mathcal{M}}_{g,n}$ these complex lines vary holomorphically to give n holomorphic line bundles $\mathcal{L}_{(i)} \rightarrow \bar{\mathcal{M}}_{g,n}$. Let $c_1(\mathcal{L}_{(i)}) \in H^2(\bar{\mathcal{M}}_{g,n}; \mathbf{R})$ be the first Chern class of $\mathcal{L}_{(i)}$. The expectation value of these classes is defined by

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle = \int_{\bar{\mathcal{M}}_{g,n}} c_1(\mathcal{L}_{(1)})^{d_1} \wedge c_1(\mathcal{L}_{(2)})^{d_2} \wedge \cdots \wedge c_1(\mathcal{L}_{(n)})^{d_n}, \quad (15)$$

where $d_1, d_2, \dots, d_n$ are non-negative integers. The dimension of $\bar{\mathcal{M}}_{g,n}$ is $6g - 6 + 2n$, so the expectation value (15) will vanish unless

$$\sum_{i=1}^n d_i = 3g - 3 + n. \quad (16)$$

To study topological gauge theory coupled to topological gravity we introduce the space $\mathcal{N}_\Sigma$ of pairs (Σ, V) , where Σ is a stable curve of genus g with n marked points $x_1, x_2, \dots, x_n$ and V is a semi-stable holomorphic $G_{\mathbf{C}}$ -bundle over Σ .

The space $\mathcal{N}_\Sigma$ fibres (non-holomorphically) over $\bar{\mathcal{M}}_{g,n}$, with the fibre over $\Sigma \in \bar{\mathcal{M}}_{g,n}$ being the space $\mathcal{R}_\Sigma$. For each $x_i \in \Sigma$ we have the complex line $T_{x_i}^* \Sigma$ which, as Σ varies over $\mathcal{N}_\Sigma$, gives n line bundles $\hat{\mathcal{L}}_{(i)} \rightarrow \mathcal{N}_\Sigma$. We can also consider the universal space $\mathcal{UN}_\Sigma$ of $\mathcal{N}_\Sigma$, defined by requiring that the fibre of $\mathcal{UN}_\Sigma$ over a point $P \in \mathcal{N}_\Sigma$ is a pair (Σ, V) in the isomorphism class denoted by P . The marked point $x_i \in \Sigma$ can be regarded as a section $x_i : \mathcal{N}_\Sigma \rightarrow \mathcal{UN}_\Sigma$ of the universal space. If we also introduce the universal bundle $\mathcal{W} \rightarrow \mathcal{UN}_\Sigma$, where $\mathcal{W}|_{(\Sigma, V)} = V$, then for each x_i we obtain a bundle $\mathcal{W}_i = x_i^* \mathcal{W}$ over $\mathcal{N}_\Sigma$ with associated Chern classes $\lambda_{i,k} = c_k(\mathcal{W}_i) \in H^{2k}(\mathcal{N}_\Sigma; \mathbf{R})$. Thus for each marked point x_i we may define natural cohomology classes $c_1(\hat{\mathcal{L}}_{(i)})^d \wedge \lambda_{i,k}$ in $H^*(\mathcal{N}_\Sigma; \mathbf{R})$, where d is a non-negative integer. Given n non-negative integers $d_1, \dots, d_n$ we can define the expectation value of these classes to be :

$$\langle \mathcal{O}_{d_1, k_1}(x_1) \cdots \mathcal{O}_{d_n, k_n}(x_n) \rangle = \int_{\mathcal{N}_\Sigma} c_1(\hat{\mathcal{L}}_{(1)})^{d_1} \wedge \lambda_{1, k_1} \wedge \cdots \wedge c_1(\hat{\mathcal{L}}_{(n)})^{d_n} \wedge \lambda_{n, k_n} \quad (17)$$

The dimension of $\mathcal{N}_\Sigma$ is $\dim \mathcal{N}_\Sigma = \dim \bar{\mathcal{M}}_{g,n} + \dim \mathcal{R}_\Sigma = 2(3g - 3 + n + (g - 1)\dim(G))$. Thus (17) vanishes unless

$$2 \sum_{i=1}^n d_i k_i = 3g - 3 + n + (g - 1)\dim(G). \quad (18)$$

This gives the model independent definition of the correlation functions of the quantum field theory defined by the Lagrangian (18) coupled to two-dimensional topological gravity.

