BNL-45804
CERN-TH-6018/91

ON THE DEFINITION OF THE Z -BOSON MASS

S. Willenbrock^{*}

*Physics Department
Brookhaven National Laboratory, Upton, NY 11973*

and

G. Valencia

*Theory Division, CERN
CH-1211, Geneva 23*

ABSTRACT

We propose a physical definition of the Z -boson mass, the real part of the pole in the Z -boson propagator, which is field-redefinition invariant and gauge invariant to all orders. The physical Z -boson mass lies approximately 26 MeV below the commonly-used "on-shell" mass.

To appear in Phys. Lett. B

BNL-45804
CERN-TH-6018/91
February 1991

^{*} This manuscript has been authored under contract number DE-AC02-76 CH 00016 with the U.S. Department of Energy. Accordingly, the U.S. Government retains a non-exclusive, royalty-free license to publish or reproduce the published form of this contribution, or allow others to do so, for U.S. Government purposes.

Measurements at *LEP* and (to a lesser extent) *SLC* of the e^+e^- cross section in the vicinity of the Z -boson resonance have allowed an accurate determination of the Z -boson mass, $M_Z = 91.177 \pm .006(\text{stat}) \pm .020 \text{ GeV}$,^[1] where the last error is due to the uncertainty in the machine energy.[†] The Z -boson mass now joins the fine-structure constant α and the Fermi constant G_μ as the most precisely measured electroweak parameters.

The extraction of the Z -boson mass from the measured resonant cross section requires a theoretical calculation which is at least as accurate as the experiment itself. The remarkable experimental precision necessitates the calculation of the $O(\alpha)$ radiative corrections to the Z -boson resonant cross section.^[3] These corrections have been calculated by several groups.^[4] The largest effect is due to initial-state radiation, which has been calculated to $O(\alpha^2)$.^[5] The result of these calculations is a prediction of the Z -boson line shape, for a given Z -boson mass and width, which is believed to be accurate to at least 10 MeV in shape, *i.e.*, in the position and width of the peak, and to at least 0.3% in absolute scale, *i.e.*, in the height of the peak.

In S -matrix theory, the mass of an unstable particle is usually defined as the real part of the pole in a scattering amplitude. The pole position is a physical quantity: it has meaning independent of any theoretical framework, and is independent of the scattering process. In this letter we point out that the definition of the Z -boson mass which has been commonly used (the “on-shell” mass) does not correspond to the real part of the pole position, and that the two definitions differ within the accuracy of the theoretical calculation of the Z -boson resonant cross section.

In perturbative field theory, a pole in a scattering amplitude is induced by the propagator of a particle. Consider the part of the Z -boson propagator proportional

[†] The uncertainty in the *LEP* beam energy has recently been reduced from $\pm 30 \text{ MeV}$.^[2]

to $-ig^{\mu\nu}$,

$$D(q^2) = \frac{1}{q^2 - M_0^2 + \Pi(q^2)} \quad , \quad (1)$$

where M_0 is the bare Z -boson mass, and $\Pi(q^2)$ is the part of the Z -boson self energy proportional to $-ig^{\mu\nu}$.[‡] It is convenient to expand the real part of $\Pi(q^2)$ about the point $q^2 = M_R^2$,

$$\text{Re}\Pi(q^2) = \text{Re}\Pi(M_R^2) + \text{Re}\Pi'(M_R^2)(q^2 - M_R^2) + \dots \quad (2)$$

where the ellipses represent finite terms which we may neglect to the order we are working. Defining the renormalized mass parameter and the wavefunction renormalization factor via

$$M_R^2 \equiv M_0^2 - \text{Re}\Pi(M_R^2) \quad (3)$$

$$Z^{-1} \equiv 1 + \text{Re}\Pi'(M_R^2) \quad (4)$$

we may cast the propagator into the form

$$D(q^2) = \frac{Z}{q^2 - M_R^2 + iZ\text{Im}\Pi(q^2)} \quad . \quad (5)$$

The imaginary part of $\Pi(q^2)$ is related to the Z -boson width when evaluated at $q^2 = M_R^2$:

$$Z\text{Im}\Pi(M_R^2) = M_R\Gamma_Z \quad . \quad (6)$$

The field-theoretic definition of the mass of a particle is the real part of the pole in the particle's propagator. For a stable particle, $\text{Im}\Pi(q^2) = 0$; the parameter M_R is then the particle's mass. However, for an unstable particle, $\text{Im}\Pi(q^2) \neq 0$,

[‡] We have chosen to ignore the issue of Z -photon mixing since it doesn't affect our discussion to the order we are working.

