CERN-TH 6023/91

THE ANOMALOUS SECTOR OF THE QCD EFFECTIVE LAGRANGIAN

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Abstract

The effective action of QCD in the abnormal intrinsic parity sector is calculated to next to leading order in the Chiral Perturbation Theory expansion in the chiral limit. As expected all corrections are invariant under chiral symmetry. This calculation complements those done for the normal intrinsic parity effective action. The result is combined with the loop calculation done previously for the case of pseudoscalar decays into photon and a charged lepton pair. This allows for a physical determination of the parameter in the expansion that can be interpreted as the constituent quark mass.

An expression for the abnormal intrinsic parity effective action is derived to all orders in terms of only one set of Seeley-DeWitt coefficients.

1 Introduction

The theory for the strong interaction QCD has withstood all experimental tests. It also incorporates naturally all the symmetries which were found to be important in low energy hadronic physics. Low energy hadronic processes have, however, been most successfully described using an effective Lagrangian that incorporates those symmetries and the relevant degrees of freedom (Chiral Perturbation Theory). Going beyond the lowest order in these effective Lagrangians can be done consistently but requires the introduction of a large number of free parameters.

In fact if we want to test QCD quantitatively we have to separate out the consequences that follow purely from chiral symmetry and its spontaneous breakdown. These can be separated out most easily using Chiral Perturbation Theory. We write an effective Lagrangian to a certain order in the low energy expansion and leave its parameters free. Loops can then be calculated. For a detailed explanation of the contributions from loops and finite parts at higher order see Refs.[1, 2]. The only difference between different theories with the same spontaneously broken chiral symmetry is in the values of the parameters in the effective Lagrangian. So to test QCD at the quantitative level we have to calculate the values of these parameters. It is this that we try to do approximately in this paper for the abnormal intrinsic parity sector.

The Lagrangian to next-to-leading order in the normal intrinsic parity sector was determined completely by Gasser and Leutwyler [1]. It has two free parameters at lowest order and 10 more at next-to-leading order. These can all be fixed from experiments and have been used to calculate several processes. In this case it was found that [3, 4] several of those free parameters could be well described by direct derivation from a QCD model. A short rederivation of this result will be given later. Their method was extended by Espriu et al. [5] to include gluonic corrections and the effect of chiral symmetry breaking.

In the abnormal intrinsic parity sector of QCD the low energy effective action was first derived by Wess and Zumino [6] as a series expansion by directly integrating the anomaly consistency conditions. This Lagrangian was later obtained more directly by Witten [7] by writing an action of abnormal intrinsic parity. This paper also introduced the concept of intrinsic parity. This action only contained one free parameter that has to take integer values, this integer can then be fixed unambiguously from the QCD anomaly. This action was subsequently derived using the global form of the anomaly consistency conditions [8] or by integration of the change in the fermion measure in the functional integral when going from an effective action with quarks incorporating the anomaly or not [9, 10]. The latter method will have some relevance here. The reason for this is that for massive quarks the anomaly is only present if a direct non-derivative pseudoscalar quark coupling is present. If the pseudoscalars couple only derivatively to quarks it can be shown [9] that the anomaly is not present. Therefore it is important to use an effective model that has direct quark pseudoscalar meson couplings.

The effective action in this sector to next-to-leading order has a more patchy recent history. It was known that there were no non-analytic corrections to next-to-leading order to the decays $\pi^0, \eta \rightarrow \gamma\gamma$ [11]. This together with the fixed coefficient of the Wess-Zumino term and the Adler non-renormalization of the anomaly theorem was often used as an argument for the absence of corrections to processes of abnormal intrinsic parity. However, there is

no reason why terms not breaking any symmetries cannot contribute to these processes. This was shown by Bijnens et al. [12] and Donoghue and Wyler [13] both by explicitly constructing such terms and calculating the loops of pseudoscalars contributions. The latter were divergent for the decays $\pi^0, \eta \rightarrow \gamma l^+ l^-$ so free parameters to absorb these divergences are needed at the next order.

The full infinite part of the next order was calculated subsequently by Bijnens et al. [14] and Issler [15]. Issler also determined the full set of free parameters at this order. There are 40 free constants here. In the chiral limit there are still 29 left. The relevance of these terms to a number of physical processes was worked out in references [14, 16]. To circumvent the problem of the free parameters the approach of determining the constants by estimating resonance contributions to them was taken there. This had been shown to work well in the normal intrinsic parity sector [2, 17, 18].

In this paper we take the alternative approach of determining the constants (almost) directly from QCD. This has been shown to work well in the normal parity sector whenever gluonic corrections are small. We use the same approach here. More care is needed, however, because of the problem of integrating out chiral fermions. In section 2, I give a short overview of Chiral Perturbation Theory and the Wess-Zumino term. This section also serves to define my conventions here. In the next section, I rederive the results of Balog and Zuk in the normal intrinsic parity sector. Section 4 gives the extension to the abnormal intrinsic parity part in general. Then we proceed to calculate the main result of this paper, the anomalous effective action of QCD at order p^6 in section 5. Section 6 describes the application to $\pi^0, \eta \rightarrow \gamma l^+ l^-$ where we have an observable that can be unambiguously compared with my result. This will allow for a determination of a free parameter in the QCD model. In the last section we summarize our results. The appendices list the required Seeley-DeWitt coefficients and my Euclidean space sign conventions.

2 Chiral Perturbation Theory at lowest order

The strong interaction phenomenology at low energy can be well described by an effective Lagrangian that incorporates the $SU(3)_L \times SU(3)_R$ chiral symmetry of QCD. This symmetry is spontaneously broken to its vector subgroup. There are then 8 Goldstone bosons. These can be identified with the physical π , K and η mesons after the effect of explicit chiral symmetry breaking by the quark masses is taken into account. In this paper we work in the chiral limit and all effects of the explicit quark masses are neglected.

The Goldstone bosons can be described by a 3×3 unitary matrix Σ that transforms under $SU(3)_L \times SU(3)_R$ as

$$\Sigma \rightarrow U_L \Sigma U_R^\dagger, \quad (1)$$

where $U_{L(R)} \in SU(3)_{L(R)}$. A parametrization of Σ in terms of the physical pseudoscalar mesons is

$$\Sigma = \exp \frac{2i}{f} M \quad (2)$$

where

$$M = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta_8 \end{pmatrix}, \quad (3)$$

and f is a free constant.

The Lagrangian can be expanded in powers of external momenta and fields [1, 2]. Here vector fields and momenta are of $\mathcal{O}(p)$. At lowest order (p^2) there is only one term

$$L_2 = \frac{f^2}{8} \text{tr} D_\mu \Sigma D^\mu \Sigma^\dagger \quad (4)$$

The covariant derivative on Σ is defined in terms of the left- and right-handed external vector fields, L_μ and R_μ , as

$$D_\mu \Sigma = \partial_\mu \Sigma + iL_\mu \Sigma - i\Sigma R_\mu. \quad (5)$$

The vector fields transform under the chiral symmetry as

$$\begin{aligned} L_\mu &\rightarrow U_L L_\mu U_L^\dagger - iU_L \partial_\mu U_L^\dagger \\ R_\mu &\rightarrow U_R R_\mu U_R^\dagger - iU_R \partial_\mu U_R^\dagger. \end{aligned} \quad (6)$$

It can be easily checked that L_2 is invariant under the chiral symmetry with these transformations. We can determine the parameter f from the pion decay constant. Taking the left current from L_2

$$J_\mu^{La} = -\frac{if^2}{4} \text{tr}(T^a \partial_\mu \Sigma \Sigma^\dagger) \quad (7)$$

and the definition of the pion decay constant

$$\langle 0 | J_\mu^{L1+i2} | \pi^+ \rangle = -\frac{i}{\sqrt{2}} f_\pi P_{\pi\mu} \quad (8)$$

we get $f = f_\pi = 132 \text{ MeV}$. We have complete $SU(3)$ symmetry for the decay constants here.