We would now like to consider a connection between this two-dimensional topological gauge theory and Chern-Simons gauge theory in three dimensions. Let M be a compact three-manifold and let G be a compact simple simply connected Lie group. The Chern-Simons action is

$$S_{\text{cs}} = \frac{k}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \quad (19)$$

where A is a G -connection on M and Tr denotes an invariant bilinear form on the Lie algebra of G . The requirement that $e^{iS_{\text{cs}}}$ be single valued implies that k is quantized ; the normalization of Tr can be fixed so that k is quantized as an integer. The object of interest in Chern-Simons theory is the partition function of the theory on the manifold M :

$$Z_k(M; G) = \int \mathcal{D}A e^{iS_{\text{cs}}}. \quad (20)$$

Let us consider Z_k for Chern-Simons theory defined on the three-manifold $M = S^1 \times \Sigma$, where Σ is a compact surface. In the Hamiltonian formulation of quantum field theory the partition function is given by the trace over the physical Hilbert space of the time evolution operator. Since Chern-Simons theory is generally covariant, and hence has vanishing Hamiltonian, this operator is trivial. The partition function is, therefore, given by

$$\begin{aligned} Z_k(S^1 \times \Sigma; G) &= \text{Tr}_{\mathcal{H}_\Sigma^{(k)}}(1) \\ &= \dim \mathcal{H}_\Sigma^{(k)}. \end{aligned} \quad (21)$$

Here $\mathcal{H}_\Sigma^{(k)}$ is the physical Hilbert space associated to the surface Σ . We recall that $\mathcal{H}_\Sigma^{(k)}$ may be described in the following way [25] [26]. The canonical phase space of Chern-Simons theory may be identified with the space $\mathcal{R}_\Sigma$ of flat G -connections on Σ . The space $\mathcal{R}_\Sigma$ has a canonical symplectic structure determined by the topology of Σ [20]. Once a complex structure σ is chosen on Σ , $\mathcal{R}_\Sigma$ acquires a complex structure. The symplectic structure and

complex structure are compatible and together give a Kähler structure on $\mathcal{R}_\Sigma$, with Kähler form ω [20]. It now follows from the principles of geometric quantization that the physical Hilbert space of Chern-Simons theory is the space of holomorphic sections of a holomorphic line bundle $\mathcal{L}_1 \rightarrow \mathcal{R}_\Sigma$, where $\mathcal{L}_1$ admits a connection ∇ with curvature $R^\nabla = \omega$. The line bundle $\mathcal{L}_1$ is given by the determinant line bundle $\text{Det } \bar{\partial}$ of the $\bar{\partial}$ operator. For a general value of k we consider the holomorphic line bundle $\mathcal{L}_1^{\otimes k} \rightarrow \mathcal{R}_\Sigma$. The physical Hilbert space $\mathcal{H}_\Sigma^{(k)}$ is then the space of holomorphic sections of $\mathcal{L}_1^{\otimes k} : \mathcal{H}_\Sigma^{(k)} = H^0(\mathcal{R}_\Sigma; \mathcal{L}_1^{\otimes k})$. We now wish to determine the large k behaviour of the partition function (21), $Z_k(S^1 \times \Sigma; G) = \dim \mathcal{H}_\Sigma^{(k)}$. The desired result follows from the Riemann-Roch theorem.