and the pole position is complex. If q_0^2 is the pole position, the physical mass and width are often defined by

$$q_0^2 = (M_Z - \frac{i}{2}\Gamma_Z)^2 \quad (7)$$

i.e. the mass is defined as the real part of the pole in the energy plane. Calculating in powers of α , we find that the relationship between the physical Z -boson mass and the renormalized mass parameter, M_R , is

$$M_Z^2 = M_R^2 + \frac{1}{4}\Gamma_Z^2 - M_Z\Gamma_Z\text{Im}\Pi'(M_R^2) + \dots \quad (8)$$

where the ellipses represent terms of $O(\alpha^3)$ and higher. Thus the difference between M_R and M_Z arises at $O(\alpha^2)$.[§]

The parameter M_R is defined to be the Z -boson mass in the so-called “on-shell” renormalization scheme.^[6] The difference between M_R and M_Z is meaningful if its effect on the resonant cross section is within the accuracy of the calculation, which has been performed completely through $O(\alpha)$. Near resonance, i.e., $|q^2 - M_R^2| \lesssim M_R\Gamma_Z \sim O(\alpha)$, the amplitude begins at $O(1)$, because the denominator of the Z -boson propagator is $O(\alpha)$, and cancels the power of α coming from the Z -boson coupling to the initial and final fermions. A difference in the definition of the Z -boson mass at $O(\alpha^2)$ therefore produces a correction to the amplitude of $O(\alpha)$.

The imaginary part of the two-point function at one loop is related to the square of the $Z \rightarrow f\bar{f}$ vertex by unitarity:

$$\text{Im}\Pi(q^2) = \frac{1}{16\pi} \sum_f |M_3(Z \rightarrow f\bar{f})|^2 \quad (9)$$

Since $|M_3(Z \rightarrow f\bar{f})|^2$ is proportional to q^2 (for massless fermions), we find

$$\text{Im}\Pi'(M_R^2) = \frac{1}{M_R^2} \text{Im}\Pi(M_R^2) \quad (10)$$

§ Similarly, if the Z -boson width is defined by eq. (7), then eq. (6) is valid only to $O(\alpha^2)$.

Using this relation, along with eq. (6), we obtain from eq. (8)

$$M_Z = M_R \left(1 - \frac{3}{8} \left(\frac{\Gamma_Z}{M_R} \right)^2 + \dots \right) \quad (11)$$

where the ellipses again represent terms of $O(\alpha^3)$ and higher. Using $\Gamma_Z \approx 2.50$ GeV, we find that M_Z is less than M_R by approximately 26 MeV, to order $O(\alpha^2)$. Since it is M_R , not M_Z , which has been extracted from the *LEP* data, we obtain ¶

$$M_Z = 91.151 \pm .006(\text{stat}) \pm .020 \text{ GeV} \quad . \quad (12)$$

The shift in the mass is comparable to the experimental uncertainty.*

The real part of the pole in the propagator is a physical quantity; it is therefore field-redefinition invariant and gauge invariant.^[7] These attributes are not necessarily shared by the renormalized mass parameter M_R , defined by eq. (3), which corresponds to the zero of the real part of the inverse propagator. We have given an example elsewhere of the field-redefinition dependence of this parameter in the context of another model: the scalar sector of the standard model, *i.e.*, the Higgs boson coupled to Goldstone bosons.^[8] However, in a gauge theory, one is generally more concerned with gauge invariance. At lowest order, $\text{Im}\Pi(q^2)$ involves only the coupling of the Z -boson to fermions, which is gauge independent. Thus M_R , which is related to M_Z via eq. (8), is gauge-independent to $O(\alpha^2)$. Gauge dependence of this quantity may appear at higher order.

At present, there are no practical implications of the difference between the “on-shell” mass and the real part of the pole in the propagator; the Z -boson resonant cross section is the only quantity where a definition of the Z -boson mass

¶ In the case of a relativistic system such as the Z -boson resonance, it is perhaps more natural to define the mass via the real part of the pole in the energy-squared plane, *i.e.*, $M_Z^2 = \text{Re}q_0^2$. This removes the $\Gamma_Z^2/4$ term from eq. (8), and increases the difference between M_R and M_Z , with the latter now lying 34 MeV below the former.

* The uncertainty in the *LEP* beam energy may be reduced to as low as ± 5 MeV in the 1991 run.^[2]

to $O(\alpha^2)$ is necessary to achieve the required theoretical accuracy. For example, one may extract the top-quark mass^{**} by comparing the Z -boson mass, α , and G_μ with at least one other electroweak measurement: *e.g.*, M_W , R_ν (the ratio of neutral-current to charged-current neutrino-nucleon cross sections), or Γ_Z . The relation between these measurements and the aforementioned parameters is needed only to $O(\alpha)$, due to the experimental errors in the measurements, so either definition of M_Z will suffice. However, when the experimental errors are reduced such as to warrant an $O(\alpha^2)$ calculation, it will be necessary to choose one definition of mass, for consistency.