Electromagnetic interactions can be incorporated by the identification

$$L_\mu = R_\mu = eA_\mu Q \quad (9)$$

where e is the electromagnetic coupling constant, A_μ the photon field and Q the quark charge matrix :

$$Q = \begin{pmatrix} 2/3 & & \\ & -1/3 & \\ & & -1/3 \end{pmatrix} \quad (10)$$

This action has not only all symmetries of QCD but also intrinsic parity :

$$\begin{aligned} \Sigma &\leftrightarrow \Sigma^\dagger \\ L_\mu &\leftrightarrow R_\mu \end{aligned} \quad (11)$$

So it is not possible to describe processes like $\pi\pi\pi KK$ or $\pi^0 \rightarrow \gamma\gamma$. The first terms of this type are $\mathcal{O}(p^4)$ but they cannot be written in a compact form in 4 dimensions. They were first obtained by Wess and Zumino as a series expansion [6] by directly integrating the anomalous Ward identities. A compact form for this action written in 5 dimensions was found by Witten [7]. Here it can be easily seen that the coefficient of these terms had to be quantized. The underlying argument was that the action in 4 dimensions should not depend on the way in which the extrapolation in the fifth dimension was made. One way the fifth dimension comes in is as the integration parameter of the anomalous Ward identities. In section 4 this comes in too in the form of the extra integration over t .

The full lowest order action is given by

$$S = \int d^4x L_2 - N_C S_{WZ}, \quad (12)$$

where N_C is the number of colours and

$$S_{WZ} = \frac{i}{240\pi^2} \int d\Sigma^{ijklm} \text{tr} \left(\Sigma \partial_i \Sigma^\dagger \partial_j \Sigma^\dagger \partial_k \Sigma \partial_l \Sigma^\dagger \partial_m \Sigma^\dagger \right) + \frac{i}{48\pi^2} \int d^4x \epsilon^{\mu\nu\alpha\beta} Z_{\mu\nu\alpha\beta}. \quad (13)$$

Here $Z_{\mu\nu\alpha\beta}$ is zero when all external fields are set to zero. Its explicit expression in my conventions can be found in ref. [14]. The sign of the Wess-Zumino term is the one that reproduces the original calculation of the anomaly.

The amplitude for $\pi^0, \eta \rightarrow \gamma\gamma^*$ can be derived from the explicit form for $Z_{\mu\nu\alpha\beta}$ and is given by

$$A(M \rightarrow \gamma\gamma^*) = \frac{i\sqrt{2}e^2}{4\pi^2 f_M} C_M \epsilon^{\mu\nu\alpha\beta} \epsilon_\mu k_\nu \epsilon_\alpha^* k_\beta^*, \quad (14)$$

where $C_\pi = 1$, $C_{\eta 8} = 1/\sqrt{3}$ and ϵ and ϵ^* are the polarization vectors of the photons with momenta k and k^* , respectively.

3 The normal intrinsic parity effective Lagrangian

The general method used in this and the next section is the functional integral for fermions regulated by the proper time method. A recent review of this method can be found in the Physics Report by Ball [19]. The relevant formulae will be briefly repeated here.

The QCD action in Euclidean space can be written in the following form :

$$\mathcal{L}_{QCD} = \bar{q} \mathcal{D} q \quad (15)$$

where

$$\begin{aligned} \mathcal{D} &= \gamma_\mu D_\mu + \mathcal{M} \\ D_\mu &= \partial_\mu + ig_{QCD} G_\mu + iL_\mu P_L + iR_\mu P_R \\ \mathcal{M} &= -m_Q (\Sigma^\dagger P_L + \Sigma P_R) \\ P_L &= \frac{1}{2}(1 - \gamma_5) \quad \text{and} \quad P_R = \frac{1}{2}(1 + \gamma_5) \end{aligned} \quad (16)$$

I use hermitian gamma matrices in Euclidean space. The fields L_μ and R_μ are the same as in the previous section. Gluonic corrections are not included here and I will drop the G_μ in the remainder of this paper. The quarks are a 3 vector in the flavour indices. Under the chiral $SU(3)_L \times SU(3)_R$ they transform as

$$\begin{aligned} q &\rightarrow (U_L P_L + U_R P_R)q \\ \bar{q} &\rightarrow \bar{q}(U_L^\dagger P_L + U_R^\dagger P_R). \end{aligned} \quad (17)$$

The parameter m_Q can be interpreted as a constituent quark mass. Chiral symmetry in the QCD Lagrangian is spontaneously broken due to the presence of this term¹. It can be easily checked using the transformation properties of Σ that the terms containing m_Q are chirally invariant. They do provide a mass m_Q for the quarks. This is essentially the same model as given by Georgi and Manohar [20] with $g_A = 1$. The only difference is that this one correctly includes the effects of the anomaly.

We now define the hatted quantities

$$\begin{aligned} \hat{\mathcal{M}} &= -m_Q(\Sigma P_L + \Sigma^\dagger P_R) \\ \hat{D}_\mu &= \partial_\mu + iR_\mu P_L + iL_\mu P_R. \end{aligned} \quad (18)$$

This corresponds to interchanging P_L and P_R in all formulas. The hat notation will be used for this interchange for other quantities too. Using this we get the hermitian conjugate of the action term

$$\mathcal{D}^\dagger = -\gamma_\mu \hat{D}_\mu + \hat{\mathcal{M}}. \quad (19)$$

For later convenience we also define

$$\begin{aligned} \mathcal{D}^\dagger \mathcal{D} &= -D^2 + Y + m_Q^2 \\ Z_{\mu\nu} &= D_\mu D_\nu - D_\nu D_\mu. \end{aligned} \quad (20)$$

In terms of other quantities this can be rewritten as

$$\begin{aligned} Y &= -\gamma_\mu D_\mu \mathcal{M} + \sigma_{\mu\nu} Z_{\mu\nu} \\ \sigma_{\mu\nu} &= -\frac{1}{4}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \\ Z_{\mu\nu} &= iL_{\mu\nu} P_L + iR_{\mu\nu} P_R \\ L(R)_{\mu\nu} &= \partial_\mu L(R)_\nu - \partial_\nu L(R)_\mu + i[L(R)_\mu, L(R)_\nu] \\ D_\mu \mathcal{M} &= -m_Q(D_\mu \Sigma^\dagger P_L + D_\mu \Sigma P_R) \\ D_\alpha Z_{\mu\nu} &= iD_\alpha L_{\mu\nu} P_L + iD_\alpha R_{\mu\nu} P_R \end{aligned} \quad (21)$$

Covariant derivatives of Σ , $\Sigma^\dagger$ and $L_{\mu\nu}$, $R_{\mu\nu}$ are those defined in the standard way. Higher derivatives are also defined similarly.

The effective action at low energies is defined by integrating out the quarks from the functional integral.

$$\exp(\Gamma_E(\Sigma, L_\mu, R_\nu)) = \int [dq][d\bar{q}] \exp\left(\int d^4x \mathcal{L}_{QCD}\right) \quad (22)$$

¹This point is elaborated in ref. [5].