We recall that the Riemann-Roch theorem applies to a holomorphic vector bundle E over a compact complex manifold X (or a projective algebraic variety X). The Riemann-Roch theorem states that the dimensions of the cohomology groups $H^i(X; E)$ satisfy

$$\sum_i (-1)^i \dim H^i(X; E) = \{\text{ch}(E) \text{Td}_\mathbb{C}(X)\}[X], \quad (22)$$

where $\text{ch}(E)$ is the Chern character of E and $\text{Td}_\mathbb{C}$ is the Todd class of X . In the case of interest to us here we take $X = \mathcal{R}_\Sigma$ and $E = \mathcal{L}_1^{\otimes k}$. It follows from standard cohomological vanishing theorems that $H^i(\mathcal{R}_\Sigma; \mathcal{L}_1^{\otimes k}) = 0$, for $i > 0$. Thus we have

$$\dim H^0(\mathcal{R}_\Sigma; \mathcal{L}_1^{\otimes k}) = \{\text{ch}(\mathcal{L}_1^{\otimes k}) \text{Td}_\mathbb{C}(\mathcal{R}_\Sigma)\}[\mathcal{R}_\Sigma]. \quad (23)$$

Upon recalling that $\mathcal{L}_1^{\otimes k}$ admits a connection with curvature $R = k\omega$, we immediately obtain from (23) that

$$\dim \mathcal{H}_\Sigma^{(k)} = k^n \int_{\mathcal{R}_\Sigma} \frac{\omega^n}{n!} + \dots, \quad (24)$$

where $n = \frac{1}{2} \dim_{\mathbb{R}} \mathcal{R}_\Sigma = (g-1) \dim(G)$. The dots on the right hand side of (24) indicate terms lower order in k which may be neglected in the large k limit. We note that $\frac{\omega^n}{n!}$ is the Liouville measure on $\mathcal{R}_\Sigma$, and the volume of $\mathcal{R}_\Sigma$ with respect to this canonical symplectic volume measure is

$$\text{vol}(\mathcal{R}_\Sigma) = \int_{\mathcal{R}_\Sigma} \frac{\omega^n}{n!}. \quad (25)$$

Thus we obtain

$$\dim \mathcal{H}_\Sigma^{(k)} = k^{(g-1) \dim(G)} \text{vol}(\mathcal{R}_\Sigma) + \dots. \quad (26)$$

One interesting consequence of (26) is that it provides a means of calculating $\text{vol}(\mathcal{R}_\Sigma)$. According to the standard work on Chern-Simons theory [25] - [31] the Hilbert space $\mathcal{H}_\Sigma^{(k)}$

can be identified with the space of conformal blocks of current algebra on Σ at level k . The dimension of $\mathcal{H}_{\Sigma}^{(k)}$ can be calculated using standard methods of conformal field theory. The volume of $\mathcal{R}_{\Sigma}$ can then be deduced from the large k limit of (26). We will now give brief details of the calculation.

Under the identification of $\mathcal{H}_{\Sigma}^{(k)}$ with the space conformal blocks of current algebra at level k the dimension of $\mathcal{H}_{\Sigma}^{(k)}$ may be calculated from the fusion algebra. In general, for a rational conformal field theory on Σ with N primary fields the dimension of the space of conformal blocks $\mathcal{B}_g$ (where $g \geq 2$ is the genus of Σ) is given by [32]

$$\dim \mathcal{B}_g = \sum_{n=0}^{N-1} |S_{no}|^{-2(g-1)}, \quad (27)$$

where S_{nm} is a unitary matrix on which the modular transformation $\tau \rightarrow -1/\tau$ is represented. We are interested here in the case in which the conformal field theory is the current algebra of G . For simplicity we will restrict our attention to the case $G = SU(2)$, although the argument generalizes to the case of an arbitrary compact simple Lie group. The primary fields of $SU(2)$ current algebra at level k correspond to integrable representations of the $SU(2)$ affine Kac-Moody algebra at level k . It is well known that the $SU(2)$ affine Kac-Moody algebra at level k has $k+1$ integrable representations. It is also known [32] that the matrix S_{nm} is given by the following expression

$$S_{nm} = \left(\frac{2}{k+2} \right)^{1/2} \sin \left(\frac{(n+1)(m+1)}{k+2} \pi \right). \quad (28)$$