The question of which definition of mass to use therefore reduces to a matter of principle. The “on-shell” definition corresponds to a (relativistic) Breit-Wigner mass parameter,^{***} which the Particle Data Group has adopted as the definition of mass in the case of hadron resonances (although the pole position is also given for the N and Δ resonances, and exclusively for the $f_0(975)$).^[10] However, there is no fundamental dynamical theory which describes these resonances, so issues of field-redefinition invariance and gauge invariance do not arise.^{****} The Z boson (as well as the W boson) is the first resonance for which a fundamental dynamical theory exists, and it seems sensible to adopt a physical definition of mass, which respects, in a natural way, the invariances of the theory to all orders.

Acknowledgements

We are grateful for conversations with G. Altarelli, W. Bardeen, T. Bhattacharya, E. Blucher, D. Espriu, C. Goebel, B. Kniehl, R. Longacre, W. Marciano, M. Martinez, R. Pisarski, and S. Protopopescu.

^{**} Modulo uncertainties due to experimental errors and the unknown Higgs-boson mass.

^{***} In “model independent” analyses of the Z -boson line shape, the Z -boson mass is regarded as a Breit-Wigner resonance parameter.^[9] The resonance mass M_R is still related to the physical mass by eq. (11), since the energy dependence of the width ($\text{Im}\Pi(q^2)$) which is assumed in these analyses is the same as that which is obtained from field theory.

^{****} In principle, these resonances are thought to be described by QCD, but it is not possible to do so at the present time.

REFERENCES

1. Number reported by E. Fernandez at the *Neutrino 90 Conference*, CERN-PPE/90-151 (1990).
2. "LEP absolute energy in 1990", LEP Performance Note 12, Dec. 1990; E. Blucher, private communication.
3. W. Wetzel, *Nucl. Phys.* **B227** (1983) 1.
4. M. Consoli, W. Hollik and F. Jegerlehner; and F. A. Berends *et al.* in *Z Physics at LEP 1*, CERN Yellow Report 89-08, eds. G. Altarelli, R. Kleiss and C. Verzegnassi, Vol. 1, ps. 7 and 89, and references therein; W. Beenakker and W. Hollik, *Z. Phys.* **C40** (1988) 141; F. A. Berends, G. Burgers, W. Hollik, and W. L. van Neerven, *Phys. Lett.* **203B** (1988) 177; G. Burgers in *Polarization at LEP*, CERN Yellow Report 88-06, eds. G. Alexander *et al.*, Vol. 1, p. 121; A. A. Akhundov, D. Yu Bardin and T. Riemann, *Nucl. Phys.* **B276** (1986) 1; D. Yu. Bardin, A. Leike, T. Riemann, and M. Sachwitz, *Phys. Lett.* **206B** (1988) 539; D. Kennedy, B. Lynn, C. Im, and R. Stuart, *Nucl. Phys.* **B321** (1989) 83; D. Kennedy and B. Lynn, *Nucl. Phys.* **B322** (1989) 1.
5. E. A. Kuraev and V. S. Fadin, *Sov. J. Nucl. Phys.* **41** (1985) 446; G. Altarelli and G. Martinelli in *Physics at LEP*, CERN Yellow Report 86-02, eds. J. Ellis and R. Peccei, Vol. 1, p. 47; F. A. Berends, G. Burgers and W. L. van Neerven, *Phys. Lett.* **185B** (1987) 395; *Nucl. Phys.* **B297** (1988) 429; **B304** (1988) 921(E); *Phys. Lett.* **177B** (1986) 191; O. Nicrosini and L. Trentadue, *Phys. Lett* **196B** (1987) 551; *Z. Phys.* **C39** (1988) 479; V. S. Fadin and V. A. Khoze, *Yad. Phys.* **47** (1988) 1693.
6. D. A. Ross and J. C. Taylor, *Nucl. Phys.* **B51** (1973) 125.
7. G. 't Hooft and M. Veltman, *Nucl. Phys.* **B50** (1972) 318; B. Lee and J. Zinn-Justin, *Phys. Rev.* **D5** (1972) 3121; D. Gross, in *Methods in Field Theory*, ed. R. Balian and J. Zinn-Justin (North-Holland, New York, 1976),

- p. 141; R. Kobes, G. Kunstatter, and A. Rebhan, *Phys. Rev. Lett.* **64** (1990) 2992.
8. G. Valencia and S. Willenbrock, *Phys. Rev.* **D42** (1990) 853; S. Willenbrock and G. Valencia, *Phys. Lett.* **247B** (1990) 341.
9. A. Borelli, M. Consoli, L. Maiani, and R. Sisto, *Nucl. Phys.* **B333** (1990) 357; M. Martinez, Ll. Garrido, R. Miquel, J. Harton, and R. Tanaka, CERN-PPE/90-109 (1990).
10. Particle Data Group, *Phys. Lett.* **239B** (1990) 1.