This integral can formally be done resulting in

$$\Gamma_E = \log \det \mathcal{D} \quad (23)$$

and similarly for the complex conjugate of the action

$$\Gamma_E^* = \log \det \mathcal{D}^\dagger \quad (24)$$

From this we can naively define the real part Γ^+ and the imaginary part Γ^- . Then using the above definitions we get

$$\Gamma^+ = \frac{1}{2} \log \det \mathcal{D}^\dagger \mathcal{D} \quad (25)$$

We now use² $\log \det A = \text{tr} \log A$ and the proper time definition of the logarithm to obtain [19]

$$\Gamma^+ = -\frac{1}{2} \int d^4x \text{tr} \int_0^\infty \frac{d\tau}{\tau} \rho(\tau, \mu) \left(e^{-\tau \mathcal{D}^\dagger \mathcal{D}} - e^{-\tau} \right) \quad (26)$$

The function $\rho(\tau, \mu)$ is the regulator. For $\mu = \infty$ it is 1. For μ large, it is 1 for large τ and it must converge to zero for τ near zero fast enough to make the integral above well defined. This can be evaluated using the Seeley-DeWitt coefficients. These are defined (in the coincidence limit) by

$$\langle x | e^{-\tau(\mathcal{D}^\dagger \mathcal{D} - m_Q^2)} | x \rangle = \frac{1}{16\pi^2 \tau^2} \sum_{n=0}^\infty a_n(x, x) \tau^n \quad (27)$$

Putting this into (26) we see that regularization ambiguities only affect the terms with $n = 0, 1, 2$. All other terms are finite. Choosing a renormalization then leads to

$$\begin{aligned} \Gamma^+ = -\frac{1}{32\pi^2} \int d^4x & \left[\sum_{n \geq 3} \frac{(n-3)!}{(m_Q^2)^{n-2}} a_n(x, x) + \frac{1}{2} m_Q^4 \left(\frac{3}{2} + \log \frac{\mu^2}{m_Q^2} \right) a_0(x, x) \right. \\ & \left. - m_Q^2 \left(1 + \log \frac{\mu^2}{m_Q^2} \right) a_1(x, x) + \log \frac{\mu^2}{m_Q^2} a_2(x, x) \right] \end{aligned} \quad (28)$$

Here μ is an ultra-violet cutoff parameter.

At this point I would like to make a digression and discuss the concept of intrinsic parity. We can define a parity that only acts on space-time indices by :

$$\begin{aligned} x'_0 &= -x_0 \\ L'_0(x') &= -L_0(x) \quad R'_0(x') = -R_0(x) \end{aligned} \quad (29)$$

with all other quantities unchanged. This is **NOT** a symmetry of the QCD action. Using the definition of $\mathcal{D}$ we can see that

$$\mathcal{D}(x) \rightarrow \gamma_0 \mathcal{D}^\dagger(x') \gamma_0 \quad \text{and} \quad \mathcal{D}^\dagger(x) \rightarrow \gamma_0 \mathcal{D}(x') \gamma_0 \quad (30)$$

²The det is over all degrees of freedom so the trace is a tr over Dirac, colour and flavour indices and an integral over space.

From the definitions (23) it can be seen that the real and imaginary parts of the effective action transform as

$$\Gamma^\pm \rightarrow \pm \Gamma^\pm. \quad (31)$$

As a consequence Γ^+ must contain an even number of Levi-Civita tensors $\epsilon_{\mu\nu\alpha\beta}$ and Γ^- an odd number. Using the fact that the effective action has to be invariant under the full parity transformation we can then derive that $\Gamma^\pm$ has the same behaviour under the intrinsic parity operation as defined by Witten [7]

$$L_\mu \leftrightarrow R_\mu \quad \text{and} \quad \Sigma \leftrightarrow \Sigma^\dagger. \quad (32)$$

The effective actions also have the same behaviour under intrinsic parity independent of the value of τ and m_Q . So the integral of every a_n contains no $\epsilon_{\mu\nu\alpha\beta}$ independent of the order of m_Q . We will exploit this fact to simplify calculations in the next section.

We can now use the formulas for $a_n(x, x)$ given in appendix A and the values for Y and $Z_{\mu\nu}$ given above to get the action to $\mathcal{O}(1)$ in an expansion in m_Q^{-2} . The result coincides with that obtained previously [3, 4]. I only give the terms that depend on Σ . The trace over colours only adds the overall factor of N_C .

$$\begin{aligned} \Gamma^+ = & -\frac{N_C}{32\pi^2} \int d^4x \text{tr} \{ m_Q^2 \log \frac{\mu^2}{m_Q^2} 2D_\mu \Sigma^\dagger D_\mu \Sigma - i(D_\mu \Sigma D_\nu \Sigma^\dagger L_{\mu\nu} + D_\mu \Sigma^\dagger D_\nu \Sigma R_{\mu\nu}) \\ & + \frac{1}{3} D_\mu \Sigma^\dagger D_\mu \Sigma D_\nu \Sigma^\dagger D_\nu \Sigma - \frac{1}{6} D_\mu \Sigma^\dagger D_\nu \Sigma D_\mu \Sigma^\dagger D_\nu \Sigma - \frac{1}{3} D_\mu D_\nu \Sigma^\dagger D_\mu D_\nu \Sigma \} \end{aligned} \quad (33)$$

This can be rewritten in the notation of ref. [1] using partial integration and the equations of motion and the $SU(3)$ relation quoted in ref. [1].

At this point it is appropriate to mention that the derivative expansion coincides here with an expansion in m_Q^{-2} . The reason is that in the effective action the only dimensional quantities are the derivatives and the external vector fields. This does not coincide with the expansion in Seeley-DeWitt coefficients because Y has a term of $\mathcal{O}(m_Q)$ and one of $\mathcal{O}(m_Q^0)$. E.g., the result for the order 4 Lagrangian above has contributions from a_3 and a_4 . In ref. [5] this was used to give an improved size estimate of the terms in the effective action. The improvement was by including the factorial of the a_n expansion in the estimate of the size of the counterterms.

4 The anomalous intrinsic parity effective action.

In this section I extend the methods used in the previous section to the part of the effective action containing $\epsilon_{\mu\nu\alpha\beta}$. A direct definition in terms of the proper time expression for the determinant is not possible here because of ambiguities in the regularization procedure. We will use the definition given by Ball [19]. There it was shown exactly how to define it explicitly. I will give a quick overview of it here. It can also be shown that these terms do not depend on the specific regulator chosen. This ambiguity is only present in the normal parity action. Since we are only interested in the terms of $\mathcal{O}(m_Q^{-2})$ I will be sloppy about subtracting the terms with $m_Q \rightarrow \infty$. This is important to get the original Wess-Zumino term correctly but does not effect the terms needed here. The subscript ϵ means that only

terms containing an odd number of Levi-Civita tensors should be kept. This can always be done by subtracting the same expression with $\mathcal{D}$ and $\mathcal{D}^\dagger$ interchanged.