Hence the dimension of the space of conformal blocks at level k is

$$\dim \mathcal{H}_{\Sigma}^{(k)} = \left(\frac{k+2}{2} \right)^{g-1} \sum_{n=0}^k \left| \sin \left(\frac{n+1}{k+2} \pi \right) \right|^{-2(g-1)}. \quad (29)$$

It now follows from (26) and (29) that

$$\begin{aligned} \text{vol}(\mathcal{R}_{\Sigma}) &= \lim_{k \rightarrow \infty} (k^{3-3g} \dim \mathcal{H}_{\Sigma}^{(k)}) \\ &= \frac{2^{g-1}}{(2g-2)!} |B_{2g-2}|, \end{aligned} \quad (30)$$

where B_{2m} is the $2m$ th Bernoulli number.

We note that the result we have obtained for the large k limit of Z_k

$$Z_k(S^1 \times \Sigma; G) = k^{(g-1)\dim(G)} \text{vol}(\mathcal{R}_\Sigma) \quad (31)$$

may also be obtained by applying the stationary phase approximation to $Z_k(S^1 \times \Sigma; G)$ as defined by the path integral (20). This approach has been discussed by Witten in [33]. Witten has also given an interesting alternative method of calculating $\text{vol}(\mathcal{R}_\Sigma)$ [33]. He considers the topological action (12) defined on Σ . It is not hard to see that the partition function of (12) is the volume of $\mathcal{R}_\Sigma$. Moreover the action (12) may be deformed to the action of two-dimensional Yang-Mills theory on Σ . Two-dimensional Yang-Mills theory is exactly soluble and taking the limit in which the deformation parameter tends to zero gives the volume of $\mathcal{R}_\Sigma$.

To see the connection between topological gauge theory on Σ and Chern-Simons theory on $M = S^1 \times \Sigma$ consider the cohomology class $\kappa_{2,\gamma} \in H^2(\mathcal{R}_\Sigma; \mathbf{R})$ defined by (c.f.(11))

$$\kappa_{2,\gamma} = \int_{\gamma} c_2(\mathcal{V}_\gamma), \quad (32)$$

where $\gamma = [\Sigma]$ is the fundamental homology cycle in Σ . It follows from [20] that $\kappa_{2,\gamma} = [\omega]$, where $[\omega]$ is the Kähler class of $\mathcal{R}_\Sigma$. Thus the intersection number $\langle \kappa_{2,\gamma}^{(g-1)\dim(G)} \rangle$ defined by

$$\langle \kappa_{2,\gamma}^{(g-1)\dim(G)} \rangle = \int_{\mathcal{R}_\Sigma} \kappa_{2,\gamma}^{(g-1)\dim(G)} \quad (33)$$

satisfies

$$\langle \kappa_{2,\gamma}^{(g-1)\dim(G)} \rangle = [(g-1)\dim(G)]! \text{vol}(\mathcal{R}_\Sigma). \quad (34)$$

Combining (34) with (21) and (26) we find that

$$Z_k(S^1 \times \Sigma; G) = \frac{k^{(g-1)\dim(G)}}{[(g-1)\dim(G)]!} \langle \kappa_{2,\gamma}^{(g-1)\dim(G)} \rangle + \dots \quad (35)$$

Thus in the semiclassical limit the partition function of Chern-Simons theory on $M = S^1 \times \Sigma$ is directly related to the expectation value of two-dimensional topological Yang-Mills theory. As Chern-Simons theory is exactly soluble this relation enables us to calculate

the expectation values of two-dimensional topological gauge theory. For example, for $G = SU(2)$, (30) and (34) imply that

$$\langle \kappa_{2,\gamma}^{3g-3} \rangle = \frac{(3g-3)! 2^{g-1}}{(2g-2)!} |B_{2g-2}|. \quad (36)$$

To conclude we observe that a similar relation holds, at least formally, between $SL(2, \mathbf{R})$ Chern-Simons theory and two-dimensional topological gravity. For a detailed discussion of $SL(2, \mathbf{R})$ Chern-Simons theory see [34]; here we will just give a brief outline. The phase space of $SL(2, \mathbf{R})$ Chern-Simons theory on $\mathbf{R} \times \Sigma$ is given by the moduli space of flat $SL(2, \mathbf{R})$ connections on Σ . The maximal component of the phase space, on which we will concentrate here, may be identified with the Teichmüller space $\mathcal{T}_g$ of Σ . The Teichmüller space $\mathcal{T}_g$ has a natural Kähler structure induced by the Petersson inner product on automorphic forms. The Weil-Petersson Kähler form on $\mathcal{T}_g$ projects to a Kähler form on the moduli space $\mathcal{M}_g = \mathcal{T}_g / \Gamma_g$, where Γ_g is the mapping class group of Σ . We shall regard $\mathcal{M}_g$ as the reduced phase space of $SL(2, \mathbf{R})$ Chern-Simons theory. This corresponds to taking the symmetries of $SL(2, \mathbf{R})$ Chern-Simons theory to be $SL(2, \mathbf{R})$ gauge transformations plus mapping class transformations.

The moduli space $\mathcal{M}_g$ has a natural compactification as the moduli space $\bar{\mathcal{M}}_g$ of stable curves, and it was shown in [35] that the Weil-Petersson Kähler form on $\mathcal{M}_g$ extends to a closed form $\bar{\mathcal{M}}_g$. We shall denote this Kähler form on $\bar{\mathcal{M}}_g$ by ω_{WP} . Regarding $\bar{\mathcal{M}}_g$ as the (compactified) phase space of $SL(2, \mathbf{R})$ Chern-Simons theory, we expect that the analogue of (21) should hold

$$Z_k(S^1 \times \Sigma; SL(2, \mathbf{R})) = \dim \dot{\mathcal{H}}_{\Sigma}^{(k)}, \quad (37)$$

where $\dot{\mathcal{H}}_{\Sigma}^{(k)}$ is the physical Hilbert space associated to Σ . The Hilbert space $\dot{\mathcal{H}}_{\Sigma}^{(k)}$ is obtained by quantizing $\bar{\mathcal{M}}_g$ geometrically. It was shown in [36] that $\frac{1}{2\pi^2}[\omega_{\text{WP}}]$ is an integral cohomology class, hence there exists a holomorphic line bundle $\dot{\mathcal{L}} \rightarrow \bar{\mathcal{M}}_g$ such that $c_1(\dot{\mathcal{L}}) = \frac{1}{2\pi^2}[\omega_{\text{WP}}]$. The Hilbert space of $SL(2, \mathbf{R})$ Chern-Simons theory is then given by $\dot{\mathcal{H}}_{\Sigma}^{(k)} = H^0(\bar{\mathcal{M}}_g; \dot{\mathcal{L}}^{\otimes k})$.

The Riemann-Roch formula (22) may be used to calculate the dimension of $\dot{\mathcal{H}}_{\Sigma}^{(k)}$:

$$\begin{aligned} \dim \dot{\mathcal{H}}_{\Sigma}^{(k)} &= \{ \text{ch}(\dot{\mathcal{L}}^{\otimes k}) \text{Td}_{\mathbf{C}}(\bar{\mathcal{M}}_g) \} [\bar{\mathcal{M}}_g] \\ &= \frac{k^{3g-3}}{(2\pi^2)^{3g-3}} \text{vol}(\bar{\mathcal{M}}_g) + \dots \end{aligned} \quad (38)$$

where $\text{vol}(\bar{\mathcal{M}}_g)$ is the canonical symplectic volume determined by ω_{WP} , and the dots on the right-hand side of (38) denote terms lower order in k , which may be neglected in the

large k limit. We note in passing that taking the limit in which k is large and even in (38) immediately yields that the volume of $\bar{\mathcal{M}}_g$ is a rational multiple of π^{6g-6} .