From (23) we have

$$\Gamma^- = -(\log \det \mathcal{D}^\dagger)_\epsilon. \quad (34)$$

We can always subtract a constant from the effective action. Here we subtract the effective action with $\mathcal{D} = \gamma_\mu \partial_\mu - m_Q$. We define then a family of $\mathcal{D}(t)$ with $\mathcal{D}(1) = \mathcal{D}$ and $\mathcal{D}(0) = \gamma_\mu \partial_\mu - m_Q$. This together with $\delta \log \det A = \text{tr} \delta A A^{-1}$ leads to ³

$$\Gamma^- = \int_0^1 dt \int d^4 x \text{tr} \left(\frac{\partial \mathcal{D}^\dagger(t)}{\partial t} \mathcal{D}^{\dagger-1} \right)_\epsilon. \quad (35)$$

Using $\mathcal{D}^{\dagger-1} = \mathcal{D}(\mathcal{D}^\dagger \mathcal{D})^{-1}$ and $A^{-1} = \int_0^\infty d\tau \exp(-\tau A)$ we obtain

$$\Gamma^- = - \int_0^1 dt \int d^4 x \int_0^\infty d\tau \left\{ \text{tr} \left(\frac{\partial \mathcal{D}^\dagger}{\partial t} \mathcal{D} e^{-\tau \mathcal{D}^\dagger \mathcal{D}} \right) \right\}_\epsilon \quad (36)$$

We now call the expression in (36) $\Gamma(L_\mu, R_\mu, \mathcal{M})$. To properly reproduce the anomalous behaviour (see ref. [19]) we need to subtract the same expression with $\mathcal{M} = -\infty 1$. So the same expression with L_μ and R_μ unchanged but $\Sigma = 1$ and $m_Q \rightarrow \infty$. The correct expression for the abnormal intrinsic parity action is

$$\Gamma^- = \Gamma(L_\mu, R_\mu, \mathcal{M}) - \Gamma(L_\mu, R_\mu, -\infty 1) \quad (37)$$

The integral here is over τ and a cylinder in 5 space with 4 space as boundary. We now add and subtract the same integral with external fields $L_\mu, R_\mu \neq 0$ and $\Sigma \neq 1$ but with $m_Q \rightarrow \infty$.

$$\Gamma^- = \{\Gamma(L_\mu, R_\mu, \mathcal{M}) - \Gamma(L_\mu, R_\mu, \infty \mathcal{M})\} + \{\Gamma(L_\mu, R_\mu, \infty \mathcal{M}) - \Gamma(L_\mu, R_\mu, -\infty 1)\}. \quad (38)$$

Using the fact that we need $\epsilon_{\mu\nu\alpha\beta}$ in Γ^- we can count powers of m_Q and see that there are no positive powers of m_Q anywhere. This was, of course, needed to have sensible $m_Q \rightarrow \infty$ limits. So the first pair of brackets only contains terms that are suppressed by powers of m_Q and the second pair of brackets contains terms only of order m_Q^0 . This is the Wess-Zumino term ⁴.

Let us now look closer to the first two terms in (38). The first term is an integral from $\mathcal{D}(0)_1 = \gamma_\mu \partial_\mu - m_Q$ to $\mathcal{D}(1)_1 = \gamma_\mu D_\mu + \mathcal{M}$ while the second term is from $\mathcal{D}(0)_2 = \gamma_\mu \partial_\mu - \infty$ to $\mathcal{D}(1)_2 = \gamma_\mu D_\mu + \infty \mathcal{M}$. The difference between the expression with $\mathcal{D}(0)_1$ and $\mathcal{D}(0)_2$ does not depend on L_μ, R_μ and Σ so it can be dropped and we can integrate from $\mathcal{D}(1)_2$ to $\mathcal{D}(1)_1$. This can be done by defining $\mathcal{D}(t') = \gamma_\mu D_\mu + t'^{-1} \mathcal{M}$. So

$$\Gamma^- = \Gamma_{WZ} - \int d^4 x \int_0^1 dt' \int_0^\infty d\tau \text{tr} \left(\frac{\partial \mathcal{D}^\dagger}{\partial t'} \mathcal{D} e^{-\tau \mathcal{D}^\dagger \mathcal{D}} \right)_\epsilon \quad (39)$$

³The determinant is over all degrees of freedom so the trace is a tr over Dirac, colour and flavour indices and an integral over space.

⁴This argument has to be extended for the case of explicit chiral symmetry breaking by current quark masses.

We now replace the exponent by its Seeley-DeWitt expansion and perform the τ integration explicitly. Here it should be kept in mind that it now contains $\exp(-m_Q^2 \tau/t'^2)$. All terms in the t' integral are suppressed by at least one power of m_Q . So the t' integral is well defined and we can do the integration directly. The subscript r in (40) means terms of order m_Q^{-r} .

$$\Gamma^- = \Gamma_{WZ} + \int d^4x \sum_{n=3}^{\infty} \sum_{r=1}^{\infty} \frac{(n-2)!}{16\pi^2 r} \left[\frac{1}{m_Q^{2n-2}} \text{tr} \hat{\mathcal{M}} \mathcal{D}a_n \right]_{r\epsilon} \quad (40)$$

This equation is the main result of this section. In the remainder we will simplify this formula even further.

Here we have to look at some properties of the Seeley-DeWitt coefficients. The coefficients $\mathcal{D}a_n$ are given in terms of more elementary quantities in the coincidence limit by

$$\mathcal{D}a_n = \frac{1}{2} \gamma_\mu (\overline{D}_\mu a_n + D_\mu(a_n)) + \mathcal{M}a_n \quad (41)$$

The last term never gives rise to terms with $\epsilon_{\mu\nu\alpha\beta}$. Since $\hat{\mathcal{M}}\mathcal{M} = m_Q^2$, that only contributes like $\sum_r c_{rn} \text{tr}(a_n)_r$ to the effective action. c_{rn} just indicates that the factor in front depends on n and the order in m_Q . Γ^+ was also a sum over $\text{tr}a_n$ and the argument why Γ^+ does not contain $\epsilon_{\mu\nu\alpha\beta}$ was independent of τ and m_Q . Therefore this term does not contribute to Γ^- .

A similar but more involved argument can be given for the term containing $D_\mu a_n$. I will only sketch it here. For the terms used in the next section I have explicitly checked it. Commuting the γ_μ through $\hat{\mathcal{M}}$ and then performing a partial derivation makes that term proportional to $\text{tr}a_n \gamma_\mu D_\mu \mathcal{M}$. We can then use that (see ref. [19], appendix A)

$$a_n = -\frac{\delta}{\delta Y} \int d^4x \text{tr} a_{n+1} \quad (42)$$

and the fact that $Y = -D_\mu \mathcal{M} + \sigma_{\mu\nu} Z_{\mu\nu}$, to prove that (see below)

$$\text{tr} \gamma_\mu D_\mu(a_n) \hat{\mathcal{M}} = \sum_r d_{rn} \text{tr}(a_{n+1})_r \quad (43)$$

Here the equality is with a different coefficient for terms of a different order in m_Q . We then once more use the argument for Γ^+ to drop this term.

The final result of this section is

$$\Gamma^- = \Gamma_{WZ} + \int d^4x \sum_{n=3}^{\infty} \sum_{r=1}^{\infty} \frac{(n-2)!}{32\pi^2 r} \left[\frac{1}{m_Q^{2n-2}} \text{tr} \gamma_\mu \overline{D}_\mu a_n \hat{\mathcal{M}} \right]_{r\epsilon} \quad (44)$$

Let me repeat that the subscript r means terms of order m_Q^{-r} and the subscript ϵ means that only terms containing an odd number of Levi-Civita tensors should be kept.

Let me now prove eq. (43). We want to prove that for $Y = a + b$

$$\text{tr} \left(a \frac{\delta f(Y)}{\delta Y} \right) = \sum_n \alpha_n \text{tr} f(Y)_n \quad (45)$$

Here the α_n are constants and $f(Y)_n$ is the term of order a^n in $f(Y)$. For the terms in f of at most one Y this is obvious with $\alpha_1 = 1$ and $\alpha_0 = 0$. If we now look at the terms in f with two Y . The left-hand side does not contain any terms without a so $\alpha_0 = 0$ again. Terms

with one a are on the right-hand side first $Y = a$ and second $Y = b$ and vice versa. On the left-hand side we have first the derivative with respect to the first Y and the remaining $Y = b$ and vice versa, so again $\alpha_1 = 1$. Terms of order a^2 are now also possible. On the right-hand side there is only one with both Y set equal to a . On the left-hand side we get that term twice, first with the derivative on the first Y and the second $Y = a$ and vice versa so $\alpha_2 = 2$. The argument can now be easily generalized to higher order in Y to prove eq. (45) with $\alpha_n = n$.

One remark is in order here. The action except for the Wess-Zumino term could be written explicitly in 4 dimensions and in gauge invariant quantities showing that the Adler anomaly non-renormalization theorem has not been violated.

5 The QCD $\mathcal{O}(p^6)$ effective Lagrangian.

In this section I use the main formula from section 4 to obtain the abnormal intrinsic parity effective action at next-to-leading order. There are only contributions from $\overline{D}_\mu a_3$, $\overline{D}_\mu a_4$ and $\overline{D}_\mu a_5$. As we will see $\overline{D}_\mu a_3$ contributes to processes with at least two external vector fields and one pseudoscalar meson. $\overline{D}_\mu a_4$ contains processes with at least one external field and three pseudoscalars and $\overline{D}_\mu a_5$ to processes with at least 5 pseudoscalars.

There is another hidden source of contributions to Γ^- . In rewriting the normal parity action into the standard form we have used the equations of motion. These do contain terms of both normal and intrinsic parity⁵ and can thus produce terms of abnormal intrinsic parity if used in Γ^+ . In section 3 we derived the effective action. To rewrite it in the notation of ref.[1] we only need to use the equations of motion on the last term.

Using partial integration and commuting covariant derivatives we have

$$\text{tr} D_\mu D_\nu \Sigma^\dagger D_\mu D_\nu \Sigma = \text{tr} D^2 \Sigma^\dagger D^2 \Sigma + \text{terms with external fields} \quad (46)$$

Then we rewrite it as

$$\frac{1}{4} \text{tr} \left(\Sigma D^2 \Sigma^\dagger + D^2 \Sigma \Sigma^\dagger \right)^2 - \frac{1}{4} \text{tr} \left(\Sigma D^2 \Sigma^\dagger - D^2 \Sigma \Sigma^\dagger \right)^2 + \dots \quad (47)$$

The first term is

$$\text{tr} \left[\left(D_\mu \Sigma D_\mu \Sigma^\dagger \right)^2 \right] \quad (48)$$

The second term has to be removed by the equations of motion. These are in the chiral limit of the form

$$\Sigma D^2 \Sigma^\dagger - D^2 \Sigma \Sigma^\dagger = \epsilon_{\mu\nu\alpha\beta} A_{\mu\nu\alpha\beta} \quad (49)$$

so that the extra term contains two $\epsilon_{\mu\nu\alpha\beta}$ and is of order p^8 . So it does not contribute to the order we are interested in either. In case explicit chiral symmetry breaking is taken into account this will contribute at order p^6 to Γ^- .

⁵The full equation of motion can be derived quickly from the first variation of the action including the Wess-Zumino term calculated in ref. [14].

I will drop the subscript ϵ signifying keep only terms with an odd number of $\epsilon_{\mu\nu\alpha\beta}$ here. We only consider terms with an odd number of γ_5 in the traces.

There are no contributions from $\overline{D}_\mu a_2$. From (44) it can be seen that those terms are suppressed by $m_Q^{-2}m_Q = m_Q$, or at most by m_Q^{-1} . This is however not possible because derivatives have to come in pairs. Similarly the leading term in the term with $\overline{D}_\mu a_6$ is $m_Q^{-10}m_Q^7$ or suppressed by one factor of m_Q more than we consider here.

Let me first look at the contribution from $\overline{D}_\mu a_5$. We can replace Y here by $-\gamma_\mu D_\mu \mathcal{M}$. So the first term contains

$$\begin{aligned} \text{tr} \gamma_\mu \overline{D}_\mu a_5 \hat{\mathcal{M}} = & \frac{(-1)^5}{360} \text{tr} [2\gamma_\mu \gamma_\nu D_\nu \mathcal{M} \gamma_\alpha D_\alpha \mathcal{M} \gamma_\beta D_\beta \mathcal{M} \gamma_\gamma D_\gamma \mathcal{M} \gamma_\delta D_\delta \mathcal{M} \hat{\mathcal{M}} \\ & - 2\gamma_\mu \gamma_\delta D_\mu D_\delta \mathcal{M} \gamma_\nu D_\nu \mathcal{M} \gamma_\alpha D_\alpha \mathcal{M} \gamma_\beta D_\beta \mathcal{M} \gamma_\gamma D_\gamma \mathcal{M} \hat{\mathcal{M}} + \dots] \end{aligned} \quad (50)$$

We now use $\gamma_\alpha \mathcal{M} = \hat{\mathcal{M}} \gamma_\alpha$ and $D_\mu \mathcal{M} \hat{\mathcal{M}} = -\mathcal{M} D_\mu \hat{\mathcal{M}}$. We also rename a few of the indices and obtain

$$\begin{aligned} \text{tr} \gamma_\mu \overline{D}_\mu a_5 = & \frac{-2}{360} \text{tr} (D_\mu D_\nu \hat{\mathcal{M}} \mathcal{M} D_\alpha \hat{\mathcal{M}} D_\beta \mathcal{M} D_\gamma \hat{\mathcal{M}} D_\delta \mathcal{M} \gamma_\nu \gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\delta \\ & - D_\mu D_\nu \mathcal{M} D_\alpha \hat{\mathcal{M}} D_\beta \mathcal{M} D_\gamma \hat{\mathcal{M}} D_\delta \mathcal{M} \hat{\mathcal{M}} \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\delta) + \dots \end{aligned} \quad (51)$$

Here we have to keep in mind all quantities contain $P_{L,R}$. We can interchange all P_L and P_R in any particular term if we change its sign. This will be very useful for the later calculations. So our expression can be rewritten as

$$\frac{-2}{360} \text{tr} (D_\mu D_\nu \hat{\mathcal{M}} \mathcal{M} D_\alpha \mathcal{M} D_\beta \hat{\mathcal{M}} D_\gamma \mathcal{M} D_\delta \hat{\mathcal{M}} \mathcal{M} \{\gamma_\mu, \gamma_\nu\} \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\delta) \quad (52)$$

Here the γ algebra can be easily done. Remember that only terms containing $\epsilon_{\mu\nu\alpha\beta}$ need to be kept. We can now define new quantities $\mathcal{M}$ and $Z_{\mu\nu}$ and the like by defining a new P_L and P_R with $\text{tr} P_R = 1$, $\text{tr} P_L = -1$, $P_L P_R = 0$ and $P_{L,R}^2 = P_{L,R}$. Then we can rewrite our result as

$$\frac{-16}{360} \epsilon_{\mu\nu\alpha\beta} \text{tr} (\mathcal{M} D^2 \hat{\mathcal{M}} D_\mu \mathcal{M} D_\nu \hat{\mathcal{M}} D_\alpha \mathcal{M} D_\beta \hat{\mathcal{M}}) \quad (53)$$

We can now use the interchange of P_L and P_R to rewrite it as $\text{tr} (\mathcal{M} D^2 \hat{\mathcal{M}} - D^2 \mathcal{M} \hat{\mathcal{M}}) \dots$. This then vanishes because of the equations of motion for Σ . This calculation was carried out explicitly to show the general simplifying techniques used to obtain the remainder of this section.