The connection between $SL(2, \mathbf{R})$ Chern-Simons theory and two-dimensional topological gravity results from considering the natural cohomology classes on $\bar{\mathcal{M}}_g$ defined in [37]. These classes are defined as follows. Let $\pi : \bar{\mathcal{C}}_g \rightarrow \bar{\mathcal{M}}_g$ be the universal curve over $\bar{\mathcal{M}}_g$. The universal curve $\bar{\mathcal{C}}_g$ may be viewed as the moduli space $\bar{\mathcal{M}}_{g,1}$ of curves with a single puncture. Let $\mathcal{L}$ be the holomorphic line bundle over $\bar{\mathcal{M}}_{g,1}$ defined previously. The tautological classes on $\bar{\mathcal{M}}_g$ are defined as

$$\epsilon_r = \pi_*(c_1(\mathcal{L})^{r+1}), \quad (39)$$

where $\pi_* : H^{2r+2}(\bar{\mathcal{C}}_g) \rightarrow H^{2r}(\bar{\mathcal{M}}_g)$ is integration over the fibre of $\pi : H^{2r+2}(\bar{\mathcal{C}}_g) \rightarrow H^{2r}(\bar{\mathcal{M}}_g)$. We can consider the intersection numbers of the ϵ s:

$$\langle \epsilon_{r_1} \epsilon_{r_2} \cdots \epsilon_{r_n} \rangle = \int_{\bar{\mathcal{M}}_g} \epsilon_{r_1} \wedge \epsilon_{r_2} \wedge \cdots \wedge \epsilon_{r_n}. \quad (40)$$

It was shown in [36] that $\epsilon_1 = \frac{1}{2\pi^2}[\omega_{\text{WP}}]$. Thus

$$\langle \epsilon_1^{3g-3} \rangle = \frac{(3g-3)!}{(2\pi^2)^{3g-3}} \text{vol}(\bar{\mathcal{M}}_g). \quad (41)$$

Equation (38) for the dimension of $\mathcal{H}_\Sigma^{(k)}$ therefore takes the form

$$\dim \mathcal{H}_\Sigma^{(k)} = \frac{k^{3g-3}}{(3g-3)!} \langle \epsilon_1^{3g-3} \rangle + \cdots. \quad (42)$$

The cohomology classes ϵ_r on $\bar{\mathcal{M}}_g$, defined by (39), are closely related to the cohomology classes τ_{d_i} on $\bar{\mathcal{M}}_{g,n}$, defined by

$$\tau_{d_i} = c_1(\mathcal{L}_{(i)})^{d_i}, \quad (43)$$

which occurred in the definition of two-dimensional topological gravity discussed previously. The relation between the ϵ_r s and the τ_{d_i} s has been discussed in [16] and [38]. One finds (see [38]) that

$$\langle \epsilon_1^{3g-3} \rangle = \langle \tau_2^{3g-3} \rangle - C_1 \langle \tau_3 \tau_2^{3g-5} \rangle + C_2 \langle \tau_4 \tau_2^{3g-6} \rangle + C_3 \langle \tau_3^2 \tau_2^{3g-7} \rangle + \cdots + (-1)^{3g-4} \langle \tau_{3g-2} \rangle, \quad (44)$$

where the integers C_i are symmetry factors which can be simply determined. Hence, it follows from (37), (42) and (44), that in the semi-classical limit, the partition function

of $SL(2, \mathbf{R})$ Chern-Simons theory on $M = S^1 \times \Sigma$ is related to the expectation values of two-dimensional topological gravity on Σ .