Another formula that was used extensively is the one stating that there are no fully antisymmetric combinations of 5 indices in 4 dimensions :

$$\epsilon_{\mu\nu\alpha\beta} \delta_{\gamma\delta} - \epsilon_{\gamma\nu\alpha\beta} \delta_{\mu\delta} - \epsilon_{\mu\gamma\alpha\beta} \delta_{\nu\delta} - \epsilon_{\mu\nu\gamma\beta} \delta_{\alpha\delta} - \epsilon_{\mu\nu\alpha\gamma} \delta_{\beta\delta} = 0. \quad (54)$$

The contribution from $\overline{D}_\mu a_5$ to the effective action is

$$\begin{aligned} \Gamma^-(a_5) = & \frac{N_C}{32\pi^2} \int d^4x \frac{-\epsilon_{\mu\nu\alpha\beta}}{30} \\ & \text{tr} \left\{ \left(\Sigma D_\gamma D_\mu \Sigma^\dagger - D_\gamma D_\mu \Sigma \Sigma^\dagger \right) D_\nu \Sigma \left(D_\alpha \Sigma^\dagger D_\gamma \Sigma - D_\gamma \Sigma^\dagger D_\alpha \Sigma \right) D_\beta \Sigma^\dagger \right\} \end{aligned} \quad (55)$$

⁶These relations follow respectively from the anticommutation of γ_5 and γ_μ and the same expression for Σ .

Here it can be easily seen that this part of Γ^- describes processes with at least 5 pseudoscalars. The overall factor of N_C comes from the trace over colours. The only trace left now is the one over flavours.

A similar but more involved calculation gives the part coming from $\overline{D}_\mu a_4$. I have not rewritten it in the set of operators given by [15]. Given the sheer number of terms involved it is easier to calculate directly the different physical processes from the form of the action given here.

$$\begin{aligned}
\Gamma^-(a_4) = \frac{N_C}{32\pi^2 m_Q^2} \int d^4x \frac{i}{180} \epsilon_{\mu\nu\alpha\beta} \text{tr} \bigg\{ & 2(D_\gamma R_{\gamma\mu} + \Sigma^\dagger L_{\gamma\mu} \Sigma) D_\nu \Sigma^\dagger D_\alpha \Sigma D_\beta \Sigma^\dagger \Sigma \\
& -3(D_\gamma R_{\mu\nu} + \Sigma^\dagger D_\gamma L_{\mu\nu} \Sigma) D_\alpha \Sigma^\dagger D_\gamma \Sigma D_\beta \Sigma^\dagger \Sigma \\
& + R_{\mu\gamma} [20 D_\nu \Sigma^\dagger D_\alpha \Sigma \Sigma^\dagger D_\gamma D_\beta \Sigma - 20 D_\gamma D_\nu \Sigma^\dagger \Sigma D_\alpha \Sigma^\dagger D_\beta \Sigma \\
& - 2 D_\nu \Sigma^\dagger D_\alpha \Sigma D_\gamma D_\beta \Sigma^\dagger \Sigma + 2 \Sigma^\dagger D_\gamma D_\nu \Sigma D_\alpha \Sigma^\dagger D_\beta \Sigma \\
& - 8 D_\nu \Sigma^\dagger (\Sigma D_\gamma D_\alpha \Sigma^\dagger - D_\gamma D_\alpha \Sigma \Sigma^\dagger) D_\beta \Sigma] \\
& - L_{\mu\gamma} [20 D_\nu \Sigma D_\alpha \Sigma^\dagger \Sigma D_\gamma D_\beta \Sigma^\dagger - 20 D_\gamma D_\nu \Sigma \Sigma^\dagger D_\alpha \Sigma D_\beta \Sigma^\dagger \\
& - 2 D_\nu \Sigma D_\alpha \Sigma^\dagger D_\gamma D_\beta \Sigma \Sigma^\dagger + 2 \Sigma D_\gamma D_\nu \Sigma^\dagger D_\alpha \Sigma D_\beta \Sigma^\dagger \\
& - 8 D_\nu \Sigma (\Sigma^\dagger D_\gamma D_\alpha \Sigma - D_\gamma D_\alpha \Sigma^\dagger \Sigma) D_\beta \Sigma^\dagger] \\
& + 3 R_{\mu\nu} [D_\gamma D_\alpha \Sigma^\dagger \Sigma D_\beta \Sigma^\dagger D_\gamma \Sigma + D_\alpha \Sigma^\dagger D_\gamma D_\beta \Sigma \Sigma^\dagger D_\gamma \Sigma \\
& - D_\gamma \Sigma^\dagger \Sigma D_\gamma D_\alpha \Sigma^\dagger D_\beta \Sigma - D_\gamma \Sigma^\dagger D_\alpha \Sigma \Sigma^\dagger D_\gamma D_\beta \Sigma] \\
& - 3 L_{\mu\nu} [D_\gamma D_\alpha \Sigma \Sigma^\dagger D_\beta \Sigma D_\gamma \Sigma^\dagger + D_\alpha \Sigma D_\gamma D_\beta \Sigma^\dagger \Sigma D_\gamma \Sigma^\dagger \\
& - D_\gamma \Sigma \Sigma^\dagger D_\gamma D_\alpha \Sigma D_\beta \Sigma^\dagger - D_\gamma \Sigma D_\alpha \Sigma^\dagger \Sigma D_\gamma D_\beta \Sigma^\dagger] \\
& - 2 D^2 D_\mu \Sigma \{ R_{\nu\alpha} \Sigma^\dagger - \Sigma^\dagger L_{\nu\alpha} \Sigma D_\beta \Sigma^\dagger \} + 2 D^2 D_\mu \Sigma^\dagger \{ L_{\nu\alpha} \Sigma - \Sigma R_{\nu\alpha} \Sigma^\dagger D_\beta \Sigma \} \\
& + i 2 D_\mu D_\gamma D_\nu \Sigma [2 D_\gamma D_\alpha \Sigma^\dagger \Sigma D_\beta \Sigma^\dagger + 2 D_\beta \Sigma^\dagger \Sigma D_\gamma D_\alpha \Sigma^\dagger \\
& + \Sigma^\dagger D_\gamma D_\alpha \Sigma D_\beta \Sigma^\dagger + D_\beta \Sigma^\dagger D_\gamma D_\alpha \Sigma \Sigma^\dagger] \\
& - i 2 D_\mu D_\gamma D_\nu \Sigma^\dagger [2 D_\gamma D_\alpha \Sigma \Sigma^\dagger D_\beta \Sigma + 2 D_\beta \Sigma \Sigma^\dagger D_\gamma D_\alpha \Sigma \\
& + \Sigma D_\gamma D_\alpha \Sigma^\dagger D_\beta \Sigma + D_\beta \Sigma D_\gamma D_\alpha \Sigma^\dagger \Sigma] \bigg\} \quad (56)
\end{aligned}$$

Here we can see that this part of Γ^- contributes to processes with at least one external field and three pseudoscalars. That it is also true for the last terms can be seen from the relation

$$\begin{aligned}
& -i 2 \epsilon_{\mu\nu\alpha\beta} D_\mu D_\gamma D_\nu \Sigma = \\
& \epsilon_{\mu\nu\alpha\beta} (2 L_{\mu\gamma} D_\nu \Sigma - 2 D_\nu \Sigma R_{\mu\gamma} + D_\gamma L_{\mu\nu} \Sigma + L_{\mu\nu} D_\gamma \Sigma - D_\gamma \Sigma R_{\mu\nu} - \Sigma D_\gamma R_{\mu\nu}) \quad (57)
\end{aligned}$$

and the corresponding one for $\Sigma^\dagger$.

The last part of Γ^- is the part coming from the third Seeley-DeWitt coefficient.

$$\begin{aligned}
\Gamma^-(a_3) = \frac{N_C}{32\pi^2 m_Q^2} \int d^4x \frac{-1}{120} \epsilon_{\mu\nu\alpha\beta} \text{tr} \bigg\{ & 7 \{ D_\gamma R_{\gamma\mu}, R_{\alpha\beta} \} D_\nu \Sigma^\dagger \Sigma - 7 \{ D_\gamma L_{\gamma\mu}, L_{\alpha\beta} \} D_\nu \Sigma \Sigma^\dagger \\
& \dots \bigg\}
\end{aligned}$$

$$\begin{aligned}
& +2D_\gamma R_{\gamma\mu}(\Sigma^\dagger L_{\alpha\beta} D_\nu \Sigma - D_\nu \Sigma^\dagger L_{\alpha\beta} \Sigma) - 2D_\gamma L_{\gamma\mu}(\Sigma R_{\alpha\beta} D_\nu \Sigma^\dagger - D_\nu \Sigma R_{\alpha\beta} \Sigma^\dagger) \\
& +3\{R_{\mu\gamma}, R_{\alpha\beta}\}(\Sigma^\dagger D_\gamma D_\nu \Sigma - D_\gamma D_\nu \Sigma^\dagger \Sigma) - 3\{L_{\mu\gamma}, L_{\alpha\beta}\}(\Sigma D_\gamma D_\nu \Sigma^\dagger - D_\gamma D_\nu \Sigma \Sigma^\dagger) \\
& +4L_{\mu\gamma}(D_\gamma D_\nu \Sigma R_{\alpha\beta} \Sigma^\dagger - \Sigma R_{\alpha\beta} D_\gamma D_\nu \Sigma^\dagger) - 4R_{\mu\gamma}(D_\gamma D_\nu \Sigma^\dagger L_{\alpha\beta} \Sigma - \Sigma^\dagger L_{\alpha\beta} D_\gamma D_\nu \Sigma) \\
& +2L_{\mu\gamma}(D_\nu \Sigma D_\gamma R_{\alpha\beta} \Sigma^\dagger - \Sigma D_\gamma R_{\alpha\beta} D_\nu \Sigma^\dagger) - 2R_{\mu\gamma}(D_\nu \Sigma^\dagger D_\gamma L_{\alpha\beta} \Sigma - \Sigma^\dagger D_\gamma L_{\alpha\beta} D_\nu \Sigma) \\
& -13\{R_{\mu\gamma}, D_\gamma R_{\alpha\beta}\} D_\nu \Sigma^\dagger \Sigma + 13\{L_{\mu\gamma}, D_\gamma L_{\alpha\beta}\} D_\nu \Sigma \Sigma^\dagger \\
& +[R_{\alpha\beta}, R_{\mu\gamma}](\Sigma^\dagger D_\gamma D_\nu \Sigma + D_\gamma D_\nu \Sigma^\dagger \Sigma) \\
& -[L_{\alpha\beta}, L_{\mu\gamma}](\Sigma D_\gamma D_\nu \Sigma^\dagger + D_\gamma D_\nu \Sigma \Sigma^\dagger) \Big\} \quad (58)
\end{aligned}$$

It is obvious from the above equation that this part only contributes to processes with at least two external vector fields and one pseudoscalar.

The full order p^6 expression is the sum of equations (55,56) and (58). This is the main result of this paper. It is the equivalent of the action derived in [3, 4] for the normal parity part. Here it was necessary to pay more careful attention to the definition of the effective action to get a well-defined expression. It should be kept in mind that the above equations are in Euclidean quantities. Changing to Minkowski quantities can be done simply by multiplying by i .

6 Application to $\pi^0, \eta \rightarrow \gamma l^+ l^-$.

Now we can look at the decays $\pi^0, \eta \rightarrow \gamma l^+ l^-$. I only consider this decay and not the direct two photon decay because we work in the chiral limit here and the decay rate also gets contributions from chiral symmetry breaking terms. The slope at small lepton pair masses is at this order a pure prediction. It already contains six derivatives in its expression so any contribution involving the quark masses is at least $\mathcal{O}(p^8)$.

We will include the singlet η_1 component here as well. This can be done by adding the singlet to the matrix M of section 2. This is the nonet approximation for the couplings of the η' . That approximation works well if we add the extra mass for the singlet component to reproduce the physically observed masses.

We now set

$$M = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} + \frac{\eta_1}{\sqrt{3}} & & \\ & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} + \frac{\eta_1}{\sqrt{3}} & \\ & & -\sqrt{\frac{2}{3}}\eta_8 + \frac{\eta_1}{\sqrt{3}} \end{pmatrix}, \quad (59)$$

Now we also set

$$L_\mu = R_\mu = eA_\mu Q. \quad (60)$$

This means that all quantities in eq. (58) commute.

$$\begin{aligned}
\Gamma^-(M \rightarrow \gamma\gamma) &= \frac{-iN_C e^2}{960\pi^2 m_Q^2 f} \epsilon_{\mu\nu\alpha\beta} \left(20F_{\alpha\beta} F_{\mu\gamma} \text{tr}(Q^2 \partial_\gamma \partial_\nu M) \right. \\
&\quad \left. -10\partial_\gamma F_{\gamma\mu} F_{\alpha\beta} \text{tr}(Q^2 \partial_\nu M) + 30\partial_\gamma F_{\alpha\beta} F_{\mu\gamma} \text{tr}(Q^2 \partial_\nu M) \right) \quad (61)
\end{aligned}$$

This we can then rewrite in a simpler form using⁷ $\epsilon_{\mu\nu\alpha\beta}D_\gamma F_{\mu\nu} = 2\epsilon_{\mu\nu\alpha\beta}D_\mu F_{\gamma\nu}$ and partial derivation. The amplitude then becomes

$$\Gamma^-(M \rightarrow \gamma\gamma) = \frac{-iN_C e^2 \epsilon_{\mu\nu\alpha\beta}}{960\pi^2 m_Q^2 f} 10\partial^2 F_{\mu\nu} F_{\alpha\beta} \text{tr} Q^2 M \quad (62)$$