REFERENCES

1. E. Brezin and V. Kazakov, *Phys. Lett.*, **B236** (1990) 144.
2. M. Douglas and S. Shenker, *Nucl. Phys.*, **B335** (1990) 635.
3. D.J.Gross and A. Migdal, *Phys. Rev. Lett.*, **64** (1990) 127.
4. T. Banks, M. Douglas, N. Seiberg and S. Shenker, *Phys. Lett.*, **238B** (1990) 279.
5. D.J. Gross and A. Migdal, *Nucl. Phys.*, **B340** (1990) 333.
6. E. Brezin, M. Douglas, V. Kazakov and S. Shenker, *Phys. Lett.*, **237B** (1990) 43.
7. D. J. Gross and A. Migdal, *Phys. Rev. Lett.*, (1990) 717.
8. C. Crnković, P. Ginsparg and G. Moore, “The Ising Model, the Yang-Lee Edge Singularity and 2D Quantum Gravity”, *Yale Preprint*, **YCTP-P20-89** (1989)
9. M. Douglas, *Phys. Lett.*, **238B** (1990) 176.
10. E. Witten, *Nucl. Phys.*, **B340** (1990) 281.
11. R.Dijkgraaf and E. Witten, *Nucl. Phys.*, **B342** (1990) 486.
12. E. Verlinde and H. Verlinde, “A solution of Two-Dimensional Topological Quantum Gravity”, *IAS preprint*, **IASSNS-HEP-90/40** (1990).
13. J. Distler, *Nucl. Phys.*, **B342** (1990) 528.
14. R. Dijkgraaf, E. Verlinde and H. Verlinde, “Loop Equations and Virasoro Constraints in Non-Perturbative 2-D Quantum Gravity”, *IAS Princeton preprint*, **IASSNS-HEP-90/48, PUPT-1184** (1990).
15. G. Moore, “Geometry of String Equation”, *Yale preprint* YCTP-PA-90 (1990).
16. E. Witten, “Two Dimensional Gravity and Intersection Theory on Moduli Space”, *IAS preprint*, **IASSNS-HEP-90/45** (1990).
17. S. Donaldson, “Polynomial Invariants of Smooth Four Manifolds”, to appear in *Topology*.
18. E. Witten, *Commun. Math. Phys.*, **117** (1988) 353.
19. M. Narasimhan and S. Seshadri, *Ann. Math.*, **82** (1965) 540.

20. M.F. Atiyah and R. Bott, *Phil. Trans. R. Soc. Lond.*, **A308** (1982) 523.
21. M. F. Atiyah and I. Singer, *Proc. Natt. Acad. Sci.*, **81** (1984) 2597.
22. M. Blau and G. Thompson, "Topological Gauge Theories of Antisymmetric Tensor Fields", to appear in *Ann. Phys.*
23. E. Witten, "Surprises with topological field theories", IAS preprint IASSNS-HEP-90/37 (1990).
24. D. Birmingham, M. Blau and G. Thompson, "Geometry and Quantization of Gauge Theories", to appear in *Int. J. Mod. Phys. A*.
25. E. Witten, *Commun. Math. Phys.* **121** (1989) 351.
26. T.P. Killingback, *Phys. Lett.* **B219** (1989) 448.
27. M. Bos and V. Nair, Columbia preprint CU-TP-432 (1989).
28. G. Dunne, R. Jackiw and C. Trugenberger, MIT preprint MIT-CTP 1711 (1989).
29. S. Elitzur, G. Moore, A. Schimmer and N. Seiberg, IAS preprint IASSNS-HEP-89/20 (1989).
30. H. Murayama, Tokyo University preprint (1989).
31. E. Verlinde and H. Verlinde, IAS preprint IASSNS-HEP-90/58 (1989).
32. E. Verlinde, *Nucl. Phys.* **B300** [FS22] (1988) 360.
33. E. Witten, Lectures at the Conference on Topological Methods in Quantum Field Theory, ICTP Trieste, 11-15 June 1990.
34. T.P. Killingback, "Quantization of $SL(2, \mathbf{R})$ Chern-Simons Theory", CERN preprint CERN-TH.5698/90 (1990).
35. S. Wolpert, *Am. J. Math.* **107** (1985) 969.
36. S. Wolpert, *Ann. Math.* **118** (1983) 491.
37. D. Mumford, "Towards an Enumerative Geometry of the Moduli Space of Curves", in *Arithmetic and Geometry*, eds. M. Artin and J. Tate (Birkhauser, 1983).
38. J. Horne, "Intersection Theory and Two Dimensional Gravity at Genus 3 and 4", Princeton preprint PUPT-1185 (1990).