We have used the equations of motion several times in the previous section. Here they correspond to $\partial^2 M = 0$ so we also drop these terms. We now rewrite it in Minkowski space. This amounts to multiplying by i . Looking at the decay with one photon off-shell ($k^{*2} \neq 0$) and adding the lowest order contribution this leads to

$$A(M \rightarrow \gamma\gamma^*) = \frac{i\sqrt{2}e^2}{4\pi^2 f_M} C_M \epsilon^{\mu\nu\alpha\beta} \epsilon_\mu k_\nu \epsilon_\alpha^* k_\beta^* \left[1 + \frac{k^{*2}}{12m_Q^2} \right] \quad (63)$$

We can now compare this value with the experimental values for the slope. The measured slope in $\pi^0 \rightarrow \gamma e^+ e^-$ is negative with a rather large error [21].

$$\frac{1}{12m_Q^2} = -1.6 \pm 1.6 \text{ GeV}^{-2}. \quad (64)$$

This would lead to the nonsense result of an imaginary constituent quark mass. Therefore we only take the value of the slope from the corresponding decay $\eta \rightarrow \gamma\mu^+\mu^-$ which is better known [22]. The slope from our formula is the same for η_8 and η_1 . In the limit of N_C large our result is the leading term and this gives :

$$\frac{1}{12m_Q^2} = 1.9 \pm 0.4 \text{ GeV}^{-2}. \quad (65)$$

This translates in a value for m_Q of

$$m_Q = 210 \pm 25 \text{ MeV}. \quad (66)$$

Including the effect of the loops (which are non-leading in N_C) changes this value to

$$m_Q = 225 \pm 25 \text{ MeV}. \quad (67)$$

The same result can be obtained directly by comparing our result with the result from vector meson dominance.⁸

$$m_Q = \frac{m_\rho}{\sqrt{12}} \quad (68)$$

This has to be compared with other determinations of m_Q . In ref. [5] an estimate from the vacuum expectation values from QCD sum rules was given, $m_Q \approx 320 \text{ MeV}$. This is rather higher than the value we obtain but both estimates have rather large errors. The gluonic corrections to the slope could be sizable. The number obtained is, however, a reasonable value when compared to the other determination.

⁷The same relation holds true for $L_{\mu\nu}$ and $R_{\mu\nu}$ in general and was used in the previous section to simplify formulas.

⁸This was noted already a long time ago where this contribution to the slope was calculated using Feynman diagrams[23].

7 Conclusions

In this paper we have calculated from a QCD model the corrections to the abnormal intrinsic parity low energy effective Lagrangian. As expected on general grounds this could be written in a manifestly chirally invariant form. As a byproduct we have rederived the formula for the effective action given by Ball [19] and simplified it considerably. This formula can be extended to other dimensions than four in a straightforward way.

We have then used the derived effective action to obtain a value for the parameter m_Q in the QCD model used. The value obtained was in reasonable agreement with other determinations of it though a bit smaller.

Acknowledgements

The author would like to thank A. Pich and E. de Rafael for encouragement and useful conversations.

Appendix A Seeley-DeWitt coefficients.

In this appendix I give the expressions for the coincidence limits of the Seeley-DeWitt coefficients used here. I only give the coincidence limit. They are in terms of the Y and $Z_{\mu\nu}$ defined in the text. For the a_i I only give the terms up to cyclicity of the trace and total derivatives. This is in fact the only part needed for section 3.

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= -Y \\
a_2 &= \frac{1}{2}Y^2 + \frac{1}{12}Z^2 \\
a_3 &= -\frac{1}{6}Y^3 - \frac{1}{12}D_\mu Y D_\mu Y + \mathcal{O}(Y) \\
a_4 &= \frac{1}{24}Y^4 + \mathcal{O}(Y^3)
\end{aligned} \tag{69}$$

For the abnormal intrinsic parity part we only need the⁹ $\overline{D}_\mu a_n$ with $n = 3, 4, 5$.

$$\begin{aligned}
\overline{D}_\mu a_3 &= \frac{1}{12}[Y^2, D_\mu Y] - \frac{1}{60} \left([Y, D_\mu D^2 Y] - 2[D_\mu Y, D^2 Y] - [D_\mu D_\nu Y, D_\nu Y] \right) \\
&\quad + \frac{1}{30} (2\{Y, \{D_\nu Y, Z_{\mu\nu}\}\} + \{Z_{\mu\nu}|Y|D_\nu Y\}) \\
&\quad - \frac{1}{60} (3\{D_\nu Z_{\nu\mu}, Y^2\} + 4Y D_\nu Z_{\nu\mu} Y) + \mathcal{O}(Y)
\end{aligned} \tag{70}$$

$$\begin{aligned}
\overline{D}_\mu a_4 &= -\frac{1}{120} (3[Y^3, D_\mu Y] + [Y^2|D_\mu Y|Y]) + \frac{1}{120}[Y^2, D_\mu D^2 Y] \\
&\quad + \frac{1}{360} (6[Y|D^2 Y|D_\mu Y] + 5[D^2 Y|Y|D_\mu Y] + [D^2 Y|D_\mu Y|Y])
\end{aligned}$$

⁹This is not true for the Wess-Zumino term, see section 4.

$$\begin{aligned}
& + \frac{1}{180} (3[D_\nu Y D_\nu Y, D_\mu Y] + 2[Y|D_\nu Y|D_\mu D_\nu Y] + [D_\nu Y|Y|D_\mu D_\nu Y]) \\
& \quad - \frac{1}{180} (4Y\{D_\nu Y, Z_{\mu\nu}\}Y + 3\{Y^2, \{D_\nu Y, Z_{\mu\nu}\}\} \\
& \quad \quad + 2\{Y, \{D_\nu Y|Y|Z_{\mu\nu}\}\} + \{D_\nu Y|Y^2|Z_{\mu\nu}\}) \\
& \quad + \frac{1}{180} (2\{D_\nu Z_{\nu\mu}, Y^3\} + 3\{Y|D_\nu Z_{\nu\mu}|Y^2\}) + \mathcal{O}(Y^2)
\end{aligned} \tag{71}$$

$$\overline{D_\mu a_5} = \frac{1}{360} (2[Y^4, D_\mu Y] + [Y^3|D_\mu Y|Y]) + \mathcal{O}(Y^4) \tag{72}$$

Here I used the notation

$$[A|B|C] = ABC - CBA \quad \text{and} \quad \{A|B|C\} = ABC + CBA. \tag{73}$$

Appendix B Conventions for Euclidean expressions

In this appendix I give a short overview of the sign conventions used for gamma matrices and the relation to Minkowski quantities. The γ matrices are hermitian and satisfy

$$\{\gamma_\mu, \gamma_\nu\} = \delta_{\mu\nu} \tag{74}$$

The γ_5 is hermitian, anticommutes with all gamma matrices and has as defining trace

$$\text{tr} \gamma_5 \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta = 4\epsilon_{\mu\nu\alpha\beta} \tag{75}$$

The relation between Euclidean and Minkowski space parameters is

$$x_{0E} = ix_M^0 \quad x_{iE} = x_M^i \tag{76}$$

$$A_{0E} = -iA_{0M} \quad A_{iE} = -A_M^i. \tag{77}$$

In Euclidean space we have $e_{0123} = 1$ and for the Minkowski one I use $e^{0123} = 1$. Together with the above transformation for vector indices this leads to

$$(\epsilon_{\mu\nu\alpha\beta} A_\mu A_\nu A_\alpha A_\beta)_E = -i (\epsilon^{\mu\nu\alpha\beta} A_\mu A_\nu A_\alpha A_\beta)_M \tag{78}$$

Using these formulas it can be checked that these conventions give the correct sign and magnitude for the Wess-Zumino term.

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